Pacific Journal of Mathematics

HOMOMORPHISMS ON NORMED ALGEBRAS

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Vol. 8, No. 2 April 1958

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1. **Introduction** Let B_1 and B be real normed Q-algebras (not necessarily complete) and T be a homomorphism of B_1 into B. Our main object is to show that, for certain algebras B, T will always be either continuous or closed if the range $T(B_1)$ contains "enough" of B. If B is the algebra of all bounded linear operators on a Banach space $\mathfrak{X}$ and $T(B_1)$ contains all finite-dimensional operators then T is continuous. If B is primitive with minimal one-sided ideals, $T(B_1)$ is dense in B and intersects at least one minimal ideal of B then T is closed. Other examples are given. In these results we can obtain the conclusion for ring homomorphism as well as algebra homomorphism if we assume that $\rho(T(x))\leqq\rho(x)$, $x\in B_1$, where $\rho(x)$ is the spectral radius of x. Note that this is a necessary condition for real-homogeneity. For the application of these results it is desirable to have examples of algebras which are Q-algebras in all possible normed algebra norms. Examples are given in § 2. For previous work on the continuity of homomorphisms and the homogeneity of isomorphisms on Banach algebras see [8], [9], [11], [12] and [14].

2. **Normed Q-algebras and continuity of homomorphisms.** For the algebraic notions used see [6]. Let B be a normed algebra over the real field (completeness is not assumed). As in [8], [11] a complex number $\lambda\neq 0$ is in the spectrum of $x\in B$ if it is in the usual complex algebra spectrum of $(x, 0)$ in the complexification of B. If B is already a complex algebra then the spectrum of x in this sense is the smallest set in the complex plane symmetric with respect to the real axis which contains the spectrum of x in the complex algebra sense. Let $\rho(x)$ be the *spectral radius* of x, $\rho(x)=\sup|\lambda|$ for λ in the spectrum of x. B is called a *Q-algebra* if the set of quasi-regular elements of B is open. Every regular maximal one-sided or two-sided ideal in a Q-algebra is closed. Hence the radical of a Q-algebra is closed and so also is any primitive ideal. See [10 ; 77].

2.1. LEMMA. *For a normed algebra B the following statements are equivalent.*

(a) *B is a Q-algebra.*

(b) $\rho(x)=\lim ||x^n||^{1/n}$, $x\in B$.

(c) $\rho(x)\leqq||x||$, $x\in B$.

Received January 14, 1958. This research was supported in part by the National Science Foundation, research grant NSF-G 2573.

Suppose (a). Then there exists a number $c>0$ such that x is quasi-regular for all x, $\|x\|<c$. Set $k=[(1+c)^{1/2}-1]^{-1}$. Let $x\in B$ and $\lambda=a+bi$ be any complex number $\neq 0$ where $|\lambda|>k\|x\|$. Then

$$|\lambda|^{-2}\,\|2ax-x^2\| \leqq |\lambda|^{-2}(2|\lambda|\ \|x\|+\|x\|^2) < 2k^{-1}+k^{-2}<c$$

This shows that $\rho(x)\leqq k\|x\|$. Thus

$$\rho(x)=\rho(x^n)^{1/n}\leqq k^{1/n}\|x^n\|^{1/n}$$

for every positive integer n. Letting $n\to\infty$ we see that $\rho(x)\leqq\lim \|x^n\|^{1/n}$. But $\lim\|x^n\|^{1/n}=\rho(x\,|\,B^c)$, the spectral radius of x in the completion B^c of B. Hence $\rho(x)\leqq\rho(x\,|\,B^c)$. Since $\rho(x\,|\,B^c)\leqq\rho(x)$, (b) follows. Clearly (b) implies (c). Suppose that (a) is false. Then there exists a sequence $\{x_n\}$, $x_n\to 0$ where x_n is not quasi-regular. Then $\rho(x_n)\geqq 1$ for each n and (c) is false.

Let $\mathfrak{X}$ be a Banach space and let $\mathfrak{E}(\mathfrak{X})$ be the Banach algebra of all bounded linear operators on $\mathfrak{X}$ in the uniform topology. Let $\mathfrak{F}(\mathfrak{X})$ be the ideal of all elements of $\mathfrak{E}(\mathfrak{X})$ with finite dimensional range.

2.2. Lemma. *Let j be an idempotent in a normed algebra B. Then the non-zero spectrum of an element in jBj is the same whether computed in jBj or B.*

This is given in [9 ; 375] in the complex case. The real case offers no new difficulty.

2.3. Theorem. *Let U be a ring homomorphism or anti-homomorphism of a normed Q-algebra B_1 into $\mathfrak{E}(\mathfrak{X})$ where $U(B_1)\supset\mathfrak{F}(\mathfrak{X})$ and $\rho[U(V)]\leqq\rho(V)$, $V\in B_1$. Then U is continuous.*

Suppose that U is not continuous. By the additivity of U (see [2; 54]) there exists a sequence $\{T_n\}$ in B_1 such that $\|T_n\|_1\to 0$ and $\|U(T_n)\|\to\infty$ where $\|T\|_1$ is the norm in B_1 and $\|T\|$ is the usual norm in $\mathfrak{E}(\mathfrak{X})$. Consider any idempotent J of $\mathfrak{E}(\mathfrak{X})$ such that $J\mathfrak{E}(\mathfrak{X})$ is a minimal right ideal of $\mathfrak{E}(\mathfrak{X})$. By the work of Arnold [1] these elements J are the linear operators on $\mathfrak{X}$ of the form $J(x)=x^*(x)y$ where $x^*\in\mathfrak{X}^*$, $y\in\mathfrak{X}$ and $x^*(y)=1$. Let $U(W)=J$ and $U(T_n)=V_n$. Since $\|WT_nW\|_1\to 0$ we have, by Lemma 2.1, $\rho(WT_nW)\to 0$ and therefore $\rho(JV_nJ)\to 0$. By Lemma 2.2 and the Gelfand-Mazur theorem, $\|JV_nJ\|\to 0$. Note that $JV_nJ(x)=x^*(x)\,x^*[V_n(y)]y$. Hence $x^*[V_n(y)]\to 0$. Fix $y\neq 0$ in $\mathfrak{X}$. Then $x^*[V_n(y)]\to 0$ for all $x^*\in K=\{x^*\in\mathfrak{X}^*\,|\,x^*(y)\neq 0\}$. Let $z^*\in\mathfrak{X}^*$, $z^*(y)=0$. Since z^* can be written as the sum of two elements of K, $x^*[V_n(y)]\to 0$ for all $x^*\in\mathfrak{X}^*$. Hence $\sup\|V_n(y)\|<\infty$ for each $y\in\mathfrak{X}$. By the uniform boundedness theorem, $\sup\|V_n\|<\infty$. This is a contradiction.

2.4. Theorem. *Let T be a ring homomorphism or anti-homomorphism of a normed Q-algebra onto a dense subring of a semi-simple* *finitedimensional normed algebra B where $\rho[T(x)]\leqq\rho(x)$, $x\in B_1$. Then T is continuous.*

By [7 ; 698] B is strongly semi-simple and so, by Theorem proved below, T is real-homogenous and closed. Let $||x||_1$ ($||x||$) denote the norm in $B_1(B)$. Suppose that T is not continuous. Then there exists a sequence $\{x_n\}$ in B_1 such that $||x_n||_1\to 0$ and $||T(x_n)||=1$, $n=1, 2, \cdots$. There exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $||T(y_n)-w||\to 0$ for some $w\in B$. Since $||w||=1$ we contradict the fact that T is a closed mapping.

A normed algebra B is called a *permanent Q-algebra* if it is a Q-algebra in all normed algebra norms. We say that the normed algebra B has the *spectral extension property* if the spectral radius of $x\in B$ is the same as the spectral radius of x considered as an element of any Banach algebra B_1 in which B may be algebraically imbedded. Examples of algebras with this property are B^*-algebras [13] and annihilator Banach algebras [3]. To test if a normed algebra B has this property it is sufficient to consider the completions of B in all possible normed algebra norms.

2.5. LEMMA. *A normed algebra B is a permanent Q-algebra if and only if B has the spectral extension property.*

Let B be a permanent Q-algebra, $x\in B$. Then $\lim ||x^n||^{1/n}$ has the same value $\rho(x)$, by Lemma 2.1, for any normed algebra norm for B. Thus B has the spectral extension property. If B has the latter property then for any norm $||x||$, $\rho(x)=\lim ||x^n||^{1/n}$ and B is a permanent Q-algebra by Lemma 2.1.

2.6. THEOREM. *Any two sided ideal I of $\mathfrak{E}(\mathfrak{X})$ where $I\supset\mathfrak{F}(\mathfrak{X})$ and any closed subalgebra B of $\mathfrak{E}(\mathfrak{X})$, $B\supset\mathfrak{F}(\mathfrak{X})$ have the spectral extension property.*

Let R be any such ideal I or closed subalgebra B. Let $||T||_1$ be a normed algebra norm for R and $||T||$ the usual norm. For $T\in R$ let $\rho(T)$ be its spectral radius as an element of R, $\rho_1(T)$ as an element of the completion of R in the norm $||T||_1$ and $\rho_2(T)$ as an element $\mathfrak{E}(\mathfrak{X})$. In the ideal case if $U\in R$ has a quasi-inverse V in $\mathfrak{E}(\mathfrak{X})$ then $V\in R$. In every case $\rho(T)=\rho_2(T)$.

It is enough to show the identity imbedding of R (with norm $||T||_1$) into $\mathfrak{E}(\mathfrak{X})$ (with norm $||T||$) is continuous. For then there exists $c>0$, $||T||\leqq c||T||_1$, $T\in R$, whence

$$||T^n||^{1/n}\leqq c^{1/n}||T^n||_1^{1/n}$$

for all positive integers n. Consequently $\rho(T)\leqq\rho_1(T)$. Since $\rho_1(T)\leqq\rho(T)$ we would have $\rho(T)=\rho_1(T)$.

Theorem 2.3 cannot be applied since it is not known *a priori* that R is a Q-algebra in the norm $\|T\|_1$. If, however, the imbedding is discontinuous there exists a sequence $\{T_n\}$ in R such that $\|T_n\|_1 \to 0$ and $\|T_n\| \to \infty$. By the arguments of [1], the minimal ideals of R are the same as the minimal ideals of $\mathfrak{C}(\mathfrak{X})$. For each idempotent generator J of a minimal right ideal of R, JRJ is a normed division algebra and hence has a unique norm topology by the Gelfand-Mazur theorem. Since $\|JT_nJ\|_1 \to 0$ we have $\|JT_nJ\| \to 0$. The remainder of the proof may be handled as in Theorem 2.3.

For a ring B and a subset $A \subset B$ we denote the left (right) annihilator of A by $L(A)$ $(R(A))$. Bonsall and Goldie [4] have considered topological rings called annihilator rings in which for each proper right (left) closed ideal I, $L(I) \neq (0)$ $(R(I) \neq (0))$. We consider the related purely algebraic concept of a *modular annihilator ring* which is defined to be a ring in which $L(M) \neq (0)$ $(R(M) \neq (0))$ for every regular maximal right (left) ideal. From the standpoint of algebra these rings appear to be a natural class containing H^*-algebras, etc. In view of what follows it is natural to ask if the two concepts agree for semi-simple normed Q-algebras or semi-simple Banach algebras. A affirmative answer would settle an unsolved problem in the theory of annihilator algebras.

2.7. LEMMA. *Let B be a semi-simple normed annihilator Q-algebra and I be a closed two-sided ideal in B. Then I is a modular annihilator Q-algebra.*

Thus if we had affirmative answer to the above question, any closed two-sided ideal of a semi-simple annihilator Banach algebra would also be one. The analogous result is known for dual algebras [7; 690].

Let M be a regular maximal right ideal of I. Since I is a Q-algebra (as an ideal in B), M is closed in B. Since $L(I) = R(I)$, ([4; 159]), $L(I + R(I)) = (0)$ so that $I + R(I)$ is dense. The arguments of [7; Theorem 2] show that M is a right ideal in B. We must show $L(M) \cap I \neq (0)$. Suppose the contrary. Then $I\ L(M) = (0)$ and $L(M) \subset R(I) = L(I)$. As $M \subset I$, $L(M) \supset L(I)$. Therefore $L(M) = L(I)$. $R(M)M = (0)$ since it is a nilpotent ideal in B. Thus $R(M) \subset L(M) = R(I)$. Then since $R(M) \supset R(I)$ we see that $R(M) = L(M)$. If $x \in L(M + R(M))$ then $x \in L(M) = R(M)$ and $x \in LR(M)$. Thus $x^2 = 0$ and, by semi-simplicity and the annihilator property, $M + R(M)$ is dense in B. Then $(M + R(M))\ I = (M + L(I))I \subset M$ and $BI \subset M$. Let j be a left identity for I modulo M. Then $jx - x \in M$, $x \in I$ and $jx \in M$, $x \in I$. Hence $I \subset M$ which is a contradiction.

2.8. LEMMA. *In a semi-simple modular annihilator ring, every proper right (left) ideal contains a minimal right (left) ideal. A normed* *modular annihilator algebra B has the spectral extension property.*

Since the first statement is shown by stripping the arguments of Bonsall and Goldie [4] of all topological connotations, a sketch of the argument is sufficient. As in [4, Lemma 2], if j is not right (left) quasi-regular there exists $x\neq 0$ in B where $xj=x(jx=x)$. The arguments of [4, Theorem 1] show that if M is a regular maximal right (left) ideal of B then $L(M)$ $(R(M))$ is a minimal left (right) ideal generated by an idempotent. Also the left (right) annihilator of a minimal right (left) ideal is a regular maximal left (right) ideal. Consider the socle K of B. By the reasoning of [4, Theorem 4], $L(K)=R(K)=(0)$. Let I be a proper right ideal of B. If I contained no minimal right ideals of B then, as in the proof of [4, Lemma 4], $I\subset L(K)$, which is impossible.

Let $x\in B$ and let B' be the completion of B in the normed algebra norm $||x||_1$. Consider $\lambda=a+bi\neq 0$ in $sp(x|B)$. Then $u=|\lambda|^{-2}(2ax-x^2)$ has no quasi-inverse in B. As in [3 ; p 159] there exists $y\neq 0$ such that $uy=y$ and u has no quasi-inverse in B'. Then $\rho(x|B')=\rho(x|B)$.

3. **Closure of homomorphisms and anti-homomorphisms.** Throughout this section the following notation is assumed. Let $B_1(B)$ be a real normed algebra with norm $||x||_1\,(||x||)$. T is a ring homomorphism or anti-homomorphism of B_1 *onto a dense subset of* B. T is called closed if $||x_n-x||_1\to 0$, $||T(x_n)-y||\to 0$ imply that $y\in T(B_1)$ and $y=T(x)$. By the *separating set* S of T we mean the set of all $y\in B$ such that there exists a sequence $\{x_n\}$ in B_1 where $||x_n||_1\to 0$ and $||y-T(x_n)||\to 0$. We assume that $\rho[T(x)]\leqq\rho(x)$, $x\in B_1$. Note that this condition is automatic if T is real-linear.

The next lemma is an adaptation of results of Rickart [11].

3.1. LEMMA. *T is closed and real-homogeneous if and only if $S=(0)$. S is a closed two-sided ideal in B and $T^{-1}(S)$ a closed two-sided ideal in B_1. If B_1 is a normed Q-algebra then every element of S is a topological divisor of zero in B.*

Clearly T is rational-homogeneous. Let $x\in B_1$ and $r_n\to r$ where each r_n is rational and r is real. Then $||r_nx-rx||_1\to 0$ and $||rT(x)-T(rx)-T(r_nx-rx)||\to 0$. Hence $rT(x)-T(rx)\in S$. The first statement follows by a straightforward argument.

Let $y_n\in S$, $||w-y_n||\to 0$. There exists, for each n, an element $z_n\in B_1$ such that $||y_n-T(z_n)||<n^{-1}$ and $||z_n||_1<n^{-1}$. Then $||w-T(z_n)||\to 0$ so that $w\in S$. Hence S is closed in B. Since $x\in S$ and r rational imply $rx\in S$ it follows that S is a real linear manifold. To show that S is an ideal in B it is enough to show that xy and $yx\in S$ for $x\in S$ and $y=T(z)\in T(B_1)$. This, however, is a simple matter. Suppose next that $||x_n-x||_1\to 0$ where each $x_n\in T^{-1}(S)$. For each n there exists $y_n\in B_1$ such that $||T(x_n)-T(y_n)||<n^{-1}$ and $||y_n||_1<n^{-1}$. Then $||x-(x_n-y_n)||_1\to 0$ while

$||T(x)-T[x-(x_n-y_n)]||\to 0$ whence $T(x)\in S$. Hence $T^{-1}(S)$ is closed. It is readily seen to be a two-sided ideal in B_1.

Let B^c be the completion of B where we use $||x||$ to denote the norm in B^c and $\rho(x)$ the spectral radius there. To show that $s\in S$ is a topological divisor of zero in B it is sufficient to show that it is one in B^c. Choose a sequence $\{x_n\}$ in B_1 such that $||s-T(x_n)||\to 0$ and $||x_n||_1\to 0$. If B_1 is a normed Q-algebra s is the limit of quasi-regular elements of B^c by Lemma 2.1. Hence so also is λs for any real λ. By the arguments of [11 ; 621] it suffices to rule out the possibility that both B^c has an identity 1 and that s has a two-sided inverse in B^c.

Suppose this is the case. Let S_0 be the separating set for T considered as a mapping of B_1 into B^c. Clearly $S\subset S_0$. Then as S_0 is an ideal in B^c, $S_0=B^c$ and $1\in S_0$. There exists a sequence $\{u_n\}$ in B_1 such that $||1-T(u_n)||\to 0$ and $||u_n||_1\to 0$. Since $1-T(u_n)$ and $T(u_n)$ permute we have by Lemma 2.1,

$$1=\rho(1)\leqq \rho(1-T(u_n))+\rho(T(u_n))\leqq ||1-T(u_n)||+\rho(u_n|B_1)\to 0$$

This contradication completes the argument.

If B_1 and B are Banach algebras, by the closed graph theorem [2 ; 41] $S=(0)$ will imply that T is continuous. In every case $S=(0)$ will imply real-homogeneity for T and the closure of $T^{-1}(0)$.

3.2. LEMMA. *Let B_1 be a normed Q-algebra and B be semi-simple with minimal one-sided ideals. Suppose that there exists a minimal one-sided ideal I of B_1 such that $T(B_1)\cap I\neq 0$. Then $S\cap I=(0)$.*

We consider the case where I is a right ideal and T is a homomorphism. The other cases follow by the reasoning employed. Set $I_1=T^{-1}(I)$. I_1 is a right (ring) ideal of B_1. Let $I=jB$, $j^2=j$ and consider $x_0\in I_1$ where $T(x_0)=jv\neq 0$. By the semi-simplicity of B, $jvB\neq(0)$ and, as jB is minimal, $jvB=jB$. Then $jvT(B_1)$ is dense in I. It follows that $T(I_1^2)\neq(0)$ for otherwise $[jvT(B_1)]^2=(0)$ and $I^2=(0)$. Select $x\in I_1$, $T(x)=jw\neq 0$ and $T(x^2)\neq 0$. Let R be the set of elements y in B for which $jy\in T(I_1)$. As observed, jR is dense in jB. Hence jRj is dense in jBj. But jBj is a normed division algebra and therefore, by the Gelfand-Mazur theorem, finite-dimensional in B. Thus $jRj=jBj$. There exists $z\in R$ such that $jzjwj=jwjzj=j$. For some $x_1\in I_1$, $T(x_1)j=jzj$. Then $T(x_1x)=jzjw=T((x_1x)^2)$. Set $jzjw=h$ and $x_1x=u$. Then h is a non-zero idempotent in $I\cap T(B_1)$. Clearly $hB=I$ so that hBh is a division algebra hence isomorphic to the reals, complexes or quaternions.

We show that $h\notin S$. For suppose otherwise. Then there exists a sequence $\{y_n\}$ in B_1 such that $||h-T(y_n)||\to 0$ and $||y_n||_1\to 0$. Thus $||uy_nu||_1\to 0$ and $||h-T(uy_nu)||\to 0$. By Lemma 2.2 and the fact that hBh is the reals, complexes or quaternions, $||hT(y_n)h||\to 0$. This is a contradiction as $h \neq 0$. Now $S \cap I$ is a right ideal of B, $S \cap I \neq I$. Since I is minimal, $S \cap I = (0)$.

3.3. THEOREM. *Let B_1 be a normed Q-algebra and B be primitive with minimal one-sided ideals. If $T(B_1) \cap I \neq (0)$ for a minimal one-sided ideal I of B then T is closed and real-homogeneous.*

Let K be the socle of B. If $S \neq (0)$ then $K \subset S$ by [6; 75]. Then $I \subset S$ which is impossible by Lemma 3.2.

3.4. COROLLARY. *Let B be any subalgebra of $\mathfrak{C}(\mathfrak{X})$ closed in the uniform norm $\|T\|$ where $B \supset \mathfrak{F}(\mathfrak{X})$. Let $\|T\|_1$ be any normed algebra norm for B such that the completion B^c of B in this norm is primitive. Then the two norms are equivalent.*

By Theorem 2.6 and Lemma 2.5, B is a Q-algebra in the norm $\|T\|_1$. By Theorem 2.3, there exists $c>0$ such that $\|T\| \leqq c\|T\|_1$, $T \in B$. Consider the embedding mapping I of B (with norm $\|T\|$) into B^c. B is a primitive algebra with a minimal right ideal JB, $J^2=J$. Then $I(J)I(B)I(J)$ a normed division algebra and, by the Gelfand-Mazur theorm, closed in B^c. Since $I(J)$ is an idempotent, its closure in B^c is $I(J)B^cI(J)$. Therefore $I(J)B^c$ is a minimal right ideal of B^c. From Theorem 3.3, I is closed. The closed graph theorem [2; 41] shows that I is continuous. Hence there exists $c_1>0$ such that $\|T\|_1 \leqq c_1\|T\|$, $T \in B$.

3.5. THEOREM. *Let B_1 and B be normed Q-algebras. Then S is contained in the Brown-McCoy radical of B. If B is strongly semi-simple then T is closed and real-homogeneous.*

The Brown-McCoy radical [5] coincides with the intersection of the regular maximal two-sided ideals of B. Let M be such an ideal of B. Since B is a normed Q-algebra, M is closed. Let π be the natural homomorphism of B onto B/M. Since $T(B_1)$ is dense in B, then $\pi\, T(B_1)$ is dense in B/M. Also $\rho[\pi T(x)] \leqq \rho[T(x)] \leqq \rho(x)$, $x \in B_1$. Hence our theory applies to the mapping πT.

Let S_0 be the separating set for πT. Since B/M is simple with an identity, $S_0=(0)$ by Lemma 3.1. Let $y \in S$, $\|x_n\|_1 \to 0$, $\|y - T(x_n)\| \to 0$. Then $\|\pi(y) - \pi T(x_n)\| \to 0$ or $\pi(y) \in S_0$. Therefore $S \subset M$. B is called *strongly semi-simple* if its Brown-McCoy radical is (0).

3.6. THEOREM. *Let B_1 and B be semi-simple normed Q-algebras where B_1 has a dense socle K and B has an identity Let T be real-linear. Then Tis closed.*

Let P be a primitive ideal of B and π be the natural homomorphism of B onto B/P. Since B is a Q-algebra then P is closed, π is continuous and $\pi T(B_1)$ is dense in B/P. Let S_0 be the separating set for πT

as a mapping of B_1 into B/P. We show first that $T(K)\subset P$ is impossible. Suppose $T(K)\subset P$. Since $K\subset(\pi T)^{-1}(S_0)$, by Lemma 3.1 we have $B_1=(\pi T)^{-1}(S_0)$ and $S_0=B/P$. Since B/P has an identity this is contrary to Lemma 3.1. Hence there exists a minimal right ideal jB_1 of B_1, $j^2=j$ such that $T(j)\notin P$. Set $\pi T(j)=u$, $\pi T(B_1)=B_2$. πT is an isomorphism or anti-isomorphism of the division algebra jB_1j onto uB_2u. Hence uB_2u is a normed division algebra and thus, by the Gelfand-Mazur theorem closed in B/P. Since u is an idempotent, $u(B/P)$ is a minimal right ideal of B/P. By Theorem 3.3, πT is closed from which we obtain $S\subset P$. Since B is semi-simple, $S=(0)$.

3.7. THEOREM. *Let B_1 be a normed Q-algebra and B semi-simple where either B is a modular annihilator algebra or has dense socle. If $T(B_1)$ contains the socle of B then T is closed and real-homogeneous.*

By Lemma 3.2, $S\cap I=(0)$ for every minimal one-sided ideal of B. Let I be a minimal right ideal. Then $SI=(0)$. Thus S annihilates the socle. It follows (see the proof of Lemma 2.8) that $S=(0)$ in the first case. In the second case we have $S^2=(0)$ and $S=(0)$ by semi-simplicity.

Consider further a semi-simple normed modular annihilator algebra B. B is a permanent Q-algebra by Lemma 2.5 and 2.8. From Theorem 3.7 we see that any algebraic homomorphism or anti-homomorphism of B onto B is closed no matter which two norms are used for B.

Let B be a real normed algebra. By an *involution* on B we mean a mapping $x\to x^*$ of B onto B which is a real-linear automorphism or anti-automorphism of period two. Let $H(K)$ be the set of self-adjoint (skew) elements of B with respect to the involution $x\to x^*$. B is the direct sum $H\oplus K$ of the linear manifolds H and K.

The mapping $x\to x^*$ of B onto B is subject to the above analysis. Here S is the set of all $x\in B$ for which there exists a sequence $\{x_n\}$ in B with $\|x_n\|\to 0$ and $\|x-x_n^*\|\to 0$.

3.8. LEMMA. $S=\overline{H}\cap\overline{K}$. *$S=(0)$ if and only if H and K are closed.*

Let $w\in S$. Then there exist sequences $\{h_n\}$ and $\{k_n\}$ in H and K respectively such that $\|w-(h_n-k_n)\|\to 0$ and $\|h_n+k_n\|\to 0$. Therefore $\|w-2h_n\|\to 0$ and $\|w+2k_n\|\to 0$ so $w\in\overline{H}\cap\overline{K}$. Conversely suppose that $\|z-h_n\|\to 0$, $\|z-k_n\|\to 0$ where each $h_n\in H$, $k_n\in K$. Then $\|z-(h_n+k_n)/2\|\to 0$ and $\|(h_n-k_n)/2\|\to 0$ and $z\in S$.

If H and K are closed, clearly $S=(0)$. Suppose $S=(0)$. Let $h_n\to u+v$ where $h_n\in H$, $u\in H$ and $v\in K$. Then $h_n-u\to v$ and $v\in\overline{H}\cap\overline{K}$. Then $v=0$ and H is closed. Similarly K is closed.

Let B be a semi-simple normed annihilator algebra, for example an H^*-algebra. Then it follows from the above that H and K are closed in B for any involution on B and any normed algebra norm on B. For

B^*-algebras we have been able to show only the following weaker result.

3.9. THEOREM. *Let B be a B^*-algebra with $H(K)$ as the set of self-adjoint (skew) elements in the defining involution for B. Then H and K are closed in any normed algebra norm topology for B.*

B has the spectral extension property [13] and is therefore a permanent Q-algebra by Lemma 2.5. The arguments of [14; § 3] can be adapted to show that H and K are closed in any given normed algebra norm $\|x\|_1$.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 8, No. 2 April, 1958