LOCALLY COMPACT DIVISION RINGS

Edwin Weiss and Neal Zierler
Let $K$ be a division ring with a non-discrete topology $T$ with respect to which both the additive group $K^+$ and the multiplicative group $K^*$ of $K$ are locally compact topological groups.\(^1\) If $m$ is Haar measure for $K^+$ and $a \in K$, the function $m'(E) = m(aE)$ is clearly an invariant Borel measure for $K^+$. Hence there exists a real number $\phi(a)$ such that $m'(E) = \phi(a)m(E)$ for all Borel subsets $E$ of $K^+$. The real-valued function $\phi$ on $K$ (which is essentially the Radon-Nikodym derivative of $m$ with respect to left-invariant Haar measure on $K^*$) evidently has the first two of the following three properties.

1. $\phi(a) \geq 0$; $\phi(a) = 0$ if and only if $a = 0$.
2. $\phi(ab) = \phi(a)\phi(b)$.
3. There exists $M > 0$ such that $\phi(a) \leq 1$ implies $\phi(1 + a) \leq M$.

We shall show that $\phi$ satisfies (3) also, i.e., is a valuation for $K$, and that the topology $T_\phi$ for $K$ defined by $\phi$ coincides with $T$.\(^2\) The classification of $K$ then follows from known results.

**Lemma 1.** $\phi$ is continuous.

**Proof.** Let $\varepsilon$ be a positive number and let $E$ be a compact set of positive measure. By the regularity of Haar measure we may choose an open set $U$ containing $E$ such that $m(U) - m(E) < \varepsilon m(E)$. Choose a neighborhood $V$ of $1$ with $V = V^{-1}$ and $V \cdot E \subset U$. Then for $x$ in $V$, $\phi(x) = m(xE)/m(E) \leq m(U)/m(E) < 1 + \varepsilon$; since $x^{-1} \in V$, $\phi(x) = (\phi(x^{-1}))^{-1} > (1 + \varepsilon)^{-1}$. Hence $1 - \varepsilon < \phi(x) < 1 + \varepsilon$ and the continuity of $\phi$ on $K^*$ follows.\(^2\) Now choose an open set $U$ with $m(U) < \varepsilon m(E)$ and a neighborhood $V$ of $0$ with $V \cdot E \subset U$. Then for $a$ in $V$, $\phi(a) = m(aE)/m(E) \leq m(U)/m(E) < \varepsilon$ and $\phi$ is continuous at $0$.

**Lemma 2.** $S = \{a \in K : \phi(a) \leq 1\}$ is compact.

Received February 13, 1958 and in revised form April 1, 1958, the research in this paper was supported jointly by the Army, Navy and Air Force under contract with the Massachusetts Institute of Technology. The first author is a consultant, Lincoln Laboratory, M.I.T. The second author is a staff member, Lincoln Laboratory, M.I.T.

\(^1\) Continuity of the inverse multiplicative operation need not be assumed; cf. the concluding remark. The continuity of multiplication implies that $a \rightarrow -a = (-1) \cdot a$ is continuous.

\(^2\) This idea was suggested by some work of Tate, [12].

\(^2\) Cf. Halmos [3, §60.6, p. 265].
Proof. Let $C$ be a compact neighborhood of 0 and choose a neighborhood $V$ of 0 such that $V \cdot C \subset C$. Let $a \in V \cap C$ such that $0 < \phi(a) < 1$. If $a^n S \subset C$ holds for no $n = 1, 2, \ldots$, we select for each $n$ an $s_n \in S$ such that $a^n s_n \notin C$. Since $\phi(a^n) \to 0$ and all the $a^n$ lie in the compact set $C$, $a^n \to 0$ and hence $a^n s_n \in C$ for sufficiently large $k$. We may therefore choose $k_n \geq n$ such that $a^n s_n \notin C$ but $a^{n+1} s_n \in C$. Then the sequence $\{a^n s_n\}$ of elements of the compact set $a^{-1} C$ has a cluster point $c$ in $a^{-1} C$. Hence $\phi(a^n s_n) = \phi(a^n) \phi(s_n) \leq \phi(a)^n$ has $\phi(c)$ as a cluster point by the continuity of $\phi$; thus $\phi(c) = 0$ and $c = 0$, which contradicts $a^n s_n \notin C$. It follows that $S$ is a subset of the compact set $a^{-n} C$ for some $n$ and so, being closed by virtue of the continuity of $\phi$, is compact.

COROLLARY. $\phi$ is a valuation.

Proof. $\phi(1 + S)$, the continuous image of the compact set $1 + S$, is bounded.

Lemma 3. $T_\phi = T$.

Proof. Let $V \in T - \{\phi\}$, $a \in V$ and $B_n = \{b \in K: \phi(b-a) < 2^{-n}\}$. Suppose we can choose $b_n \in B_n$ with $b_n \notin V$ for each $n = 1, 2, \ldots$. But then the points $b_n - a$, all of which lie in the compact set $S$, have a cluster point $c$ in $S$ which must be 0 since $\phi(c) = 0$. Hence $b_n \to a$ contrary to our assumption and it follows that $T \subset T_\phi$. Since the opposite inclusion is an immediate consequence of the continuity of $\phi$, the proof is complete.

If $K$ is connected, it is the real, complex or quaternion field (Pontrjagin [10]); in particular, $\phi$ is archimedean. Conversely, if $\phi$ is archimedean, the theorem of Ostrowski [8, p. 278] asserts that the center of $K$ is either the real or complex field and so $K$, not being totally disconnected, is connected.

If $K$ is totally disconnected, $\phi$ is non-archimedean (and conversely, according to the above) and results due to van Dantzig [2], Hasse [4], Hasse and Schmidt [5], Jacobson and Taussky [6] and Jacobson [7] assert that $K$ is of one of the following three types:

(i) the completion of an algebraic number field at a finite prime,
(ii) the completion of an algebraic function field in one variable.

$^3$ $K$ is either connected or totally disconnected: if the component $C$ of 0 contains $a \notin 0$ then $b a^{-1} C$ is a connected set containing 0 and $b \in K$.

$^4$ Otobe [9] shows that $a \to a^{-1}$ need not be assumed to be continuous; cf. our final remark in this connection.

$^5$ Alternatively, if $K$ is connected, it is not difficult to show that $\phi$ is archimedean; then $K$ is a vector space over the reals (Ostrowski) with $\phi$ as a norm, hence is the real, complex or quaternion field (Arens [1] Tornheim [13]), proving Pontrjagin's theorem.
over a finite field $H$,

(iii) a division ring $D$ obtained from a field $F$ of type (ii) by redefining $x$. $a=a^\sigma x$, $a \in H$, $\sigma$ a fixed non-trivial automorphism of $H$, the elements of $D$ and $F$ being regarded as power series $\sum_{i=0}^{\infty} a_i x^i$ in an indeterminate $x$ over $H$ with coefficients in $H$.

Remark. Continuity of $a \rightarrow a^{-1}$ need not be assumed, for it appears in the connected case only in the proof that $K$ is not compact in the proof of the Pontrjagin theorem [11, p. 173, Theorem 45.]. If $K$ were compact, $\phi(a) = m(aK)/m(K) \leq 1$ for all $a \in K$. But, as in the proof of the continuity of $\phi$ at 0 in Lemma 1, we can find $a \in K$ such that $0 < \phi(a) < 1$; then $\phi(a^{-1}) > 1$ and it follows that $K$ is not compact. If $K$ is totally disconnected we have only to apply to $T, K^*$ the following unpublished theorem of A. M. Gleason: Let $G$ be a group with a totally disconnnected topology $T$ under which the group operation is continuous from $G \times G$ to $G$. Then $T, G$ is a topological group.

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8. A. Ostrowski, Über einige Lösungen der Funktionalgleichung $\phi(x)\phi(y)=\phi(xy)$, Acta Math., 41 (1917), 271-284.

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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