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HOMOMORPHISMS ON NORMED ALGEBRAS

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1. Introduction Let B_1 and B be real normed Q -algebras (not necessarily complete) and T be a homomorphism of B_1 into B . Our main object is to show that, for certain algebras B , T will always be either continuous or closed if the range $T(B_1)$ contains "enough" of B . If B is the algebra of all bounded linear operators on a Banach space \mathfrak{X} and $T(B_1)$ contains all finite-dimensional operators then T is continuous. If B is primitive with minimal one-sided ideals, $T(B_1)$ is dense in B and intersects at least one minimal ideal of B then T is closed. Other examples are given. In these results we can obtain the conclusion for ring homomorphism as well as algebra homomorphism if we assume that $\rho(T(x)) \leq \rho(x)$, $x \in B_1$, where $\rho(x)$ is the spectral radius of x . Note that this is a necessary condition for real-homogeneity. For the application of these results it is desirable to have examples of algebras which are Q -algebras in all possible normed algebra norms. Examples are given in §2. For previous work on the continuity of homomorphisms and the homogeneity of isomorphisms on Banach algebras see [8], [9], [11], [12] and [14].

2. Normed Q -algebras and continuity of homomorphisms. For the algebraic notions used see [6]. Let B be a normed algebra over the real field (completeness is not assumed). As in [8], [11] a complex number $\lambda \neq 0$ is in the spectrum of $x \in B$ if it is in the usual complex algebra spectrum of $(x, 0)$ in the complexification of B . If B is already a complex algebra then the spectrum of x in this sense is the smallest set in the complex plane symmetric with respect to the real axis which contains the spectrum of x in the complex algebra sense. Let $\rho(x)$ be the *spectral radius* of x , $\rho(x) = \sup |\lambda|$ for λ in the spectrum of x . B is called a Q -algebra if the set of quasi-regular elements of B is open. Every regular maximal one-sided or two-sided ideal in a Q -algebra is closed. Hence the radical of a Q -algebra is closed and so also is any primitive ideal. See [10; 77].

2.1. LEMMA. *For a normed algebra B the following statements are equivalent.*

- (a) B is a Q -algebra.
- (b) $\rho(x) = \lim \|x^n\|^{1/n}$, $x \in B$.
- (c) $\rho(x) \leq \|x\|$, $x \in B$.

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Suppose (a). Then there exists a number $c > 0$ such that x is quasi-regular for all x , $\|x\| < c$. Set $k = [(1+c)^{1/2} - 1]^{-1}$. Let $x \in B$ and $\lambda = a + bi$ be any complex number $\neq 0$ where $|\lambda| > k\|x\|$. Then

$$|\lambda|^{-2} \|2ax - x^2\| \leq |\lambda|^{-2} (2|\lambda| \|x\| + \|x\|^2) < 2k^{-1} + k^{-2} < c$$

This shows that $\rho(x) \leq k\|x\|$. Thus

$$\rho(x) = \rho(x^n)^{1/n} \leq k^{1/n} \|x^n\|^{1/n}$$

for every positive integer n . Letting $n \rightarrow \infty$ we see that $\rho(x) \leq \lim \|x^n\|^{1/n}$. But $\lim \|x^n\|^{1/n} = \rho(x|B^c)$, the spectral radius of x in the completion B^c of B . Hence $\rho(x) \leq \rho(x|B^c)$. Since $\rho(x|B^c) \leq \rho(x)$, (b) follows. Clearly (b) implies (c). Suppose that (a) is false. Then there exists a sequence $\{x_n\}$, $x_n \rightarrow 0$ where x_n is not quasi-regular. Then $\rho(x_n) \geq 1$ for each n and (c) is false.

Let \mathfrak{X} be a Banach space and let $\mathfrak{C}(\mathfrak{X})$ be the Banach algebra of all bounded linear operators on \mathfrak{X} in the uniform topology. Let $\mathfrak{F}(\mathfrak{X})$ be the ideal of all elements of $\mathfrak{C}(\mathfrak{X})$ with finite dimensional range.

2.2. LEMMA. *Let j be an idempotent in a normed algebra B . Then the non-zero spectrum of an element in jBj is the same whether computed in jBj or B .*

This is given in [9; 375] in the complex case. The real case offers no new difficulty.

2.3. THEOREM. *Let U be a ring homomorphism or anti-homomorphism of a normed Q -algebra B_1 into $\mathfrak{C}(\mathfrak{X})$ where $U(B_1) \supset \mathfrak{F}(\mathfrak{X})$ and $\rho[U(V)] \leq \rho(V)$, $V \in B_1$. Then U is continuous.*

Suppose that U is not continuous. By the additivity of U (see [2; 54]) there exists a sequence $\{T_n\}$ in B_1 such that $\|T_n\|_1 \rightarrow 0$ and $\|U(T_n)\| \rightarrow \infty$ where $\|T\|_1$ is the norm in B_1 and $\|T\|$ is the usual norm in $\mathfrak{C}(\mathfrak{X})$. Consider any idempotent J of $\mathfrak{C}(\mathfrak{X})$ such that $J\mathfrak{C}(\mathfrak{X})$ is a minimal right ideal of $\mathfrak{C}(\mathfrak{X})$. By the work of Arnold [1] these elements J are the linear operators on \mathfrak{X} of the form $J(x) = x^*(x)y$ where $x^* \in \mathfrak{X}^*$, $y \in \mathfrak{X}$ and $x^*(y) = 1$. Let $U(W) = J$ and $U(T_n) = V_n$. Since $\|WT_nW\|_1 \rightarrow 0$ we have, by Lemma 2.1, $\rho(WT_nW) \rightarrow 0$ and therefore $\rho(JV_nJ) \rightarrow 0$. By Lemma 2.2 and the Gelfand-Mazur theorem, $\|JV_nJ\| \rightarrow 0$. Note that $JV_nJ(x) = x^*(x)x^*[V_n(y)]y$. Hence $x^*[V_n(y)] \rightarrow 0$. Fix $y \neq 0$ in \mathfrak{X} . Then $x^*[V_n(y)] \rightarrow 0$ for all $x^* \in K = \{x^* \in \mathfrak{X}^* | x^*(y) \neq 0\}$. Let $z^* \in \mathfrak{X}^*$, $z^*(y) = 0$. Since z^* can be written as the sum of two elements of K , $x^*[V_n(y)] \rightarrow 0$ for all $x^* \in \mathfrak{X}^*$. Hence $\sup \|V_n(y)\| < \infty$ for each $y \in \mathfrak{X}$. By the uniform boundedness theorem, $\sup \|V_n\| < \infty$. This is a contradiction.

2.4. THEOREM. *Let T be a ring homomorphism or anti-homomorphism of a normed Q -algebra onto a dense subring of a semi-simple*

finitedimensional normed algebra B where $\rho[T(x)] \leq \rho(x)$, $x \in B_1$. Then T is continuous.

By [7 ; 698] B is strongly semi-simple and so, by Theorem proved below, T is real-homogenous and closed. Let $\|x\|_1$ ($\|x\|$) denote the norm in $B_1(B)$. Suppose that T is not continuous. Then there exists a sequence $\{x_n\}$ in B_1 such that $\|x_n\|_1 \rightarrow 0$ and $\|T(x_n)\| = 1$, $n = 1, 2, \dots$. There exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\|T(y_n) - w\| \rightarrow 0$ for some $w \in B$. Since $\|w\| = 1$ we contradict the fact that T is a closed mapping.

A normed algebra B is called a *permanent Q-algebra* if it is a Q-algebra in all normed algebra norms. We say that the normed algebra B has the *spectral extension property* if the spectral radius of $x \in B$ is the same as the spectral radius of x considered as an element of any Banach algebra B_1 in which B may be algebraically imbedded. Examples of algebras with this property are B^* -algebras [13] and annihilator Banach algebras [3]. To test if a normed algebra B has this property it is sufficient to consider the completions of B in all possible normed algebra norms.

2.5. LEMMA. *A normed algebra B is a permanent Q-algebra if and only if B has the spectral extension property.*

Let B be a permanent Q-algebra, $x \in B$. Then $\lim \|x^n\|^{1/n}$ has the same value $\rho(x)$, by Lemma 2.1, for any normed algebra norm for B . Thus B has the spectral extension property. If B has the latter property then for any norm $\|x\|$, $\rho(x) = \lim \|x^n\|^{1/n}$ and B is a permanent Q-algebra by Lemma 2.1.

2.6. THEOREM. *Any two sided ideal I of $\mathfrak{G}(\mathfrak{X})$ where $I \supset \mathfrak{F}(\mathfrak{X})$ and any closed subalgebra B of $\mathfrak{G}(\mathfrak{X})$, $B \supset \mathfrak{F}(\mathfrak{X})$ have the spectral extension property.*

Let R be any such ideal I or closed subalgebra B . Let $\|T\|_1$ be a normed algebra norm for R and $\|T\|$ the usual norm. For $T \in R$ let $\rho(T)$ be its spectral radius as an element of R , $\rho_1(T)$ as an element of the completion of R in the norm $\|T\|_1$ and $\rho_2(T)$ as an element $\mathfrak{G}(\mathfrak{X})$. In the ideal case if $U \in R$ has a quasi-inverse V in $\mathfrak{G}(\mathfrak{X})$ then $V \in R$. In every case $\rho(T) = \rho_2(T)$.

It is enough to show the identity imbedding of R (with norm $\|T\|_1$) into $\mathfrak{G}(\mathfrak{X})$ (with norm $\|T\|$) is continuous. For then there exists $c > 0$, $\|T\| \leq c \|T\|_1$, $T \in R$, whence

$$\|T^n\|^{1/n} \leq c^{1/n} \|T^n\|_1^{1/n}$$

for all positive integers n . Consequently $\rho(T) \leq \rho_1(T)$. Since $\rho_1(T) \leq \rho(T)$ we would have $\rho(T) = \rho_1(T)$.

Theorem 2.3 cannot be applied since it is not known *a priori* that R is a Q -algebra in the norm $\|T\|_1$. If, however, the imbedding is discontinuous there exists a sequence $\{T_n\}$ in R such that $\|T_n\|_1 \rightarrow 0$ and $\|T_n\| \rightarrow \infty$. By the arguments of [1], the minimal ideals of R are the same as the minimal ideals of $\mathcal{G}(\mathfrak{X})$. For each idempotent generator J of a minimal right ideal of R , JRJ is a normed division algebra and hence has a unique norm topology by the Gelfand-Mazur theorem. Since $\|JT_nJ\|_1 \rightarrow 0$ we have $\|JT_nJ\| \rightarrow 0$. The remainder of the proof may be handled as in Theorem 2.3.

For a ring B and a subset $A \subset B$ we denote the left (right) annihilator of A by $L(A)$ ($R(A)$). Bonsall and Goldie [4] have considered topological rings called annihilator rings in which for each proper right (left) closed ideal I , $L(I) \neq (0)$ ($R(I) \neq (0)$). We consider the related purely algebraic concept of a *modular annihilator ring* which is defined to be a ring in which $L(M) \neq (0)$ ($R(M) \neq (0)$) for every regular maximal right (left) ideal. From the standpoint of algebra these rings appear to be a natural class containing H^k -algebras, etc. In view of what follows it is natural to ask if the two concepts agree for semi-simple normed Q -algebras or semi-simple Banach algebras. A affirmative answer would settle an unsolved problem in the theory of annihilator algebras.

2.7. LEMMA. *Let B be a semi-simple normed annihilator Q -algebra and I be a closed two-sided ideal in B . Then I is a modular annihilator Q -algebra.*

Thus if we had affirmative answer to the above question, any closed two-sided ideal of a semi-simple annihilator Banach algebra would also be one. The analogous result is known for dual algebras [7; 690].

Let M be a regular maximal right ideal of I . Since I is a Q -algebra (as an ideal in B), M is closed in B . Since $L(I) = R(I)$, ([4; 159]), $L(I + R(I)) = (0)$ so that $I + R(I)$ is dense. The arguments of [7; Theorem 2] show that M is a right ideal in B . We must show $L(M) \cap I \neq (0)$. Suppose the contrary. Then $I \cap L(M) = (0)$ and $L(M) \subset R(I) = L(I)$. As $M \subset I$, $L(M) \supset L(I)$. Therefore $L(M) = L(I)$. $R(M)M = (0)$ since it is a nilpotent ideal in B . Thus $R(M) \subset L(M) = R(I)$. Then since $R(M) \supset R(I)$ we see that $R(M) = L(M)$. If $x \in L(M + R(M))$ then $x \in L(M) = R(M)$ and $x \in LR(M)$. Thus $x^2 = 0$ and, by semi-simplicity and the annihilator property, $M + R(M)$ is dense in B . Then $(M + R(M))I = (M + L(I))I \subset M$ and $BI \subset M$. Let j be a left identity for I modulo M . Then $jx - x \in M$, $x \in I$ and $jx \in M$, $x \in I$. Hence $I \subset M$ which is a contradiction.

2.8. LEMMA. *In a semi-simple modular annihilator ring, every proper right (left) ideal contains a minimal right (left) ideal. A normed*

modular annihilator algebra B has the spectral extension property.

Since the first statement is shown by stripping the arguments of Bonsall and Goldie [4] of all topological connotations, a sketch of the argument is sufficient. As in [4, Lemma 2], if j is not right (left) quasi-regular there exists $x \neq 0$ in B where $xj = x(jx = x)$. The arguments of [4, Theorem 1] show that if M is a regular maximal right (left) ideal of B then $L(M)$ ($R(M)$) is a minimal left (right) ideal generated by an idempotent. Also the left (right) annihilator of a minimal right (left) ideal is a regular maximal left (right) ideal. Consider the socle K of B . By the reasoning of [4, Theorem 4], $L(K) = R(K) = (0)$. Let I be a proper right ideal of B . If I contained no minimal right ideals of B then, as in the proof of [4, Lemma 4], $I \subset L(K)$, which is impossible.

Let $x \in B$ and let B' be the completion of B in the normed algebra norm $\|x\|_1$. Consider $\lambda = a + bi \neq 0$ in $sp(x|B)$. Then $u = |\lambda|^{-2} (2ax - x^2)$ has no quasi-inverse in B . As in [3 ; p 159] there exists $y \neq 0$ such that $uy = y$ and u has no quasi-inverse in B' . Then $\rho(x|B') = \rho(x|B)$.

3. Closure of homomorphisms and anti-homomorphisms. Throughout this section the following notation is assumed. Let $B_1(B)$ be a real normed algebra with norm $\|x\|_1$ ($\|x\|$). T is a ring homomorphism or anti-homomorphism of B_1 onto a dense subset of B . T is called closed if $\|x_n - x\|_1 \rightarrow 0, \|T(x_n) - y\| \rightarrow 0$ imply that $y \in T(B_1)$ and $y = T(x)$. By the *separating set* S of T we mean the set of all $y \in B$ such that there exists a sequence $\{x_n\}$ in B_1 where $\|x_n\|_1 \rightarrow 0$ and $\|y - T(x_n)\| \rightarrow 0$. We assume that $\rho[T(x)] \leq \rho(x), x \in B_1$. Note that this condition is automatic if T is real-linear.

The next lemma is an adaptation of results of Rickart [11].

3.1. LEMMA. *T is closed and real-homogeneous if and only if $S = (0)$. S is a closed two-sided ideal in B and $T^{-1}(S)$ a closed two-sided ideal in B_1 . If B_1 is a normed \mathbb{Q} -algebra then every element of S is a topological divisor of zero in B .*

Clearly T is rational-homogeneous. Let $x \in B_1$ and $r_n \rightarrow r$ where each r_n is rational and r is real. Then $\|r_n x - r x\|_1 \rightarrow 0$ and $\|r T(x) - T(r x) - T(r_n x - r x)\| \rightarrow 0$. Hence $r T(x) - T(r x) \in S$. The first statement follows by a straightforward argument.

Let $y_n \in S, \|w - y_n\| \rightarrow 0$. There exists, for each n , an element $z_n \in B_1$ such that $\|y_n - T(z_n)\| < n^{-1}$ and $\|z_n\|_1 < n^{-1}$. Then $\|w - T(z_n)\| \rightarrow 0$ so that $w \in S$. Hence S is closed in B . Since $x \in S$ and r rational imply $r x \in S$ it follows that S is a real linear manifold. To show that S is an ideal in B it is enough to show that xy and $yx \in S$ for $x \in S$ and $y = T(z) \in T(B_1)$. This, however, is a simple matter. Suppose next that $\|x_n - x\|_1 \rightarrow 0$ where each $x_n \in T^{-1}(S)$. For each n there exists $y_n \in B_1$ such that $\|T(x_n) - T(y_n)\| < n^{-1}$ and $\|y_n\|_1 < n^{-1}$. Then $\|x - (x_n - y_n)\|_1 \rightarrow 0$ while

$\|T(x) - T[x - (x_n - y_n)]\| \rightarrow 0$ whence $T(x) \in S$. Hence $T^{-1}(S)$ is closed. It is readily seen to be a two-sided ideal in B_1 .

Let B^c be the completion of B where we use $\|x\|$ to denote the norm in B^c and $\rho(x)$ the spectral radius there. To show that $s \in S$ is a topological divisor of zero in B it is sufficient to show that it is one in B^c . Choose a sequence $\{x_n\}$ in B_1 such that $\|s - T(x_n)\| \rightarrow 0$ and $\|x_n\|_1 \rightarrow 0$. If B_1 is a normed Q -algebra s is the limit of quasi-regular elements of B^c by Lemma 2.1. Hence so also is λs for any real λ . By the arguments of [11; 621] it suffices to rule out the possibility that both B^c has an identity 1 and that s has a two-sided inverse in B^c .

Suppose this is the case. Let S_0 be the separating set for T considered as a mapping of B_1 into B^c . Clearly $S \subset S_0$. Then as S_0 is an ideal in B^c , $S_0 = B^c$ and $1 \in S_0$. There exists a sequence $\{u_n\}$ in B_1 such that $\|1 - T(u_n)\| \rightarrow 0$ and $\|u_n\|_1 \rightarrow 0$. Since $1 - T(u_n)$ and $T(u_n)$ permute we have by Lemma 2.1,

$$1 = \rho(1) \leq \rho(1 - T(u_n)) + \rho(T(u_n)) \leq \|1 - T(u_n)\| + \rho(u_n|_{B_1}) \rightarrow 0$$

This contradiction completes the argument.

If B_1 and B are Banach algebras, by the closed graph theorem [2; 41] $S = (0)$ will imply that T is continuous. In every case $S = (0)$ will imply real-homogeneity for T and the closure of $T^{-1}(0)$.

3.2. LEMMA. *Let B_1 be a normed Q -algebra and B be semi-simple with minimal one-sided ideals. Suppose that there exists a minimal one-sided ideal I of B_1 such that $T(B_1) \cap I \neq 0$. Then $S \cap I = (0)$.*

We consider the case where I is a right ideal and T is a homomorphism. The other cases follow by the reasoning employed. Set $I_1 = T^{-1}(I)$. I_1 is a right (ring) ideal of B_1 . Let $I = jB$, $j^2 = j$ and consider $x_0 \in I_1$ where $T(x_0) = jv \neq 0$. By the semi-simplicity of B , $jvB \neq (0)$ and, as jB is minimal, $jvB = jB$. Then $jvT(B_1)$ is dense in I . It follows that $T(I_1^2) \neq (0)$ for otherwise $[jvT(B_1)]^2 = (0)$ and $I^2 = (0)$. Select $x \in I_1$, $T(x) = jw \neq 0$ and $T(x^2) \neq 0$. Let R be the set of elements y in B for which $jy \in T(I_1)$. As observed, jR is dense in jB . Hence jRj is dense in jBj . But jBj is a normed division algebra and therefore, by the Gelfand-Mazur theorem, finite-dimensional in B . Thus $jRj = jBj$. There exists $z \in R$ such that $jzjwj = jwjzj = j$. For some $x_1 \in I_1$, $T(x_1)j = jzj$. Then $T(x_1x) = jzjw = T((x_1x)^2)$. Set $jzjw = h$ and $x_1x = u$. Then h is a non-zero idempotent in $I \cap T(B_1)$. Clearly $hB = I$ so that hBh is a division algebra hence isomorphic to the reals, complexes or quaternions.

We show that $h \notin S$. For suppose otherwise. Then there exists a sequence $\{y_n\}$ in B_1 such that $\|h - T(y_n)\| \rightarrow 0$ and $\|y_n\|_1 \rightarrow 0$. Thus $\|uy_nu\|_1 \rightarrow 0$ and $\|h - T(uy_nu)\| \rightarrow 0$. By Lemma 2.2 and the fact that hBh is the reals, complexes or quaternions, $\|hT(y_n)h\| \rightarrow 0$. This is a

contradiction as $h \neq 0$. Now $S \cap I$ is a right ideal of B , $S \cap I \neq I$. Since I is minimal, $S \cap I = (0)$.

3.3. THEOREM. *Let B_1 be a normed Q -algebra and B be primitive with minimal one-sided ideals. If $T(B_1) \cap I \neq (0)$ for a minimal one-sided ideal I of B then T is closed and real-homogeneous.*

Let K be the socle of B . If $S \neq (0)$ then $K \subset S$ by [6 ; 75]. Then $I \subset S$ which is impossible by Lemma 3.2.

3.4. COROLLARY. *Let B be any subalgebra of $\mathfrak{C}(X)$ closed in the uniform norm $\|T\|$ where $B \supset \mathfrak{F}(X)$. Let $\|T\|_1$ be any normed algebra norm for B such that the completion B^c of B in this norm is primitive. Then the two norms are equivalent.*

By Theorem 2.6 and Lemma 2.5, B is a Q -algebra in the norm $\|T\|_1$. By Theorem 2.3, there exists $c > 0$ such that $\|T\| \leq c \|T\|_1$, $T \in B$. Consider the embedding mapping I of B (with norm $\|T\|$) into B^c . B is a primitive algebra with a minimal right ideal JB , $J^2 = J$. Then $I(J)I(B)I(J)$ a normed division algebra and, by the Gelfand-Mazur theorem, closed in B^c . Since $I(J)$ is an idempotent, its closure in B^c is $I(J)B^cI(J)$. Therefore $I(J)B^c$ is a minimal right ideal of B^c . From Theorem 3.3, I is closed. The closed graph theorem [2 ; 41] shows that I is continuous. Hence there exists $c_1 > 0$ such that $\|T\|_1 \leq c_1 \|T\|$, $T \in B$.

3.5. THEOREM. *Let B_1 and B be normed Q -algebras. Then S is contained in the Brown-McCoy radical of B . If B is strongly semi-simple then T is closed and real-homogeneous.*

The Brown-McCoy radical [5] coincides with the intersection of the regular maximal two-sided ideals of B . Let M be such an ideal of B . Since B is a normed Q -algebra, M is closed. Let π be the natural homomorphism of B onto B/M . Since $T(B_1)$ is dense in B , then $\pi T(B_1)$ is dense in B/M . Also $\rho[\pi T(x)] \leq \rho[T(x)] \leq \rho(x)$, $x \in B_1$. Hence our theory applies to the mapping πT .

Let S_0 be the separating set for πT . Since B/M is simple with an identity, $S_0 = (0)$ by Lemma 3.1. Let $y \in S$, $\|x_n\|_1 \rightarrow 0$, $\|y - T(x_n)\| \rightarrow 0$. Then $\|\pi(y) - \pi T(x_n)\| \rightarrow 0$ or $\pi(y) \in S_0$. Therefore $S \subset M$. B is called *strongly semi-simple* if its Brown-McCoy radical is (0) .

3.6. THEOREM. *Let B_1 and B be semi-simple normed Q -algebras where B_1 has a dense socle K and B has an identity. Let T be real-linear. Then T is closed.*

Let P be a primitive ideal of B and π be the natural homomorphism of B onto B/P . Since B is a Q -algebra then P is closed, π is continuous and $\pi T(B_1)$ is dense in B/P . Let S_0 be the separating set for πT

as a mapping of B_1 into B/P . We show first that $T(K) \subset P$ is impossible. Suppose $T(K) \subset P$. Since $K \subset (\pi T)^{-1}(S_0)$, by Lemma 3.1 we have $B_1 = (\pi T)^{-1}(S_0)$ and $S_0 = B/P$. Since B/P has an identity this is contrary to Lemma 3.1. Hence there exists a minimal right ideal jB_1 of B_1 , $j^2 = j$ such that $T(j) \notin P$. Set $\pi T(j) = u$, $\pi T(B_1) = B_2$. πT is an isomorphism or anti-isomorphism of the division algebra jB_1j onto uB_2u . Hence uB_2u is a normed division algebra and thus, by the Gelfand-Mazur theorem closed in B/P . Since u is an idempotent, $u(B/P)$ is a minimal right ideal of B/P . By Theorem 3.3, πT is closed from which we obtain $S \subset P$. Since B is semi-simple, $S = (0)$.

3.7. THEOREM. *Let B_1 be a normed \mathbb{Q} -algebra and B semi-simple where either B is a modular annihilator algebra or has dense socle. If $T(B_1)$ contains the socle of B then T is closed and real-homogeneous.*

By Lemma 3.2, $S \cap I = (0)$ for every minimal one-sided ideal of B . Let I be a minimal right ideal. Then $SI = (0)$. Thus S annihilates the socle. It follows (see the proof of Lemma 2.8) that $S = (0)$ in the first case. In the second case we have $S^2 = (0)$ and $S = (0)$ by semi-simplicity.

Consider further a semi-simple normed modular annihilator algebra B . B is a permanent \mathbb{Q} -algebra by Lemma 2.5 and 2.8. From Theorem 3.7 we see that any algebraic homomorphism or anti-homomorphism of B onto B is closed no matter which two norms are used for B .

Let B be a real normed algebra. By an *involution* on B we mean a mapping $x \rightarrow x^*$ of B onto B which is a real-linear automorphism or anti-automorphism of period two. Let $H(K)$ be the set of self-adjoint (skew) elements of B with respect to the involution $x \rightarrow x^*$. B is the direct sum $H \oplus K$ of the linear manifolds H and K .

The mapping $x \rightarrow x^*$ of B onto B is subject to the above analysis. Here S is the set of all $x \in B$ for which there exists a sequence $\{x_n\}$ in B with $\|x_n\| \rightarrow 0$ and $\|x - x_n^*\| \rightarrow 0$.

3.8. LEMMA. $S = \overline{H} \cap \overline{K}$. $S = (0)$ if and only if H and K are closed.

Let $w \in S$. Then there exist sequences $\{h_n\}$ and $\{k_n\}$ in H and K respectively such that $\|w - (h_n - k_n)\| \rightarrow 0$ and $\|h_n + k_n\| \rightarrow 0$. Therefore $\|w - 2h_n\| \rightarrow 0$ and $\|w + 2k_n\| \rightarrow 0$ so $w \in \overline{H} \cap \overline{K}$. Conversely suppose that $\|z - h_n\| \rightarrow 0$, $\|z - k_n\| \rightarrow 0$ where each $h_n \in H$, $k_n \in K$. Then $\|z - (h_n + k_n)/2\| \rightarrow 0$ and $\|(h_n - k_n)/2\| \rightarrow 0$ and $z \in S$.

If H and K are closed, clearly $S = (0)$. Suppose $S = (0)$. Let $h_n \rightarrow u + v$ where $h_n \in H$, $u \in H$ and $v \in K$. Then $h_n - u \rightarrow v$ and $v \in \overline{H} \cap \overline{K}$. Then $v = 0$ and H is closed. Similarly K is closed.

Let B be a semi-simple normed annihilator algebra, for example an H^* -algebra. Then it follows from the above that H and K are closed in B for any involution on B and any normed algebra norm on B . For

B^* -algebras we have been able to show only the following weaker result.

3.9. THEOREM. *Let B be a B^* -algebra with $H(K)$ as the set of self-adjoint (skew) elements in the defining involution for B . Then H and K are closed in any normed algebra norm topology for B .*

B has the spectral extension property [13] and is therefore a permanent Q -algebra by Lemma 2.5. The arguments of [14; § 3] can be adapted to show that H and K are closed in any given normed algebra norm $\|x\|_1$.

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