Pacific Journal of Mathematics

IN THIS ISSUE—

Vol. 8, No. 2 **April, 1958**

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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SECOND ORDER COMPLEX DIFFERENTIAL EQUATIONS WITH A REAL INDEPENDENT VARIABLE

JOHN H. BARRETT

Introduction. This paper is concerned with the oscillation and boundedness properties of solutions of the complex differential equation

$$(1) \qquad (p(x)\dot{y})^{\cdot}+f(x)y=0 , \qquad a\leqq x<\infty,$$

where $p(x)=p_1(x)+ip_2(x)\neq 0$, $f(x)=f_1(x)+if_2(x)$ and each of the functions $p_1(x)$, $p_2(x)$, $f_1(x)$ and $f_2(x)$ is a continuous real function on the half-line $a\leqq x<\infty$.

Such differential equations have many interpretations and applications. For example, if $p(x)=1$ and the real and imaginary parts of equation (1) are separated the resulting system of two real equations can be interpreted as equations of motion in the y_1y_2-plane, where $y=y_1+iy_2$, as in [4, 9]. If in (1) x is replaced by the complex variable z, and the coefficients are required to be analytic functions of z the resulting completely complex equation can be reduced to one of the type (1) by considering certain analytic paths in the z-plane. This procedure has been used effectively by Taam [9] and others to find zero-free regions for the completely complex equation. Also, Hille [**5**] has made an extensive study of the behavior of solutions of a special case of (1), where $p(x)=1$ and $f(x)=\lambda F(x)$, $F(x)$ real and positive and λ a complex parameter, and has used these results in his study of Cauchy's problem for a generalized heat equation.

The present study of equation (1) begins with consideration of the special case

$$(2) \qquad (\dot{y}/q(x))^{\cdot}+\bar{q}(x)y=0 , \qquad a\leqq x<\infty,$$

where $q(x)=q_1(x)+iq_2(x)\neq 0$, $q_1(x)$ and $q_2(x)$ are real and continuous on $a\leqq x<\infty$ and $\bar{q}$ is the complex conjugate of q. For $q(x)$ *real* a fundamental set of solutions consists of $\sin \int_a^x q$ and $\cos \int_a^x q$. This suggests an investigation of the corresponding complex solutions, $s[a, x ; q]$ and $c[a, x ; q]$, of (2) when q is complex. These "trigonometric" functionals satisfy identities and inequalities analogous to those of the real sines

Received June 5, 1957. This work was partially supported by the National Science Foundation (NSF-G-1825) through the University of Delaware and was begun while the author was a lecturer at Yale University.

Presented to the American Mathematical Society December 1956.

and cosines. For example, the sum of the squares of the magnitudes is identically one and, hence, all solutions of equation (2) are bounded on $a \leqq x < \infty$. This boundedness property is the main point of departure from the analytic definition of trigonometric functions of a complex variable z, where $\sin \int_a^x q$ is unbounded if $\int_a^x q_2$ is unbounded. The boundedness property is useful in the applications of the last section.

An additional advantage in considering the special case is that for a rather large class of coefficient functions, $q(x)$, equation (2) can be solved *explicitly*, thus providing a new set of much needed examples to give insight into the oscillatory behavior of solutions of (1). Furthermore, for a still larger class of coefficient functions the oscillatory behavior of solutions of (2) is determined. An interesting result is that the zero separation properties, true for the real case, are often violated for the complex equation (2). For example, a class of functions, $q(x)$, is found for which the "sine" $s[a, x; q]$ *oscillates* (has infinitely many zeros on $a \leqq x < \infty$) and the "cosine" $c[a, x; q]$ has *no zero* on $a \leqq x < \infty$.

The final section shows that although equation (2) is a special case of (1), all oscillatory behavior patterns of equations of the type (1) are present in those of type (2). In particular, for each non-trivial solution $y(x)$ of (1), for which $y(a)=0$, and each non-zero function $w(x)$ of class C' there exist a continuous complex coefficient function $q(x)$ and a non-zero "amplitude" function $\rho(x)$ of class C' such that

$$(3) \qquad y(x)=\rho(x)s[a, x; q], \quad p(x)\dot{y}(x)=\overline{w}(x)\overline{\rho}(x)\overline{c}[a, x; q] .$$

For $w(x)$, $p(x)$ and $f(x)$ real; $q(x)$ and $\rho(x)$ are real and (3) reduces to the modified Prüfer transformation [1, 6]

$$(4) \qquad y=\rho(x)\sin\theta(x), \quad p\dot{y}=w(x)\rho(x)\cos\theta(x), \; \theta(x)=\int_a^x q ,$$

which has been useful in establishing real oscillation and boundedness theorems. The author [3] has developed a Prüfer transformation for equations of the form of (2) with both coefficients being square symmetric matrices and a similar, but less useful, transformation of type (3) (that is $y=\rho s, p\dot{y}=\rho c$) can be obtained as a corollary of the matrix theorems.

Since the "amplitude" $\rho(x)$ is non-zero and the "sine" $s[a, x; q]$ is bounded, the Prüfer-type transformation (3) separates boundedness considerations from those of oscillation, as does (4) for the real case. Applications of (3) yield bounds on solutions of (1) of the Liapounoff-Birkhoff-Levinson type. For the special case, $p(x)=1$ and $w(x)$ a positive real constant, these exponential bounds reduce to those of Taam [7]. It it noted that Taam failed to achieve a "symmetric" form because he specialized $p(x)$ to be real and, in particular, $p(x)=1$.

Further study of the relation of $q(x)$ to the original coefficients, $p(x)$ and $f(x)$, and of the oscillatory properties of the functionals of the first section should lead to new oscillation theorems for (1).

1. Complex trigonometry. Let $q(x)=q_1(x)+iq_2(x)$, q_1 and q_2 be continuous functions on $a \leqq x < \infty$ and *define* $c=c(x)=c[a, x; q]$, $s=s(x)=s[a, x; q]$ to be a solution (pair) of the complex first order system

$$\dot{s}=q\bar{c}\,, \quad s(a)=0\,, \qquad \dot{c}=-q\bar{s}\,, \quad c(a)=1\,. \tag{5}$$

If, in addition, $q(x)\neq 0$ it is easily seen that s and c are solutions of the second order equation

$$(\dot{y}/q)^{\cdot}+\bar{q}y=0\,, \tag{2}$$

with initial conditions

$$s(a)=0\,,\ c(a)=1\,, \qquad \dot{s}(a)=q(a)\,,\ \dot{c}(a)=0\,. \tag{6}$$

Note that $s=\sin\int_a^x q$ and $c=\cos\int_a^x q$, if $q(x)$ is real and, furthermore, if $q\neq 0$ both solutions oscillate (have infinitely many zeros on $a\leqq x<\infty$) only if $\int_a^\infty |q|=\infty$.

Boundedness is retained for complex $q(x)$, as is seen by:

LEMMA 1.1. $|s|^2+|c|^2=1$.

Proof. Differentiate $s\bar{s}+c\bar{c}$ and note its value at $x=a$.

There is also an extension of the properties that the real sine function is odd and the cosine is even. This result is useful in carrying out the details of the proof of Theorem 1.2.

LEMMA 1.2. *If k is a complex number such that $|k|=1$ then* $s[a, x; kq]=ks[a, x; q]$ and $c[a, x; kq]=c[a, x; q]$ *on* $a\leqq x<\infty$.

Proof. Let $m(x)=s[a, x; kq]-ks[a, x; q]$ and $n(x)=c[a, x; kq]-c[a,x; q]$.
Then $m(a)=0$, $n(a)=0$ and $\dot{m}=kq\bar{n}$, $\dot{n}=-kq\bar{m}$, whose only solution is $m\equiv n\equiv 0$ on $a\leqq x<\infty$, thus completing the proof.

Consider now the polar form of solutions of (5) and (2) in terms of the polar components of the coefficients; i.e., suppose

$$q(x)=r(x)\exp(i\theta(x))\,, \qquad a\leqq x<\infty, \tag{7}$$

where $r(x)$ is real, continuous and positive and $\theta(x)$ is real and of class C'. These conditions ensure a polar form for the complex trigonometric functionals as is seen by the following.

LEMMA 1.3. *Under the above hypotheses on $q(x)$, there exist on $a \leqq x < \infty$ real functions $h(x)$ and $\alpha(x)$ such that h, $\dot{h}/r$, α and $h\dot{\alpha}/r$ are of class C' and, furthermore*

$$s[a, x; q] = h(x) \exp(i\alpha(x)) . \tag{8}$$

Proof. Let $s(x) = s[a, x; q]$. Using a technique similar to that employed by Taam [9], define the real function

$$g(x) = \begin{cases} \Im(\dot{s}/rs) , & \text{if } s(x) \neq 0 \\ \dot{\theta}/2r , & \text{if } s(x) = 0. \end{cases} \tag{9}$$

Note that $g(x)$ is continuous on $a \leqq x < \infty$, since computation by means of L'Hopitals rule shows

$$\lim_{x \to x_0} \Im(\dot{s}/rs) = \dot{\theta}(x_0)/2r(x_0), \text{ if } s(x_0) = 0 .$$

Let

$$\alpha(x) = \theta(a) + \int_a^x r(t)g(t)dt , \tag{10}$$

then $\alpha(x)$ is of class C' on $a \leqq x < \infty$. Let

$$h(x) = s(x) \exp(-i\alpha(x)) , \tag{11}$$

then $h(x)$ is of class C', $h(a) = 0$ and

$$\dot{h}(x) = (\dot{s} - i\dot{\alpha}s) \exp(-i\alpha) .$$

Thus, $\dot{h}(a) = \dot{s}(a) \exp(-i\alpha(a)) = r(a) > 0$.
The next step is to prove that $h = h_1(x) + ih_2(x)$ is real, that is, $h_2(x) \equiv 0$. Suppose $h_2(x) \not\equiv 0$, then there exist numbers $t_0 < t_1$ such that $h_2(t_0) = 0$, $h_2(x) \neq 0$ on $t_0 < x < t_1$ and $s(x) \neq 0$ on $t_0 < x < t_1$. Then on $t_0 < x < t_1$

$$\frac{\dot{h}}{h} = \frac{\exp(i\alpha)\dot{h}}{s} = \frac{\dot{s}}{s} - i\dot{\alpha} ,$$

Also,

$$\Im\left(\frac{\dot{h}}{h}\right) = \frac{\Im(\bar{h}\dot{h})}{|s|^2} = 0 ,$$

or

$$h_1\dot{h}_2 - h_2\dot{h}_1 = 0 .$$

Therefore there exists a real constant k such that $h_1(x) = kh_2(x)$ on $t_0 < x < t_1$. Hence $h_1(t_0) = h_1(t_0+) = kh_2(t_0+) = 0$ and $s(t_0) = 0$. Suppose $t_0 = a$, then, since $h(a)$ is real, $\dot{h}_2(t_0) = 0$ and $\dot{h}_1(t_0) = \dot{h}_1(t_0+) = k\dot{h}_2(t_0)$. But this requires that $\dot{s}(t_0) = 0$, which contradicts the fact that $s(x)$ is non-trivial. In a similar manner and by use of an induction argument it is easily seen that t_0 cannot be any zero of $s(x)$. Thus, $h_2(x) \equiv 0$ and $h(x)$ as defined by (11) is real.

Recall that $\dot{s} = q\bar{c}$ and $\dot{c} = -q\bar{s}$ where $s = s[a, x; q]$ and $c = c[a, x; q]$ and $s = h(x)\exp(i\alpha(x))$, where $q = r(x)\exp(i\theta(x))$. By differentiating this polar form of s and simplifying it follows that

$$\frac{\dot{h}}{r} + i\frac{h\dot{\alpha}}{r} = \bar{c}\exp(i(\theta - \alpha))$$

and, since the right hand side is of class C', that the real components, $\dot{h}/r$ and $h\dot{\alpha}/r$ are likewise of class C'. Furthermore, by transposing the exponential factor to the left hand side and differentiating we obtain

$$\left(\frac{\dot{h}}{r}\right)^{\cdot} + i\left(\frac{h\dot{\alpha}}{r}\right)^{\cdot} + i(\dot{\alpha} - \dot{\theta})\frac{\dot{h}}{r} - (\dot{\alpha} - \dot{\theta})\frac{h\dot{\alpha}}{r} = -rh .$$

Separation into real and imaginary parts yields the system

$$\left(\frac{\dot{h}}{r}\right)^{\cdot} + \left(1 - \frac{\dot{\alpha}(\dot{\alpha} - \dot{\theta})}{r^2}\right)rh = 0 , \tag{12}$$

and

$$\left(\frac{h\dot{\alpha}}{r}\right)^{\cdot} + \frac{\dot{\alpha} - \dot{\theta}}{r}\dot{h} = 0 , \tag{13}$$

thus completing the proof of Lemma 1.3.

Integration of equation (13) gives

$$\frac{h^2(x)\dot{\alpha}(x)}{r(x)} = \int_a^x \frac{h\dot{h}\dot{\theta}}{r} . \tag{13}$$

Finally, integration by parts establishes the following.

LEMMA 1.4. *If, in addition to the hypotheses of Lemma* 1.3, *the quotient $\dot{\theta}/r$ is of class C', then on $a \leqq x < \infty$*

$$\frac{h^2(x)}{r(x)}\left(\dot{\alpha}(x)-\frac{\dot{\theta}(x)}{2}\right)=-\frac{1}{2}\int_a^x h^2(t)\left(\frac{\dot{\theta}(t)}{r(t)}\right)^{\cdot}dt\ . \tag{14}$$

Furthermore, $\dot{\alpha}/r$ *is of class* C', *whenever* $s\neq 0$.

The preceding discussion suggests special consideration of the quotient $\dot{\theta}/r$. The following theorem is a compilation of the results thus far.

THEOREM 1.1. *If on* $a\leqq x<\infty$, $q=r(x)\exp(i\theta(x))$, $r(x)$ *is real, positive and continuous,* $\theta(x)$ *is real and of class* C', $b(x)=\dot{\theta}(x)/2r(x)$, *then there exist real functions* $h(x)$ *and* $\alpha(x)$ *both of class* C', *as well as the quotient* $\dot{h}/r$, *such that*

$$s[a, x\,;\,q]=h(x)\exp(i\alpha(x))\ , \tag{8}$$

$$\left(\frac{\dot{h}}{r}\right)^{\cdot}+\left((1+b^2)-\left(\frac{\dot{\alpha}}{r}-b\right)^2\right)rh=0\ , \tag{12'}$$

$$\frac{h^2(x)\dot{\alpha}(x)}{r(x)}=2\int_a^x h\dot{h}b\ . \tag{13'}$$

Furthermore, if $b(x)$ *is of class* C' *then*

$$h^2(x)\left(\frac{\dot{\alpha}(x)}{r(x)}-b(x)\right)=-\int_a^x h^2(t)\dot{b}(t)dt\ . \tag{14}$$

and $\dot{\alpha}/r$ *is of class* C', *whenever* $s\neq 0$.

Application of the Sturmian comparison theorem to (12′) gives the following.

COROLLARY 1.1.1. *(Non-oscillation theorem) If the real second-order equation*

$$(\dot{y}/r)^{\cdot}+(1+b^2)ry=0\ , \qquad a\leqq x<\infty .$$

is non-oscillatory (i. e., non-trivial solution has infinitely many zeros on $a\leqq x<\infty$) *then* $s[a, x\,;\,q]$ *is non-oscillatory.*

Equation (14) *shows the following.*

COROLLARY 1.1.2. (*Non-oscillation theorem*) *If, in addition to the hypotheses of Theorem* 1.1, $b(x)$ *is of class* C' *and* $b'(x)\neq 0$ *on* $a<x<\infty$ *then* $s[a, x\,;\,q]$ *has no zeros of* $a<x<\infty$.

EXAMPLE 1.1. Let $q(x)=1+ix$, then $r(x)=\sqrt{1+x^2}$, $\theta(x)=\mathrm{Tan}^{-1}x$,

$b(x)=-(x^2+1)^{-3/2}$ and $\dot{b}(x)>0$. Corollary 1.1.2 then establishes that $s[a, x; q]$ has no zeros on $a<x<\infty$.

This example shows that the latter non-oscillation theorem is not a special case of the following.

THEOREM T (Taam [8]). *If* $p(x)=p_1(x)+ip_2(x)\neq 0, f(x)=f_1(x)+if_2(x)$, *where* p_1, p_2, f_1 *and* f_2 *are real continuous functions on* $a\leqq x\leqq b$ *and there exist constants* j *and* k *and a real function* $m(x)$ *of class* C' *on* $a\leqq x\leqq b$ *such that* $jp_1(x)+kp_2(x)>0$
and

$$\dot{m}+m^2/(jp_1+kp_2)\leqq-(jf_1+kf_2) \qquad \text{on } a\leqq x\leqq b$$

then the complex equation $(p(x)\dot{y})^{\cdot}+f(x)y=0$ *is disconjugate* (*i. e. no solution has two zeros on* $a\leqq x\leqq b$).

For $q(x)=1+ix$, as in Example 1.1, consider the equation $(\dot{y}/q)+\bar{q}y=0$. Then

$$p=\frac{1}{q}=\frac{\bar{q}}{|q|^2}=\frac{1-ix}{1+x^2}\ .$$

and $f(x)=\bar{q}=1-ix$. There exist constants j, k with $k<0$ such that

$$jp_1+kp_2=\frac{j+|k|x}{1+x^2}>0\ ,$$

for $x>j/k$. Consider the real second order equation

$$\left(\frac{j+|k|x}{1+x^2}\,\dot{y}\right)^{\cdot}+(j+|k|x)y=0\ , \qquad x>j/k. \tag{15}$$

Now, $(j+|k|x)^2\leqq(k^2+j^2)(1+x^2)$ and, hence,

$$\frac{j+|k|x}{1+x^2}<\frac{k^2+j^2}{j+|k|x}\ , \qquad x>j/k.$$

With the use of this inequality to increase the leading coefficient, equation (15) is altered to

$$\left(\frac{k^2+j^2}{j+|k|x}\,\dot{y}\right)^{\cdot}+(j+|k|x)y=0 \tag{15'}$$

whose fundamental solutions are

$$\sin\int_a^x \frac{j+|k|t}{\sqrt{j^2+k^2}}\,dt \text{ and } \cos\int_a^x \frac{j+|k|t}{\sqrt{j^2+k^2}}\,dt\ ,$$

and all solutions of (15′) oscillate on $j/k<a\leqq x<\infty$. By the Sturmian comparison theorem equation (15) is also oscillatory and hence for some $b>a$ there exists no function $m(x)$ required by the Riccati inequality of Theorem T. But by Corollary 1.1.2, $s[a, x; q]$ has no zeros on $a<x<\infty$ and hence the complex equation $(\dot{y}/q)^{\cdot}+\bar{q}y=0$ is disconjugate on $a\leqq x>\infty$. Therefore Example 1.1 is non-oscillatory but does not satisfy the hypotheses of Theorem T.

REMARK. A similar polar form for the "cosine" functional

$$c[a, x; q]=k(x)\exp(i\beta(x))$$

can be obtained for which k and β replace h and α, respectively, in equations (12′) and (13′). However, since $k(a)=1$, equation (14) must be replaced by

$$k^2(x)\left(\frac{\dot{\beta}(x)}{r(x)}-b(x)\right)=-b(a)-\int_a^x k^2(t)\dot{b}(t)dt\ . \tag{14′}$$

Of course, $c[a, x; q]$ can be calculated from a known $s[a, x; q]$ by the derivative formula, $\dot{s}=q\bar{c}$, which is the process actually used in the succeeding discussion.

THEOREM 1.2. *If, in addition to the hypotheses of Theorem* 1.1, $b=\dot{\theta}(x)/2r(x)$ *is constant then explicit solutions of the system* (5) *are*

$$s[a, x; q]=\frac{1}{\sqrt{b^2+1}}\exp\left(i\frac{\theta(x)+\theta(a)}{2}\right)\sin\phi(x)\ , \tag{16}$$

$$c[a, x; q]=\exp\left(i\frac{\theta(x)-\theta(a)}{2}\right)\left\{\cos\phi(x)-\frac{ib}{\sqrt{b^2+1}}\sin\phi(x)\right\}\ , \tag{17}$$

where $\phi(x)=\sqrt{b^2+1}\int_a^x r(t)dt$. *Furthermore,* $s[a, x, q]$ *oscillates (i. e., has infinitely many zeros on* $a\leqq x<\infty$) *if and only if* $\int_a^\infty |r(t)|dt=\infty$. *Finally, if* $b\neq 0$, $c[a, x; q]$ *has no zeros on* $a\leqq x<\infty$.

Note that this means that there exists a second order complex equation $(\dot{y}/q)^{\cdot}+\bar{q}y=0$ such that the zeros of one solution do not separate those of a linearly independent solution and the zeros of a solution are not separated by zeros of its derivative. Before this theorem is proved consider the following special cases.

EXAMPLE 1.2. Let $q(x)$ be real and positive, then $q(x)=r(x)$, $\theta(x)=0$, $b=0$ and

$$s[a, x\,; q]=\sin\int_a^x q(t)dt, \quad c[a, x\,; q]=\cos\int_a^x q(t)dt\,.$$

EXAMPLE 1.3. Let $q(x)=q_1(x)+iq_2(x)$, $q_2(x)=kq_1(x)$, $k=$constant, and $q_1(x)>0$ then $r(x)=\sqrt{1+k^2}\ q_1(x)$, $\theta(x)=\mathrm{Tan}^{-1}k$, $\dot\theta=0$, $b=0$ and

$$s[a, x\,; q]=\frac{1+ik}{\sqrt{1+k^2}}\sin\left(\sqrt{1+k^2}\int_a^x q_1(t)dt\right),$$

$$c[a, x\,; q]=\qquad\cos\left(\sqrt{1+k^2}\int_a^x q_1(t)dt\right).$$

EXAMPLE 1.4. Let $q(x)=\exp(ix)$, then $r(x)=1$, $\theta(x)=x$, $b=1/2$ and

$$s[a, x\,; q]=\frac{2}{\sqrt{5}}\exp\frac{i}{2}(x+a)\sin\frac{\sqrt{5}}{2}(x-a)$$

$$c[a, x\,; q]=\exp\frac{i}{2}(x-a)\left(\cos\frac{\sqrt{5}}{2}(x-a)-\frac{i}{\sqrt{5}}\sin\frac{\sqrt{5}}{2}(x-a)\right).$$

Note that there do not exist constants j, k such that $jq_1-kq_2>0$, and hence the hypotheses of Theorem T are not satisfied for $b-a>\pi$. Of course, in this case $(\dot y/q)^{\cdot}+\bar q y=0$ is oscillatory on $a\leqq x<\infty$.

PROOF OF THEOREM 1.2. In order to simplify computations let $\theta(a)=0$. There is no loss of substance decause of this assumption since Lemma 1.2 assures that if $q(x)$ is multiplied by the constant $\exp(-i\theta(a))$ then the resulting "sine" functional must be multiplied by that number and the "cosine" is unchanged. Therefore equation (14) gives

$$\dot\alpha(x)=br(x)=\frac{\dot\theta(x)}{2}\text{and }\ \alpha(x)=\frac{\theta(x)}{2}\,, \tag{18}$$

and equation (12′) becomes

$$\left(\frac{\dot h}{r\sqrt{1+b^2}}\right)^{\cdot}+r\sqrt{1+b^2}\ h=0\,. \tag{19}$$

Since $h(a)=0$ and $\dot h(a)=r(a)$ we have

$$h(x)=\frac{1}{\sqrt{b^2+1}}\sin\left(\sqrt{1+b^2}\int_a^x r(t)dt\right). \tag{20}$$

By combination of (8), (18) and (20) and the use of Lemma 1.2 and $\dot s=q\bar c$ the explicit solutions (16) and (17) are obtained. Finally, if $b\neq 0$, (17) gives

$$|c|\geqq 1-\frac{b^2}{b^2+1}=\frac{1}{b^2+1}>0\,.$$

and the theorem is proved.

Note that in Theorem 1.2, if $b=0$ then $c[a, x; q]$ oscillates if and only if $s[a, x; q]$ oscillates and the zeros of one functional separate those of the other. But if $b \neq 0$ then $s[a, x; q]$ may oscillate but $c[a, x; q]$ has no zeros on $a \leqq x < \infty$, thus violating a "Rolle's Theorem" for complex functions.

In the next section it will be shown that every complex equation of the form $(p\dot{y})^{\cdot}+fy=0$ can be transformed into the "special" form $(\dot{y}/q)^{\cdot}+\bar{q}y=0$.

2. A complex Prüfer transformation. Consider the complex general linear second-order equation

$$(1) \qquad (p\dot{y})^{\cdot}+fy=0\,, \qquad a \leqq x < \infty,$$

where $p=p_1(x)+ip_2(x)\neq 0, f=f_1(x)+if_2(x)$ and p_1, p_2, f_1, f_2 are all real continuous functions on $a \leqq x < \infty$. Suppose $y(x)$ is a non-trivial solution of (1) such that $y(a)=0$ and there exist complex functions $\rho(x)\neq 0$, $w(x)\neq 0$ of class C' and $q(x)$ continuous, such that

$$(21) \qquad \begin{aligned} y(x)&=\rho(x)s[a, x; q]\,, \\ p(x)\dot{y}(x)&=\overline{w}(x)\bar{\rho}(x)\bar{c}[a, x; q]\,. \end{aligned}$$

Then by differentiating both equations of (21) and combining with (1) we obtain

$$\dot{\rho}s+\rho q\bar{c}=\frac{\overline{w}}{p}\bar{\rho}\bar{c}\,,$$

$$\dot{\rho}c-\rho q\bar{s}=-\frac{\bar{f}}{w}\bar{\rho}\bar{s}-\frac{\dot{w}}{w}\rho c\,.$$

Hence, solving for $\dot{\rho}$ and q and recalling that $|c|^2+|s|^2=1$ we have

$$(22) \qquad \dot{\rho}=\overline{\rho s c}\left(\frac{\overline{w}}{p}-\frac{\bar{f}}{w}\right)-\frac{\dot{w}}{w}|c|^2\rho, \qquad \bar{\rho}(a)=\frac{p(a)\dot{y}(a)}{\overline{w}(a)}\neq 0$$

$$(23) \qquad q=\frac{\bar{\rho}}{\rho}\left(\frac{\overline{w}}{p}|c|^2+\frac{\bar{f}}{w}|s|^2\right)+\frac{\dot{w}}{w}sc$$

For p, f, y, ρ, q real; $s=\sin\int_a^x q$, $c=\cos\int_a^x q$ and (21), (22), (23) reduce to the modified real Prüfer transformation of [1]. The transformation of (1) given by $y=\rho s$ and $p\dot{y}=\rho c$ results in the differential-integral system $\dot{\rho}=\rho(1/p)c\bar{s}-f s\bar{c})$ and $q=(1/p)c^2+fs^2$ and can be obtained as a direct application of the matrix Prüfer transformation in [3] for (matrix) order 2. However, the form (21) seems to be more useful, e.g., see Corollary 2.1.

Consider next the question of existence of $\rho(x)$ and $q(x)$, that is the solution pair of the system (22), (23). The method is that of successive approximations and the following lemmas establish a Lipschitz condition for the system.

LEMMA 2.1. *If $q(x)$ and $q^*(x)$ are continuous complex functions on $a \leqq x < \infty$ and $s^* = s[a, x; q^*]$, $c^* = c[a, x; q^*]$* then

$$\left.\begin{matrix} |s-s^*| \\ |c-c^*| \end{matrix}\right\} \leqq 4\int_a^x |q-q^*|, \qquad a \leqq x < \infty. \tag{24}$$

Proof. By subtracting the differential equations (5) obtain the system

$$(s-s^*)^{\cdot} = q(\overline{c-c^*}) + (q-q^*)\bar{c}^*,$$
$$(c-c^*)^{\cdot} = -q(\overline{s-s^*}) - (q-q^*)\bar{s}^*,$$

which can be expressed in the vector-matrix form

$$\dot{\alpha} = Q(x)\alpha + \beta(x),\ \alpha = \begin{pmatrix} s-s^* \\ c-c^* \end{pmatrix},\ Q = \begin{pmatrix} 0 & q(x) \\ -\bar{q}(x) & 0 \end{pmatrix},\ \beta = \begin{pmatrix} (q-q^*)\bar{c}^* \\ \overline{(q-q^*)s^*} \end{pmatrix} \tag{25}$$

$$\alpha(a) = 0.$$

Let $Y(x)$ be the matrix solution of the homogeneous equation:

$$\dot{Y} = QY,\ Y(a) = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$Y(x) = \begin{pmatrix} c & s \\ -\bar{s} & \bar{c} \end{pmatrix},\ \text{and}\ Y^{-1} = \begin{pmatrix} \bar{c} & -s \\ \bar{s} & c \end{pmatrix}.$$

By elementary methods the solution of (21) is

$$\alpha(x) = \int_a^x Y(x)Y^{-1}(t)\beta(t)dt. \tag{26}$$

Hence, by taking norms (square root of sum of square of absolute values), we have

$$\|\alpha(x)\| \leqq 2\int_a^x \|\beta(t)\|dt < 4\int_a^x |q-q^*|,$$

from which the conclusion of the lemma follows.

LEMMA 2.2. *Assume the hypotheses of Lemma 2.1 and let*

$$r[q]=\sum_{i=1}^{n} k_i(x)s^{\alpha_i}(\bar{s})^{\beta_i}c^{\gamma_i}(\bar{c})^{\delta_i} ,$$

where $k_i(x)$ $(i=1, 2, \cdots, n)$ are complex continuous function on $a\leqq x\leqq b$ and α_i, β_i, γ_i, δ_i are non-negative integers. Then there exists a positive constant K_0 (independent of q and q^*) such that

$$|r[q]-r[q^*]|\leqq K_0\int_a^x |q-q^*| , \qquad a\leqq x\leqq b.$$

The proof based on Lemma 2.1 is simple and is omitted.

LEMMA 2.3. *If $u(x)$ and $v(x)$ are complex continuous functions on $a\leqq x<\infty$, the complex differential equation* (*see equation* (22))

$$\dot{\rho}=u(x)\bar{\rho}+v(x)\rho \tag{27}$$

has exactly one solution for a prescribed value of $\rho(a)$.

The proof parallels that for real linear equations and, consequently, is not given here.

LEMMA 2.4. *Let $\rho(x)$ be a solution of* (27), *where u and v are the corresponding coefficients of* (22), *and $m(x)=\bar{\rho}/\rho$. Then* : (i) $|m|\equiv 1$, (ii) *m satisfies the complex Riccati equation*

$$\dot{m}=\bar{u}-(v-\bar{v})m-um^2 \tag{28}$$

and (iii) *if m^* is the corresponding function when q is replaced by q^*, u by u^* and v by v^* then there exists a real constant K_1 (independent of q and q^*) such that*

$$|m-m^*|\leqq K_1\int_a^x |q(t)-q^*(t)|dt , \qquad a\leqq x<x_1<\infty. \tag{29}$$

Proof.
$$(m-m^*)^{\cdot}+\{(v-\bar{v})+u(m+m^*)\}(m-m^*)$$
$$=\overline{(u-u^*)}+\{(\bar{v}-v)-(\bar{v}^*-v^*)\}m^*-(u-u^*)m^{*2}$$

or

$$(m-m^*)^{\cdot}+n(x)(m-m^*)=r(x), \quad m(a)-m^*(a)=0 ,$$

and hence,

$$m(x)-m^*(x)=e^{-\int_a^x n}\int_a^x e^{\int_a^t n}\, r(t)dt .$$

Therefore, there exists a real constant K_1 such that

$$|m(x)-m^*(x)|\leqq K_1\int_a^x |q-q^*| , \qquad a\leqq x\leqq x_1<\infty.$$

LEMMA 2.5. *There exists a unique solution pair* $\rho(x), q(x)$ *of the system* (22, 23).

Proof. It follows easily from Lemmas 2.2 and 2.4 that the system ((22), (23)) satisfies a Lipschitz condition. Let $q_0(x)$ and $\rho_0(x)$ be complex continuous functions on $a \leqq x \leqq b$ and for each non-negative integer n

$$\rho_{n+1}(x)=\rho(a)+\int_a^x\left\{\bar{\rho}_n\bar{s}_n\bar{c}_n\left(\frac{\overline{w}}{p}-\frac{\overline{f}}{w}\right)-\frac{\dot{w}}{w}|c_n|^2\rho_n\right\},$$

$$q_{n+1}(x)=\frac{\bar{\rho}_n}{\rho_n}\left\{\frac{\overline{w}}{p}|c_n|^2+\frac{\overline{f}}{w}|s_n|^2\right\}+\frac{\dot{w}}{w}s_nc_n,$$

where $s_n=s[a, x ; q_n]$ and $c_n=[a, x ; q_n]$.
By the usual successive approximation arguments it follows that the sequences $\{\rho_n(x)\}$ and $\{q_n(x)\}$ converge uniformly on $a \leqq x \leqq b$ to continuous limit functions, $\rho(x)$ and $q(x)$, respectively, which form a solution pair of (22) and (23).

THEOREM 2.1. *If* $y(x)$ *is a non-trivial solution of* (1), *such that* $y(a)=0$, *and* $w(x)$ *is an arbitrary non-zero function of class* C' *then there exist a non-zero function* $\rho(x)$ *of Class* C' *and a continuous function* $q(x)$ *such that* (21) *is satisfied. Furthermore,* (22) *and* (23) *are satisfied.*

Proof. From Lemma 2.5, there exists a unique solution pair $\rho(x)$ and $q(x)$. Let $u(x)=\rho(x)s[a, s ; q]$, then $u(a)=0=y(a)$,

$$\dot{u}=\dot{\rho}s+\rho q\bar{c}=\frac{\overline{\rho w}}{p}\bar{c}, \text{ and } \dot{u}(a)=\frac{\bar{\rho}(a)\bar{w}(a)}{p(a)}=\dot{y}(a).$$

Finally,

$$(p\dot{u})^{\cdot}=-\bar{\rho}\bar{w}\bar{q}s+\dot{\bar{w}}\bar{\rho}\bar{c}+\dot{\bar{\rho}}\bar{w}\bar{c}=-f\rho s=-fu,$$

therefore

$$y(x)\equiv u(x)=\rho(x)s[a, x ; q].$$

Equation (22) yields the following bounds on solutions

COROLLARY 2.1. (i) $|\rho|=\sqrt{|y|^2+\left|\frac{p\dot{y}}{w}\right|^2}$.

(30) (ii) $$|y(x)| \leqq |\rho(x)| \leqq |\rho(a)|\exp\int_a^x\left\{\frac{1}{2}\left|\frac{\overline{w}}{p}-\frac{\overline{f}}{w}\right|+\left|\frac{\dot{w}}{w}\right|\right\}.$$

(iii) *if* $w=k$, *a real positive constant, then*

(31) $$|y(x)|\leqq\rho(a)|\exp\frac{1}{2}\int_a^x\left|\frac{k}{p}-\frac{\overline{f}}{k}\right|\leqq$$

$$|\rho(a)|\exp\frac{1}{2}\int_a^x\left\{\left|\frac{kp_1}{|p|^2}-\frac{f_1}{k}\right|+\left|\frac{kp_2}{|p|^2}-\frac{f_2}{k}\right|\right\}.$$

Proof. (i) is obvious and (ii) follows directly from (22). (iii) results from an application of (ii) and a simple inequality about complex numbers. Note that if p is real then $p_2=0$ and (31) becomes

(31′) $$|y(x)|\leqq|\rho(a)|\exp\frac{1}{2}\int_a^x\left\{\left|\frac{k}{p}-\frac{f_1}{k}\right|+\left|\frac{f_2}{k}\right|\right\},$$

which is the " non-symmetric " bound given by Taam [10].

Finally, other choices of $w(x)$ give other bounds on solutions as was found for real second-order differential equations in [1].

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REMARKS ON THE MAXIMUM PRINCIPLE FOR PARABOLIC EQUATIONS AND ITS APPLICATIONS

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Introduction. In [3] Nirenberg has proved maximum principles, both weak and strong, for parabolic equations. In § 1 of this paper we give a generalization of his strong maximum principle (Theorem 1). Hopf [2] and Olainik [4] have proved that if $Lu \geqq 0$ and L is a linear elliptic operator of the second order, if the coefficient of u in L is non-positive, and if u ($\not\equiv$const.) assumes its positive maximum at a point P^0 (which necessarily belongs to the boundary) then $\partial u/\partial \nu < 0$, where ν is the inwardly directed normal. In § 2 we extend this result to parabolic operators (Theorem 2). A further discussion of the assumptions made in Theorem 2 is given in § 3. Application of Theorem 2 to the Neumann problem is given in § 4. In § 5 we apply the weak maximum principle to prove a uniqueness theorem for certain nonlinear parabolic equations with nonlinear boundary conditions, and thus extend the special case considered by Ficken [1]. An even more special case arises in the theory of diffusion (for references, see [1]).

1. Consider the operator

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x, t)\frac{\partial u}{\partial x_i} + a(x, t)u - \frac{\partial u}{\partial t} \tag{1}$$

with $a(x, t) \leqq 0$. Here, $(x, t) = (x_1, \cdots, x_n, t)$ varies in the closure $\bar{D}$ of a given $(n+1)$-dimensional domain D. Assume that L is parabolic in $\bar{D}$, that is, for every real vector $\lambda \neq 0$ and for every $(x, t) \in \bar{D}$ we have

$$\sum a_{ij}(x, t)\lambda_i \lambda_j > 0 .$$

All the coefficients of L are assumed to be continuous in $\bar{D}$ and u is assumed to be continuous in $\bar{D}$ and to have a continuous t-derivative and continuous second x-derivatives in D. From [3; Th. 5] it follows that, under the above assumptions, *if* $Lu \geqq 0$ *and if* u *assumes its positive maximum at an interior point* P^0, *then* $u \equiv$ *const. in* $S(P^0)$. Here, $S(P^0)$ denotes the set of all points Q in D which can be connected to P^0 by a simple continuous curve in D along which the coordinate t is non-decreasing from Q to P^0. In the following theorem we consider the case

Received October 29, 1957. Prepared under ONR Contract Nonr-908 (09), NR 041 037 with Indiana University.

in which P^0 is a boundary point of D. We may assume that P^0 is the origin. Let $t=\varphi(x)$ be the equation of the boundary of D near P^0. Assume that $t=0$ is the tangent hyperplane to the boundary of D at P^0. Therefore $\partial\varphi/\partial x_i|_{P^0}=0$. Let D be on the side $t<\varphi(x)$.

THEOREM 1. *If $Lu\geqq 0$ in D, if u assumes its positive maximum M at P^0, if*

$$\lim_{P\to P^0}\frac{\partial u(P)}{\partial x_i}=0,\ \lambda\equiv\lim_{P\to P^0}\sum a_{ij}(P)\frac{\partial^2 u(P)}{\partial x_i\partial x_j}\leqq 0 \qquad P\in D \tag{2}$$

and if

$$1+\sum a_{ij}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\bigg|_{P^0}>0 \qquad \varphi\in C'' \tag{3}$$

then $u\equiv M$ in $S(P^0)$.

REMARK 1. Without making any use of (3) one can deduce the following:

Put $\mu\equiv\limsup_{P\to P^0}\dfrac{\partial u(P)}{\partial t}$ $(P\in D)$, then $\mu\geqq 0$ since $\mu<0$ will contradict $u(P^0)\geqq u(P)$. Letting $P\to P^0$ in $Lu(P)\geqq 0$ and using (2), we obtain $\lambda+a(P^0)M-\mu\geqq 0$, from which it follows that $\lambda\geqq 0$. Since, by (2), $\lambda\leqq 0$, we conclude that $\lambda=0$. Hence $a(P^0)M-\mu\geqq 0$, from which it follows that $\mu\leqq 0$ and, therefore, (since $\mu\geqq 0$) $\mu=0$. We also get $a(P^0)=0$.

REMARK 2. The assumptions (2) and (3) can be verified if we assume that $\varphi(x)=o(|x|^2)$ and that u belongs to C'' in the closure of the domain $V\cap\{t<0\}$, where V is some neighborhood of P^0. Indeed, by making an appropriate orthogonal transformation we can assume that $a_{ij}(P^0)=\delta_{ij}$. By the mean value theorem we have

$$u(x,t)-u(0,0)=\sum x_i\frac{\partial}{\partial x_i}u(\tilde{x},\tilde{t})+t\frac{\partial}{\partial t}u(\tilde{x},\tilde{t})\ .$$

Taking $(x,t)\in\overline{D}\cap V\cap\{t<0\}$ such that $|t|=o(|x|)$ and noting that $u(x,t)\leqq u(0,0)$, one can show that $\partial u(P^0)/\partial x_i=0$. Noting that $\varphi(x)=o(|x|^2)$ and expanding $[u(x,t)-u(0,0)]$ in terms of the first and second derivatives of u, one can show that $\partial^2 u(P^0)/\partial x_i^2\leqq 0$, and (2) is thereby proved. The proof of (3) is immediate.

PROOF OF THEOREM 1. For simplicity we shall prove the theorem only in case $n=1$; the proof of the general case is analogous. Lu takes the form

$$Lu\equiv A\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}+cu-\frac{\partial u}{\partial t} \qquad c\leqq 0,\ A>0\ . \tag{4}$$

From the strong maximum principle [3; Th. 5] it follows that all we need to prove is that $u(P)\equiv M$ if $P\in V'\cap S(P^0)$ where V' is some neighborhood of P^0.

There are two possibilities: Either there exists a sequence $\{P^k\}$ such that $P^k\in S(P^0)$, $P^k\to P^0$, $u(P^k)=M$, or there exists a neighborhood $V=\{x^2+t^2<R^2\}$ of P^0 such that $u(P)<M$ for all $P\in V\cap S(P^0)$, $P\neq P^0$. In the first case we can use [3; Th. 5] to conclude that $u(P)\equiv M$ if $P\in V'\cap S(P^0)$ where V' is some neighborhood of P^0 (since $u(P)=M$ for all $P\in S(P^k)$).

It remains therefore to consider the case in which $u(P)<M$ for all $P\in V\cap S(P^0)$, $P\neq P^0$. We shall prove that this case cannot occur by deriving a contradiction. Writing

$$\varphi(x)=Kx^2+o(x^2)\ ,$$

we define a domain D_δ ($\delta>0$) as the intersection of $S(P^0)$ with the set of points (x, t) in V for which

$$t<\tilde{\varphi}(x)=(K-\delta)x^2\ .$$

If $K<0$ then, because of (3), we can choose δ sufficiently small such that

$$1+A\frac{\partial^2}{\partial x^2}\tilde{\varphi}(x)|_{x=0}>0\ . \tag{5}$$

If $K\geqq 0$, we can obviously take δ such that $K-\delta<0$ and such that (5) holds.
We now can take R sufficiently small such that $\tilde{\varphi}(x)<\min(0, \varphi(x))$ for all (x, t) in D_δ, $x\neq 0$. Consequently, $u(x, t)<M$ if $t=\tilde{\varphi}(x)$, $x\neq 0$. The function $h(x, t)=-t+\tilde{\varphi}(x)$ vanishes on $t=\tilde{\varphi}(x)$ and is positive in D_δ. Therefore, if $\varepsilon>0$ is sufficiently small, then $v=u+\varepsilon h$ is smaller than M at all points on the boundary of D_δ with the exception of P^0, where $v(P^0)=M$. Noting that $\tilde{\varphi}'(0)=0$ and using (5), we conclude that

$$Lh=1+A\tilde{\varphi}''(x)+a\tilde{\varphi}'(x)+ch>0$$

if R has been chosen sufficiently small. Hence, $Lv=Lu+\varepsilon Lh>0$. It follows that v cannot assume its positive maximum at interior points of D_δ and, therefore, it assumes its maximum M at P^0. We thus obtain $\partial v/\partial t\geqq 0$ at P^0 and, consequently,

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}-\varepsilon\frac{\partial h}{\partial t}\geqq\varepsilon>0$$

(Here

$$\frac{\partial g}{\partial t}=\liminf_{t\to 0}\frac{g(0, 0)-g(0, t)}{-t}\ .$$

On the other hand, letting in (4) $P \to P^0$ in an appropriate way and using (2) and the inequality $Lu(P) \geqq 0$, we get

$$0 \leqq \lim A(P)\frac{\partial^2 u(P)}{\partial x^2} + \lim a(P)\frac{\partial u(P)}{\partial x} + C(P^0)M - \limsup \frac{\partial u(P)}{\partial t} \leqq$$
$$-\limsup \frac{\partial u(P)}{\partial t} .$$

We have thus obtained

$$\limsup_{P \to P^0} \partial u(P)/\partial t \leqq 0 < \varepsilon \leqq \partial u/\partial t.$$

This is however a contradiction (since

$$\frac{\partial u}{\partial t} = \lim_{t_k \to 0} \frac{\partial u(0, t_k)}{\partial t} \leqq \limsup_{P \to P^0} \frac{\partial u(P)}{\partial t}$$

for an appropriate sequence $\{t_k\}$), and the proof is completed.

REMARK (a) Consider the following example: $n=1$, $P^0=(0, 0)$ and D defined by

$$x^2+t^2<R,\ t<\gamma_1 x,\ t<\gamma_2 x \qquad \gamma_1>0>\gamma_2.$$

The function $u(x, t)=(t-\gamma_1 x)(\gamma_2 x-t)$ satisfies the following properties: $u<0$ in D, $u=0$ at P^0, and

$$Lu \equiv A\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}=-2A\gamma_1\gamma_2+0(|x|+|t|) \geqq 0 ,$$

provided R is sufficiently small. Consequently, (3) and the second assumption in (2) are not satisfied and also the assertion of Theorem 1 is false.

REMARK (b). Consider now the case in which the tangent hyperplane at P^0 is not of the form $t=$const.. We shall prove that in this case Theorem 1 is false. Take $n=1$ and consider first the case in which D is defined by

$$x>0,\ x^2+t^2<R^2 .$$

If $Lu \equiv \partial^2 u/\partial x^2 - \partial u/\partial t$, then the function $u(x, t)=-x$ takes its maximum in $\bar{D}$ at $P^0=(0, 0)$, $Lu=0$, but $u \neq 0$ in $S(P^0)$.

Consider next the case in which $\bar{D}$ is defined by

$$x>\alpha t,\ x^2+t^2<R^2 .$$

and take $Lu=\partial^2 u/\partial x^2-\alpha \partial u/\partial x-\partial u/\partial t$.

The transformation $t'=t$, $x'=x-\alpha t$ carries the present case into the previous one.

Note that if the tangent hyperplane H at P^0 is not the plane $t=0$ and the axes are rotated so as to give H the equation $t'=0$ (in new x', t' coordinate), then Lu loses the form (1), for $u_{x't'}$ and $u_{t't'}$ will appear in it.

REMARK (c). If in Theorem 1 the domain D is on the side $t>\varphi(x)$, then the theorem is false. Indeed, as a counter-example take $u=-t$, and D bounded from below by $t=0$.

2. Consider the linear operator

$$L'u\equiv\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i,j=1}^{m}b_{ij}(x,t)\frac{\partial^2 u}{\partial t_i\partial t_j}+\sum_{i=1}^{n}a_i(x,t)\frac{\partial u}{\partial x_i}+\sum_{i=1}^{m}b_i(x,t)\frac{\partial u}{\partial t_i}+a(x,t)u \qquad a(x,t)\leqq 0, \tag{6}$$

where $x=(x_1,\cdots,x_n)$ and $t=(t_1,\cdots,t_m)$ vary in the closure of a given $(n+m)$-dimensional domain D. We assume that L' is elliptic in the variables x and parabolic in the variables t, that is, for every real vector $\lambda\neq 0$,

$$\sum a_{ij}\lambda_i\lambda_j>0, \quad \sum b_{ij}\lambda_i\lambda_j\geqq 0\,. \tag{7}$$

All the coefficients appearing in (6) are assumed to be continuous in $\bar{D}$ and u is assumed to be continuous in $\bar{D}$ and to have a continuous t-derivative and continuous second x-derivatives in D. Under these assumptions, Nirenberg [3 ; Th. 2] has proved a weak maximum principle from which it follows that, *if $L'u\geqq 0$ in D then u must assume its positive maximum on the boundary.*

Let $P^0=(x^0,t^0)$ be a point on the boundary of D such that $u(P^0)=M>0$ is the maximum of u in $\bar{D}$. Assume that there exists a neighborhood $V: |x-x^0|^2+|t-t^0|^2<R_0^2$ of P^0 such that $u(x,t)<M$ in $V\cap D$. We then can prove the following theorem.

THEOREM 2. *If there exists a sphere $S: |x-x'|^2+|t-t'|^2<R^2$ passing through P^0 and contained in $\bar{D}$, and if $x^0\neq x'$ then, under the assumptions made above (in particular, $L'u\geqq 0$, $u(x,t)<M$ in $V\cap D$), every non-tangential derivative $\partial u/\partial\tau$ at (x^0,t^0), understood as the limit inferior of $\Delta u/\Delta\tau$ along a non-tangential direction τ, is negative.*

By a non-tangential direction we mean a direction from P^0 into the interior of the sphere S.

REMARK (a). If $a(x, t) \equiv 0$ then the assumption $M>0$ is superflous.

REMARK (b). In § 3 we shall show that the assumption $x^0 \neq x'$ is essential. We shall also discuss the case in which $u(x, t)$ is not smaller than M at all the points of $V \cap D$.

Proof. For simplicity we give the proof in the case $m=n=1$, so that

$$(8) \qquad L'u \equiv A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} + cu \quad A>0,\ B \geqq 0,\ c \leqq 0;$$

the proof of the general case is quite similar. Without loss of generality we can take $(x', t')=(0, 0)$ and $x^0>0$. Furthermore, we may assume that, with the exception of P^0, S lies in $V \cap D$, so that $u(x, t)<M$ in $S-P^0$. Denote by C the intersection of S with the plane $x>\delta$, where $0<\delta<x^0$. The function

$$h(x, t) = \exp(-\alpha(x^2+t^2)) - \exp(-\alpha R^2)$$

satisfies the following properties: $h=0$ on the boundary of S, $h \geqq 0$ in C; if α is large enough, then

$$L'h = \exp(-\alpha(x^2+t^2))[4\alpha^2(Ax^2+Bt^2) - 2\alpha(A+B+ax+bt)+c] \\ -c\exp(-\alpha R^2)>0.$$

(Here we used $x>\delta>0$, $c \leqq 0$.)

If ε is sufficiently small, then the function $v=u+\varepsilon h$ is smaller than M at all points of the boundary of C with the exception of P^0, where $v(P^0)=M$. Since $L'v=L'u+\varepsilon L'h>0$, v cannot assume its positive maximum in $\overline{C}$ at the interior of C (since, otherwise, at such interior points $L'v$ would be non-positive). Hence, v assumes its maximum at P^0 and, consequently, $\partial v/\partial\tau = \liminf(\Delta v/\Delta\tau) \leqq 0$. Since along the normal ν (i. e., along the radius through P^0) $\partial h/\partial\nu>0$ and since along the tangential direction σ $\partial h/\partial\sigma=0$, it follows that $\partial h/\partial\tau>0$. Using the definition of v, we conclude that $\partial u/\partial\tau = \partial v/\partial\tau - \varepsilon\partial h/\partial\tau<0$, and the proof is completed.

Added in proof. Theorem 2 was recently and independently proved also by R. Viborni, *On properties of solutions of some boundary value problems for equations of parabolic type*, Doklody Akad. Nauk SSSR, 117 (1957), 563–565.

3. **From now on we shall consider** only parabolic operators of the form (1). Suppose the assumption $u<M$ in $V \cap D$, made in Theorem 2, is replaced by $u \leqq M$. If there exists a sequence of points $\{P^k\}$ such

that $P^k \to P^0$, $P^k \in D$, $P^k=(x^k, t^k)$ and $t^k \geqq t^0$, $u(P^k)=M$, then, by [3 ; Th. 5], $u \equiv M$ in $S(P^k)$. Hence, if the boundary of D near P^0 is sufficiently smooth, $u \equiv M$ in some set $V' \cap D$ where V' is some neighborhood of P^0. Consequently $\partial u/\partial\tau=0$ for every τ.

If $u(P) \leqq M$ for all $P \in V \cap D$, if $u(P)$ is not strictly smaller than M for all $P \in V \cap D$, $P \neq P^0$, and if the previous situation does not arise, then one and only one of the following cases must occur:

(i) $u<M$ at all points (x, t) in $V \cap D$ with $t \geqq t^0$. Using [3; Th. 5] one can easily conclude that there exists a neighborhood V' of P such that $u<M$ in $V' \cap D$, and Theorem 2 remains true.

(ii) $u<M$ at all points (x, t) in $V \cap D$ with $t>t_0$ and $u \equiv M$ at all points (x, t) in $V \cap D$ with $t \geqq t_0$. We then consider only those directions τ along which $u<M$. We claim that *Theorem* 2 *is not true for the present situation.* To prove this, consider the following simple counter-example:

$$P^0=(0,0),\ M=0,\ Lu=\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\ ,\quad u(x, t)=\begin{cases} -t^2 & \text{if} \quad t>0 \\ 0 & \text{if} \quad t<0\ . \end{cases}$$

u satisfies $Lu \geqq 0$ and assumes its maximum 0 for $t \leqq 0$. But, the derivative $\partial u/\partial\tau$ at $P^0=(0, 0)$, along any direction τ, is zero.

As another counter-example (with $Lu=0$) one can take a fundamental solution of the heat equation.

Note that the preceding counter-examples are valid without any assumptions on the behavior of the boundary of D near P^0.

We shall now consider the case $x^1=x^0$ which was excluded by the assumptions of Theorem 2. We shall assume that at $P^0=(0, 0)$ there passes a tangent hyperplane $t=0$. If D is above this hyperplane, then the preceding counter-examples show that Theorem 2 is not true. It remains to consider the case in which D is "essentially" below $t=0$, that is, if we denote by $t=\varphi(x)$ the equation of the boundary of D near P^0, then D is on the side $t<\varphi(x)$. In this case, however, Theorem 1 tells us that in general we cannot assume both $u(P^0)=\max u(P)>0$ $(P \in \overline{D})$ and $u<u(P^0)$ in $V \cap D$.

The example in § 1 Remark (a) can also serve as a counter-example to Theorem 2 in case P^0 is a vertex-point. Indeed, along the t-direction

$$\left.\frac{\partial u}{\partial t}\right|_{P^0}=\left.\frac{\partial}{\partial t}[(t-\gamma_1 x)(\gamma_2 x-t)]\right|_{x=0,\,t=0}=0\ .$$

By a small modification of this counter-example one can get a counter-example to the analogue of Theorem 2 for elliptic operators [2] [4] in case P^0 is a vertex. Indeed, define D by

$$x^2+y^2<R^2,\ y<\gamma_1 x,\ y>\gamma_2 x \qquad \gamma_1>0>\gamma_2,$$

and take $Lu=\partial^2u/\partial x^2+A\partial^2u/\partial y^2$, where $A>|\gamma_1\gamma_2|$. The function $u(x, y)=(y-\gamma_1x)(y-\gamma_2x)$ satisfies : $u<0$ in D, $u=0$ at the origin, $Lu=2\gamma_1\gamma_2+2A>0$. But along any direction τ within D, $\partial u/\partial\tau|_{x=0, y=0}=0$.

4. **Let *D* be a domain bounded** by the two hyperplanes $t=0$, $t=T>0$ and a surface B between them. Assume that the intersection $\{t=T\}\cap\bar{D}$ is the closure of an open set on $t=T$, and denote by A the boundary of D on $t=0$. The Neumann problem for the parabolic equation $Lu=0$ consists in finding a solution to the equation $Lu=0$ which satisfies the following initial and boundary conditions :

$$u=f \text{ on } A, \quad \frac{\partial u}{\partial\nu}=g \text{ on } B$$

(f, g are given functions).

From Theorem 2 and from the strong maximum principle [3 ; Th. 5] we conclude: *If for every point $P^0=(x^0, t^0)$ of B (i) there exists a sphere with center (x', t'), $x'\neq x^0$, passing through P^0 and contained in $\bar{D}$, and (ii) $\overline{S(P^0)}$ contains interior points of A, then the Neumann problem has at most one solution.* Clearly, this uniqueness property holds also for the more general problem where $\partial u/\partial\nu$ is replaced by $\partial u/\partial\tau$ and τ is a non-tangential direction which varies on B.

As another application to Theorem 2, one can deduce the positivity of $\partial G/\partial\nu$, where G is the Green's function of $Lu=0$.

5. **Let *D* be a domain bounded** by $t=0$, $t=T\,(0<T\leqq\infty)$ and surfaces Γ_k, $0\leqq k\leqq m$, Γ_0 being the outer boundary. Suppose further that the intersection of each Γ_k with $t=t_0$ $(0\leqq t_0<T)$ is a simple closed curve $\gamma_k(t_0)$ which belongs to $C^{(3)}$ and does not reduce to a single point. Write $u_{x_i}=\partial u/\partial x_i$, $u_t=\partial u/\partial t$. We shall consider the following problem P:

$$\sum_{i,j=1}^{n} a_{ij}(x, t)u_{x_ix_j}-u_t=c(x, t, u, \nabla u) \tag{9}$$

(where ∇u denotes the vector $\partial u/\partial x_i$),

$$\frac{\partial u}{\partial\tau}\equiv\sum_{i=1}^{n}\alpha_i(x, t)u_{x_i}+\alpha(x, t)u_t=\varphi(x, t, u) \quad (x, t)\in\Gamma=\sum_{k=0}^{m}\Gamma_k \tag{10}$$

$$u(x, 0)=\psi(x) \text{ on } A \qquad A=\bar{D}\cap\{t=0\} \tag{11}$$

We make the following assumptions :

(a) $a_{ij}(x, t)$ is continuous in $\bar{D}$; $c(x, t, u, \nabla u)$ and it first derivatives with respect to u, ∇u are continuous for $(x, t)\in\bar{D}$ and for all values of u, ∇u.

(b) φ and $\partial\varphi/\partial u$ are continuous for all $(x, t)\in\Gamma$ and for all u.
(c) $\alpha_i(x, t)$, $\alpha(x, t)$ are continuous for $(x, t)\in\Gamma$; $\psi(x)$ is continuous in A.
(d) (9) is parabolic in $\bar{D}$, that is, there exists a positive constant δ such that

$$\sum a_{ij}(x, t)\xi_i\xi_j \geqq \delta\sum\xi_i^2 \tag{12}$$

holds for all real ξ and for all $(x, t)\in\bar{D}$.
(e) On each surface Γ_k $(k=0, 1, \cdots, m)$ either all the directions $\tau=(\alpha_i, \alpha)$ are exterior or all are interior, and in the exterior case $\alpha\geqq 0$ and the directions $(\alpha_i, 0)$ are exterior while in the interior case $\alpha\leqq 0$ and the directions $(\alpha_i, 0)$ are interior.

Denote by $\sum$ the class of functions $u(x, t)$ defined and continuous in $\bar{D}$ and satisfying the following conditions:
(α) $\partial u/\partial t$, $\partial u/\partial x_i$, $\partial^2 u/\partial x_i\partial x_j$ are continuous in D;
(β) For every $R>0$, $\partial u/\partial x_i$ is bounded in $D\cap\{|x|^2+t^2<R^2\}$.

THEOREM 3. *Under the assumptions* (a)—(e) *the problem P cannot have two different solutions in the class* $\sum$.

We shall need the following lemma.

LEMMA. *There exists a function* $\zeta(x)$ *defined in A and having the following properties*: (i) ζ *has continuous first derivatives in A and continuous second derivatives in the interior of A*; (ii) $\partial\zeta/\partial\nu=-1$ *and* $\partial\zeta/\partial\mu=0$ *on* $\gamma_0(0), \cdots, \gamma_m(0)$, *where* $\partial/\partial\nu$ *and* $\partial/\partial\mu$ *denote the derivatives with respect to the interior normal and to any tangential direction, respectively.*

PROOF OF THE LEMMA. It will be enough to construct a function $\chi_0(x)$ which is C'' in A, which vanishes in a neighborhood of $\gamma_i(0)$ $(i=1, \cdots, m)$ and for which $\partial\chi_0/\partial\nu=-1$, $\partial\chi_0/\partial\mu=0$ along $\gamma_0(0)$; constructing $\gamma_1(x)$ in a similar manher, we can then take $\zeta(x)=\sum\chi_1(x)$. Since $\gamma_0(0)$ belongs to $C^{(3)}$, the normals issuing from $\gamma_0(0)$ and inwardly directed cover in a one-to-one manner a small inner neighborhood of $\gamma_0(0)$, call it A_0. To each point x in A_0 there corresponds a unique point x^0 on the boundary of $\gamma_0(0)$, such that x lies on the normal through x^0. Denote by $\sigma(x)$ the distance $|x-x^0|$. It is elementary to show that $\sigma(x)$ has continuous second derivatives in A_0. Denote by A_1 the domain $0\leqq\sigma\leqq\varepsilon_0$, where $\varepsilon_0>0$ is small enough so that $A_1\subset A_0$. The function

$$\chi_0(x)=\begin{cases}\dfrac{1}{3\varepsilon_0^2}(\varepsilon_0-\sigma(x))^3 & \text{if } x\in\bar{A}_1\\[2ex] 0 & \text{if } x\in A-A_1\end{cases}$$

belongs to C'' in A and satisfies: $\partial\chi_0/\partial\nu=\partial\chi_0/\partial\sigma=-1$ and $\partial\chi_0/\partial\nu=0$ on $\gamma_0(0)$, and χ_0 vanishes near $\gamma_k(0)$, $(1\leqq k\leqq m)$; the proof is completed.

PROOF OF THEOREM 3. We first consider the case $n>1$. We may suppose that the vectors (α_i, α) are exterior directions on $\Gamma_0, \cdots, \Gamma_q$ and that (α_i, α) are interior directions on $\Gamma_{q+1}, \cdots, \Gamma_m$. Suppose now that u and v are two solutions in $\sum$ of the problem P, and define $w=v-u$. Writing

$$C(x, t, u, v)=\int_0^1 \frac{\partial}{\partial u} c(x, t, u+\lambda w, \nabla u+\lambda \nabla w)d\lambda$$

$$C_i(x, t, u, v)=\int_0^1 \frac{\partial}{\partial u_{x_i}} c(x, t, u+\lambda w, \nabla u+\lambda \nabla w)d\lambda$$

$$\Phi(x, t, u, v)=\int_0^1 \frac{\partial}{\partial u} \varphi(x, t, u+\lambda w)d\lambda$$

and using (9), (10) and (11), we obtain for w the following system:

(13) $$\sum a_{ij} w_{x_i x_j} - w_t = Cw + \sum C_i w_{x_i}$$

(14) $$\frac{\partial w}{\partial \tau} = \sum \alpha_i w_{x_i} + \alpha w_t = \Phi w$$

(15) $$w(x, 0)=0 .$$

Substituting $w(x, t)=z(x, t)\exp(Kt+M\zeta(x))$, where $\zeta(x)$ is the function constructed in the lemma and K, M are constant to be determined later, we get for z the following system:

(13′) $$\sum a_{ij} z_{x_i x_j} - z_t = -M \sum a_{ij} \zeta_{x_i x_j} z - M^2 \sum a_{ij} \zeta_{x_i} \zeta_{x_j} z$$
$$-2M \sum a_{ij} \zeta_{x_i} z_{x_j} + Kz + Cz + M \sum C_i \zeta_{x_i} z + \sum C_i z_{x_i}$$

(14′) $$\frac{\partial z}{\partial \tau} = \sum \alpha_i z_{x_i} + \alpha z_t = -M \sum \alpha_i \zeta_{x_i} z - \alpha K z + \Phi z$$

(15′) $$z(x, 0)=0 .$$

If $0 \leqq k \leqq q$, $\alpha \geqq 0$ and $\sum \alpha_i(x, 0)\zeta_{x_i}(x)>0$ on $\gamma_k(0)$, since the angle between the vectors (α_i) and grad ζ is $<\pi/2$. By continuity we get $\sum \alpha_i(x, t)\zeta_{x_i}(x) \geqq \eta>0$ on $\gamma_k(t)$, provided $0 \leqq t \leqq T'$ and T' is sufficiently small. Hence, we can choose M sufficiently large such that

(16) $$-M \sum \alpha_i \zeta_{x_i} - \alpha K + \Phi < 0$$

holds on $\gamma_k(t)$, provided $K \geqq 0$ and $0 \leqq t \leqq T'$.

If $q+1 \leqq k \leqq m$, $\alpha \leqq 0$ and $\sum \alpha_i(x, 0)\zeta_{x_i}(x)<0$, since the angle between (α_i) and $-$grad ζ is $<\pi/2$. Again, if $K \geqq 0$ and M is sufficiently large, then

(17) $$-M \sum \alpha_i \zeta_{x_i} - \alpha K + \Phi > 0$$

on $\gamma_k(t)$, $0 \leqq t \leqq T'$.

Having fixed M, we now choose K sufficiently large so that the coefficient of z on the right side of (13') becomes positive in the domain $D_{T'} = D \cup \{0 < t < T'\}$. We claim that $z \equiv 0$ in $D_{T'}$. Indeed, if this is not the case then, using (15') and the weak maximum principle [3 ; Th. 2] we conclude that z assumes either its positive maximum or its negative minimum on the boundary $\sum_{k=0}^{m} \gamma_k(t)$, $0 \leqq t \leqq T'$, of $D_{T'}$. It will be enough to consider the case in which z assumes its positive maximum at a point P^0 on $\gamma_k(t)$. If $0 \leqq k \leqq q$, then $\partial z/\partial \tau \geqq 0$ since τ is outwardly directed. On the other hand, using (14') and (16) we get $\partial z/\partial \tau < 0$, which is a contradiction. If $q+1 \leqq k \leqq m$, then $\partial z/\partial \tau \leqq 0$ since τ is inwardly directed. On the other hand, using (14') and (17) we get $\partial z/\partial \tau > 0$ which is a contradiction. We have thus proved that $z \equiv w \equiv 0$ in $D_{T'}$. We can now apply a classical procedure of continuation and thus complete the proof of the theorem for the case $n > 1$.

In the case $n=1$, $\Gamma = \Gamma_0$ is composed of two curves Γ_{01} aud Γ_{02}. Suppose Γ_{0k} intersects $t=0$ at a_k, $a_1 < a_2$. The function

$$\zeta(x) = \frac{(x-a_1)(x-a_2)}{a_2 - a_1}$$

can be used in the preceding proof. Note that it is not necessary to make any assumptions on the smoothness of the curves Γ_{0k}.

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AN INVERSION OF THE STIELTJES TRANSFORM

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A generalized Lambert transform, or L-transform, is an integral of the form

$$H(x)=\int_0^\infty \sum_{k=1}^\infty a_k e^{-kxt}\psi(t)dt .$$

In this paper we shall invert the integral transform

$$G(x)=\int_0^\infty \frac{\psi(t)t\,dt}{x^2+t^2} \qquad 0<x<\infty \tag{1}$$

by reducing it by means of a certain summation to an L-transform and then applying an inversion theorem for L-transforms.

From this we deduce an inversion formula for the Stieltjes transform. This is given in Theorem 3.

1. The inversion of the transform (1). We shall need the following theorem on L-transforms which is the case $r=1$ of Theorem 7.7 in [1].

THEOREM 1. *Let* $\{a_k\}_{k=1}^\infty$ *be a bounded sequence of non-negative numbers with* $a_1>0$. *Let* $\{b_n\}_{n=1}^\infty$ *be the (unique) sequence such that*

$$\sum_{d|m} a_d b_{m/d}=\begin{cases}1, & m=1\\ 0, & m=2,\ 3,\ \cdots,\end{cases}$$

the summation running over all divisors d *of* m. *If the* b_n *are also bounded and if*

1. $K(t)=\sum_{k=1}^\infty a_k e^{-kt}$ $(0<t<\infty)$
2. $H(x)=\int_0^\infty K(xt)\psi(t)dt$ *converges for some* $x>0$
3. $\int_0^1 \frac{|\psi(t)\log t|}{t}dt<\infty$

then

$$\lim_{p\to\infty}\frac{(-1)^p}{p\,!}\left(\frac{p}{t}\right)^{p+1}\sum_{n=1}^\infty b_n n^p H^{(p)}\left(\frac{np}{t}\right)=\psi(t) \text{ almost everywhere } (0<t<\infty) .$$

Received March,21 1958.

Now let

$$G(x)=\int_0^\infty \frac{\psi(t)t\,dt}{x^2+t^2}$$

where we assume

$$\int_0^\infty \frac{|\psi(t)|}{t}dt<\infty \quad\text{and}\quad \int_0^1 \frac{|\psi(t)\log t|}{t}dt<\infty .$$

To reduce $G(x)$ to an L-transform we define

$$H_N(x)=\frac{1}{x}\left[\frac{G(0)}{2}+\sum_{k=1}^{N}(-1)^k G\left(\frac{k\pi}{x}\right)\right] \qquad N=1,\ 2,\ \cdots$$

Then

$$(2)\qquad H_N(x)=\int_0^\infty \psi(t)\left[\frac{1}{2xt}+\sum_{k=1}^{N}\frac{(-1)^k xt}{(xt)^2+(k\pi)^2}\right]dt .$$

For $N=1,\ 2,\ \cdots$ we have

$$\left|\sum_{k=1}^{N}\frac{(-1)^k xt}{(xt)^2+(k\pi)^2}\right|<\frac{xt}{(xt)^2+\pi^2}<\frac{1}{xt} \qquad (0<xt<\infty).$$

(This is because the terms of the sum alternate in sign and decrease in absolute value so that the modulus of the sum is less than that of its first term.) Hence for any $x>0$

$$\int_0^\infty \left|\psi(t)\left[\frac{1}{2xt}+\sum_{k=1}^{N}\frac{(-1)^k xt}{(xt)+(k\pi)^2}\right]\right|dt\leqslant\frac{3}{2x}\int_0^\infty\frac{|\psi(t)|}{t}dt<\infty .$$

This, by dominated convergence, allows us to let N become infinite under the integral sign in (2) and we obtain

$$H(x)\equiv\lim_{N\to\infty}H_N(x)=\int_0^\infty \psi(t)\left[\frac{1}{2xt}+\sum_{k=1}^{\infty}\frac{(-1)^k xt}{(xt)^2+(k\pi)^2}\right]dt \qquad (x>0).$$

But for $z>0$

$$\frac{1}{2z}+\sum_{k=1}^{\infty}\frac{(-1)^k z}{z^2+(k\pi)^2}=\frac{\operatorname{cosech} z}{2}=\frac{1}{e^z-e^{-z}}=\sum_{k=1}^{\infty}e^{-(2k-1)z},$$

(see [3 ; 113]). Thus

$$(3)\qquad H(x)=\int_0^\infty\sum_{k=1}^{\infty}e^{-(2k-1)xt}\psi(t)dt=\int_0^\infty K(xt)\psi(t)dt$$

where $K(t)=\sum_{k=1}^{\infty}a_k e^{-kt}$ and

(4) $$a_{2k-1}=1\ ,\ a_{2k}=0\ , \qquad k=1,\ 2,\ \cdots$$

It was shown in [2; 556] that the sequence $\{b_n\}_{n=1}^{\infty}$ defined in Theorem 1 corresponding to the a_k in (4) is

(5) $$b_{2n-1}=\mu_{2n-1}\ ,\ b_{2n}=0\ , \qquad n=1,\ 2,\ \cdots$$

Here the μ_n are the Moebius numbers defined as $\mu_1=1$, $\mu_n=(-1)^s$ if n is the product of s distinct primes and $\mu_n=0$ if n is divisible by a square factor. The b_n are bounded, so that we may apply Theorem 1 (with the a_k and b_n as in (4) and (5)) to invert the L-transform (3) and obtain $\psi(t)$ for almost all $t>0$. These results are summarized in Theorem 2.

THEOREM 2. *Let*

$$G(x)=\int_0^{\infty}\frac{\psi(t)t\,dt}{x^2+t^2}$$

where

$$\int_0^{\infty}\frac{|\psi(t)|}{t}dt<\infty$$

and

$$\int_0^{1}\frac{|\psi(t)\log t|}{t}dt<\infty$$

Then

$$H(x)=\lim_{N\to\infty}\frac{1}{x}\left[\frac{G(0)}{2}+\sum_{k=1}^{\infty}(-1)^kG\left(\frac{k\pi}{x}\right)\right]$$

exists for all positive x and

$$H(x)=\int_0^{\infty}\sum_{k=1}^{\infty}e^{-(2k-1)xt}\psi(t)dt\ .$$

Moreover

$$\lim_{p\to\infty}\frac{(-1)^p}{p\,!}\left(\frac{p}{t}\right)^{p+1}\sum_{n=1}^{\infty}\mu_{2n-1}(2n-1)^pH^{(p)}\left[\frac{(2n-1)p}{t}\right]=\psi(t)$$

almost everywhere $(0<t<\infty)$.

2. The inversion of the Stieltjes transform. Let

(6) $$F(x)=\int_0^{\infty}\frac{\varphi(t)}{x+t}dt$$

where

$$\int_0^\infty \frac{|\varphi(t)|}{t}\,dt<\infty \quad \text{and} \quad \int_0^1 \frac{|\varphi(t)\log t|}{t}\,dt<\infty .$$

Let $G(x)=\frac{1}{2}F(x^2)$, $\psi(t)=\varphi(t^2)$. Then

$$G(x)=\frac{1}{2}F(x^2)=\frac{1}{2}\int_0^\infty \frac{\varphi(t)}{x^2+t}\,dt=\int_0^\infty \frac{\varphi(t^2)t\,dt}{x^2+t^2}=\int_0^\infty \frac{\psi(t)t\,dt}{x^2+t^2} ;$$

also

$$\int_0^\infty \frac{|\psi(t)|}{t}\,dt=\int_0^\infty \frac{\varphi(t^2)}{t}\,dt=\frac{1}{2}\int_0^\infty \frac{\varphi(t)}{t}\,dt<\infty ;$$

similarly

$$\int_0^1 \frac{|\psi(t)\log t|}{t}\,dt<\infty .$$

The assumptions of Theorem 2 thus hold. We can therefore use Theorem 2 to obtain $\psi(t)=\varphi(t^2)$ for almost all $t>0$. This gives us $\varphi(t)$ for almost all $t>0$ and thus effects an inversion of the Stieltjes transform (6).

THEOREM 3. *Let*

$$F(x)=\int_0^\infty \frac{\varphi(t)}{x+t}\,dt$$

where

$$\int_0^\infty \frac{|\varphi(t)|}{t}\,dt<\infty$$

and

$$\int_0^1 \frac{|\varphi(t)\log t|}{t}\,dt<\infty .$$

Let

$$G(x)=\frac{1}{2}F(x^2)$$

and

$$H(x)=\frac{1}{x}\left[\frac{G(0)}{2}+\sum_{k=1}^\infty(-1)^k G\left(\frac{k\pi}{x}\right)\right]$$

(the sum converging by Theorem 2). *Then*

$$\lim_{p\to\infty}\frac{(-1)^p}{p\,!}\left(\frac{p}{\sqrt{t}}\right)^{p+1}\sum_{n=1}^{\infty}\mu_{2n-1}(2n-1)^p H^{(p)}\left[\frac{(2n-1)p}{\sqrt{t}}\right]=\varphi(t)$$

almost everywhere $(0<t<\infty)$.

Of course, the Stieltjes transform has been inverted under less restrictive conditions on $\varphi(t)$. We believe the interest of this note lies in the use of the μ_n as an inverting device.

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ON THE PERIODICITY OF THE SOLUTION OF A CERTAIN NONLINEAR INTEGRAL EQUATION

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In the following paper we will study the nonlinear integral equation

$$(1) \qquad E(t)=F(t)-\int_0^t G(t-\tau)N\{E(\tau)\}\,d\tau$$

where $F(t)$ is a known periodic real function and $G(t)$ and $N(x)$ are known real functions. In particular we will investigate the behaviour of the solution $E(t)$ of the equation (1) for large values of t.

We assume that $G\in L[0,\ \infty]$ and that $N(x)$ is bounded almost everywhere and Borel-measurable in $[-\infty,\ \infty]$. Furthermore $N(x)$ is assumed expressible in the form

$$(2) \qquad N(x)\sim N(0)+\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda x}-1}{i\lambda}d\lambda$$

with $\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda<\infty$ and with finite $N(0)$. This representation is to be valid almost everywhere in $[-\infty,\ \infty]$

Because $N(x)$ is Borel-measurable in $[-\infty,\ \infty]$ and $|N(0)|<\infty$, the measurability of x implies the measurability of $N(x)$. The following four classes of $N(x)$-functions are distinguished:

$$(3) \qquad \begin{array}{llll} N\in K_{11} & \text{if} \quad x\in L[0,\ 1] & \text{implies} \quad N(x)\in L[0,\ 1] \\ N\in K_{1\infty} & \text{if} \quad x\in L[0,\ 1] & \text{implies} \quad N(x)\in L[0,\ \infty] \\ N\in K_{\infty 1} & \text{if} \quad x\in L[0,\ \infty] & \text{implies} \quad N(x)\in L[0,\ 1] \\ N\in K_{\infty\infty} & \text{if} \quad x\in L[0,\ \infty] & \text{implies} \quad N(x)\in L[0,\ \infty] \end{array}$$

The space of measurable and bounded functions defined on the finite interval $[0,\ A]$ will be denoted by $M[0,\ A]$. The norm of $x\in M[0,\ A]$ is defined, as usual, by

$$||x||=\inf_{E}\left\{\sup_{t\in[0,A]-E}|x|\right\}$$

where E ranges over the sets of measure zero in $[0,\ A]$, and the distance of $x\in M[0,\ A]$ and $y\in M[0,\ A]$ by $||x-y||$. The space $M[0,\ 1]$ is complete.

The proofs in this paper will be based on the following theorem by Tihonov (see for instance [1]) which is valid in $M[0,\ A]$: Let the operator B map $M[0,\ A]$ into itself and let $||B(x)-B(y)||\leqq\beta||x-y||$ for all x and

Received March 5, 1957. The preparation of this paper was sponsored by the Office of Naval Research and the Office of Ordnance Research, U.S. Army.

y in $M[0, A]$, where $\beta<1$. Then the equation $y=B(y)$ has a unique solution $\bar{y}$ in $M[0, A]$. The function $\bar{y}$ may be obtained by iteration:

$$\bar{y}=\lim_{n\to\infty} y_n$$

where $y_n=B(y_{n-1})$ and where y_0 may be taken arbitrarily from $M[0, A]$.

We will prove the following theorem.

THEOREM. *Suppose that $F(t)$ is a periodic function in $[0, \infty]$ with period T, and that $F\in M[0, T]$. Furthermore suppose that $G\in L[0, \infty]$, $N\in K_{1\infty}$ and*

$$\left(\int_0^\infty |G(u)|du\right)\left(\int_{-\infty}^{+\infty} |S(\lambda)|\, d\lambda\right)<1 .$$

If $E(t)$ is the solution of

$$E(t)=F(t)-N(0)\int_0^t G(u)du-\int_0^t G(t-\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda E(\tau)}-1}{i\lambda}d\lambda d\tau \tag{4}$$

then lim $E(nT+u)=v(u)$ *exists, as $n\to\infty$ through integer values. The convergence is uniform. Moreover, $v(u)$ has the period T, and satisfies*

$$v(u)=F(u)-N(0)\int_0^\infty G(u)du-\int_0^\infty G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\ \frac{e^{i\lambda v(u-\tau)}-1}{i\lambda}d\lambda d\tau \tag{5}$$

This equation can be solved by iteration stating with any element of $M[0, T]$. The solution of (5) *is unique.*

In order to prove the theorem, we will first prove two lemmas. Put

$$H[\Delta(u+mT)]=\int_0^{t_0} G(\tau)d\tau\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(mT+u-\tau)}\frac{e^{i\lambda\Delta(mT+u-\tau)}-1}{i\lambda}d\lambda$$

where $\Delta(u+mT)=E(u+nT)-E(u+mT)$ and $0\leqq u\leqq T$. Here T is a finite positive real number, t_0 a positive real number which may be finite or infinite and m and n positive integers. $E(u+nT)\in M[0, T]$ and $E(u+mT)\in M[0, T]$ implies $\Delta(u+mT)\in M[0, T]$. The operator H will play an important role in the following considerations. For this reason we will first establish some of its properties. We will write more briefly $H(\Delta(mT+u))=H(\Delta)$.

LEMMA 1. *Suppose that $G\in L[0, \infty]$, and suppose that the function $N(x)$ belongs to one of the classes K_{11} and $K_{1\infty}$. Then $\Delta\in M[0, T]$ implies $H(\Delta)\in M[0, T]$ and*

$$||H(\Delta_1)-H(\Delta_2)||\leqq\beta||\Delta_1-\Delta_2||$$

where

$$\beta=\left(\int_0^\infty |G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|\,d\lambda\right).$$

Now

$$\int_0^t G(\tau)[N\{E_1(t-\tau)\}-N\{E_2(t-\tau)\}]d\tau$$
$$=\int_0^t G(\tau)I(t_0-\tau)[N\{E_1(t-\tau)\}-N\{E_2(t-\tau)\}]d\tau$$

where

$$N(E)=\int_{-\infty}^{+\infty}S(\lambda)\frac{e^{i\lambda E}-1}{i\lambda}d\lambda$$

and

$$I(t_0-t)=\begin{cases}1 & \text{if } t\leqq t_0\\ 0 & \text{if } t_0<t\end{cases}$$

$G\in L[0,\ \infty]$ implies $G(\tau)I(t_0-\tau)\in \mathrm{L}[0,\ \infty]$. Furthermore, from $x\in M[0,\ T]$ and the properties of $N(x)$ follows that $N(x)\in M[0,\ T]$. Consequently $N(x)\in L[0,T]$. From known properties of the convolution follows now that

$$\int_0^t G(\tau)I(t_0-\tau)[N\{E_1(t-\tau)\}-N\{E_2(t-\tau)\}]d\tau\in L[0,\ T]\ .$$

Hence $H(\Delta)\in L[0,\ T]$. Now, as is easily seen,

$$||H(\Delta)||\leqq\left(\int_{-\infty}^{+\infty}|S(\lambda)|\,d\lambda\right)||\int_0^t|G(u)||\Delta|(t+mT-u)|du||\leqq\beta||\Delta||$$

which implies the boundedness of $H(\Delta)$. The function $H(\Delta)$ is thus measurable and bounded in $[0,\ T]$, $H(\Delta)\in M[0,\ T]$. Furthermore

$$||H(\Delta_2)-H(\Delta_1)||=||\int_0^{t_0}G(\tau)d\tau\int_{-\infty}^{+\infty}S(\lambda)e^{i\lambda E(u+mT-\tau)}\frac{e^{i\lambda\Delta_2(u+mT-\tau)}-e^{i\lambda\Delta_1(u+mT-\tau)}}{2}d\lambda||$$
$$\leqq\left(\int_{-\infty}^{+\infty}|S(\lambda)|\,d\lambda\right)||\int_0^{t_0}|G(\tau)||\Delta_2(u+mT-\tau)-\Delta_1(u+mT-\tau)|d\tau||$$
$$\leqq\beta||\Delta_2-\Delta_1||\ ,$$

which completes the proof.

We will now consider the norm

$$(6)\qquad ||E(u+nT)-E(u+mT)+\int_0^{f(m)}G(\tau)[N\{E(u+nT-\tau)\}-N\{E(u+mT-\tau)\}]d\tau||=Q$$

where m and n are positive integers, $f(m)$ an arbitrary function of m,

T a finite positive number and $E \in M[0, T]$. Furthermore it will be assumed that $G \in L[0, \infty]$ and $N \in K_{1\infty}$ and that they satisfy the condition

$$\left(\int_{-\infty}^{+\infty} |S(\lambda)|\, d\lambda\right)\left(\int_0^\infty |G(u)| du\right) < 1 \,.$$

The following lemma holds.

LEMMA 2. *For every $\varepsilon > 0$ there exists an integer m_0 such that $m \geqq m_0$ and $n \geqq m_0$ imply $Q < \varepsilon$, if and only if, with $v(u)$ from $M[0, T]$, $||E(u+pT) - v(u)|| \to 0$ as $p \to \infty$ through positive integral values.*

Suppose first that $||E(u+pT) - v(u)|| \to 0$, as $p \to \infty$, where E and v are in $M[0, T]$. Now

$$||\int_0^{f(m)} G(\tau)[N\{E(u+nT-\tau)\} - N\{E(u+mT-\tau)\}]d\tau$$

$$= ||\int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\, \frac{e^{i\lambda E(u+nT-\tau)} - e^{i\lambda E(u+mT-\tau)}}{i\lambda}\, d\lambda d\tau||$$

$$\leqq \left(\int_0^\infty |G(u)| du\right)\left(\int_{-\infty}^{+\infty} |S(\lambda)| d\lambda\right) ||E(u+nT-\tau) - E(u+mT-\tau)||$$

and consequently

$$||E(u+nT) - E(u+mT) + \int_0^{f(m)} G(\tau)[N\{E(u+nT-\tau)\} - N\{E(u+mT-\tau)\}]d\tau||$$

$$\leqq \left[1 + \left(\int_0^\infty |G(u)| du\right)\left(\int_{-\infty}^{+\infty} |S(\lambda)|\, d\lambda\right)\right] ||E(u+nT) - E(u+mT)||$$

where $\left(\int_0^\infty |G(u)| du\right)\left(\int_{-\infty}^{+\infty} |S(\lambda)| d\lambda\right) < 1$. Because $||E(u+pT) - v(u)|| \to 0$, as $p \to \infty$, there exists for every $\varepsilon > 0$ an integer m_1 such that $m_1 \leqq m < n$ implies

$$||E(u+nT) - E(u+mT)|| \leqq \frac{\varepsilon}{1+\beta}$$

from which the first part of the lemma follows.

Suppose now that (6) is valid for m and n greater than a given integer m_2. The inequality (6) may be written

$$\begin{aligned} &||\Delta(u+mT) \\ (7) \qquad &+ \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda) e^{i\lambda E(u+mT-\tau)}\, \frac{e^{i\lambda \Delta(u+mT-\tau)} - 1}{i\lambda}\, d\lambda d\tau|| \leqq \varepsilon \end{aligned}$$

where $\Delta(u+mT) = E(u+nT) - E(u+mT)$

Now let h be a function in $M[0, T] \cap S(\varepsilon, 0)$ where $S(\varepsilon, 0)$ is the sphere with radius ε and center at $h = 0$. Put

$$\text{(8)}\qquad \Delta(u+mT) + \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(u+mT-\tau)}\,\frac{e^{i\lambda\Delta(u+mT-\tau)}-1}{i\lambda}\,d\lambda d\tau = h(u)\,.$$

The functions Δ obtained by solving (8) for all $h\in M[0,\ T]\cap S(\varepsilon,\ 0)$ are those which satisfy (7). $E(u+mT)$ is a known function.

The equation

$$\text{(9)}\qquad \begin{aligned}\Delta(u+mT) &= h(u) - \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(u+mT-\tau)}\,\frac{e^{i\lambda\Delta(u+mT-\tau)}-1}{i\lambda}\,d\lambda d\tau \\ &= h(u) - H[\Delta(u+mT)]\end{aligned}$$

where H is the operator defined on page 3, may be solved by iteration.

Indeed, by Lemma 1 the operator H is defined in $M[0,\ T]$, $\Delta\in M[0,\ T]$ implies $H(\Delta)\in M[0,\ T]$ and

$$\|h(u)-H(\Delta_1)-(h(u)-H(\Delta_2))\| = \|H(\Delta_2)-H(\Delta_1)\| \leqq \beta\|\Delta_2-\Delta_1\|$$

where $\beta=\left(\int_0^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)<1.$

The conditions of the Tihonov's theorem are thus satisfied. We begin the iteration process with an h from $M[0,\ T]\cap S(\varepsilon,\ 0)$:

$$\Delta_1(u+nT) = h(u) - \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(u+nT-\tau)}\,\frac{e^{i\lambda h(u-\tau)}-1}{i\lambda}\,d\lambda d\tau$$

and generally

$$\Delta_{p+1}(u+nT) = h(u) - \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(u+nT-\tau)}\,\frac{e^{i\lambda\Delta_p(u+mT-\tau)}-1}{i\lambda}\,d\lambda d\tau$$

The unique solution of (9) is then $\lim\limits_{k\to\infty}\Delta_k(u+nT)=\Delta(u+nT)$ where $\Delta(u+mT)$ is in $M[0,\ T]$.

Now

$$\begin{aligned}\|\Delta_{p+1}\| &\leqq \|h\| + \left\|\int_0^{f(m)} |G(\tau)|\int_{-\infty}^{+\infty} |S(\lambda)|\,|e^{i\lambda E(u+mT-\tau)}|\,\left|\frac{e^{i\lambda\Delta_p}-1}{i\lambda}\right|d\lambda d\tau\right\| \\ &\leqq \varepsilon + \left(\int_0^{\infty}|G(\lambda)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)\|\Delta_p\| \leqq \varepsilon+\beta\|\Delta_p\|\end{aligned}$$

From this inequality one obtains now, remembering that $\|\Delta_0\|=\|h\|\leqq\varepsilon$ and that $\beta<1$,

$$\|\Delta_{p+1}\| \leqq (1+\beta+\beta^2+\cdots+\beta^{p+1})\varepsilon \leqq \frac{\varepsilon}{1-\beta}$$

This inequality holds true for all p. Consequently

$$||\Delta(u+nT)||\leqq\frac{\varepsilon}{1-\beta}$$

or, in view of the definition of $\Delta(u+nT)$,

$$||E(u+nT)-E(u+mT)||\leqq\frac{\varepsilon}{1-\beta}$$

for m and n greater than m_2. But such m_2 exists for every $\varepsilon>0$. From this and from the completeness of the space $M[0, T]$ follows that there exists a $v_1\in M[0, T]$ such that

$$||E(u+pT)-v_1(u)||\longrightarrow 0$$

as $p\to\infty$ through integral values.

We now proceed to prove the Theorem.

Because of the periodicity of $F(t)$ one obtains from (1)

$$E(u+nT)+\int_0^{u+nT}G(\tau)N\{E(u+nT-\tau)\}d\tau$$
$$=E(u+mT)+\int_0^{u+mT}G(\tau)N\{E(u+mT-\tau)\}d\tau$$

where $0\leqq u\leqq T$ and where m and n are positive integers.

Suppose that $m<n$ and $t_0\leqq mT$. Then

$$E(u+nT)-E(u+mT)+\int_0^{t_0}G(\tau)[N\{E(u+nT-\tau)\}-N\{E(u+mT-\tau)\}]d\tau$$
$$=\int_{t_0}^{u+mT}G(\tau)N\{E(u+mT-\tau)\}d\tau-\int_{t_0}^{u+nT}G(\tau)N\{E(u+nT-\tau)\}d\tau$$

and

$$||E(u+nT)-E(u+mT)+\int_0^{t_0}G(\tau)[N\{E(u+nT-\tau)\}-N\{E(u+mT-\tau)\}]d\tau||$$
$$\leqq||\int_{t_0}^{u+nT}|G(\tau)||N\{E(u+nT-\tau)\}|d\tau+\int_{t_0}^{u+mT}|G(\tau)||N\{E(u+mT-\tau)\}|d\tau||$$
$$\leqq\left(||\int_{t_0}^{u+nT}|G(\tau)|d\tau||+||\int_{t_0}^{u+mT}|G(\tau)|\,d\tau||\right)||N||\leqq 2||N||\int_{t_0}^{\infty}|G(\tau)|d\tau$$

Because $G\in L[0, \infty]$, there exists a positive integer m_3 for every $\varepsilon>0$ such that for $t_0=m_3T$

$$\int_{m_3T}^{\infty}|G(\lambda)|du\leqq\frac{\varepsilon}{2||N||}$$

But $m_3\leqq m<n$. Consequently, for every $\varepsilon>0$ there exists a positive integer m_3 such that $m_3\leqq m<n$ implies

$$||E(u+nT)-E(u+mT) + \int_0^{m_2 T} G(\tau)[N\{E(u+nT-\tau)\}-N\{E(u+mT-\tau)\}]d\tau|| \leqq \varepsilon$$

By Lemma 2 it follows now that there exists a $v \in M[0, T]$ such that $||E(u+pT)-v(u)|| \to 0$ as $p \to \infty$ through positive integral values. Consequently $E(u+pT)$ converges uniformly to $v(u)$ in $[0, T]$. That $v(u)$ is periodic with period T is immediate.

We substitute now

$$E(u+nT)=v(u)+H_n(u)$$

where $H_n \in M[0, T]$ and $||H_n|| \to 0$, as $n \to \infty$ and where $0 \leqq u \leqq T$, into (1) and obtain

$$v(u)+H_n(u)=F(u)-N(0)\int_0^{u+nT} G(\tau)d\tau - \int_0^{u+nT+} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda v(u-\tau)}e^{i\lambda H_n(u-\tau)}-1}{i\lambda}d\lambda d\tau$$

As is seen at once, this may be rewritten as follows:

$$v(u)-F(u)+\int_0^{\infty} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda v(u-\tau)}-1}{i\lambda}d\lambda d\tau + N(0)\int_0^{\infty} G(\tau)d\tau$$
$$+\int_0^{\infty} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda v(u-\tau)}\frac{e^{i\lambda H_n(u-\tau)}-1}{i\lambda}d\lambda d\tau + H_n(u) +$$
$$-\int_{nT+u}^{\infty} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda v(u-\tau)}e^{i\lambda H_n(u-\tau)}-1}{i\lambda}d\lambda d\tau - N(0)\int_{nT+u}^{\infty} G(\tau)d\tau = 0$$

which yields the inequality

$$||v(u)-F(u)+\int_0^{\infty} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda v(u-\tau)}-1}{i\lambda}d\lambda d\tau||$$
$$\leqq ||H_n(u)|| + \left(\int_0^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)||H_n(u)||$$
$$+\left(\int_{nT+u}^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)||v(u)+H_n(u)|| + N(0)\int_{nT+u}^{\infty}|G(u)|du$$
$$=(1+\beta)||H_n(u)|| + \left(\int_{nT}^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)(||v(u)||+||H_n(u)||)$$
$$+N(0)\int_{nT}^{\infty}|G(u)|d\lambda .$$

But β, $\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda$, $||v(u)||$ and $N(0)$ are finite, $||H_n|| \to 0$ as $n \to \infty$ and $\int_{nT}^{\infty}|G(u)|du \to 0$ as $n \to \infty$. Consequently

$$||v(u)-\Big(F(u)-N(0)\int_0^\infty G(u)du-\int_0^\infty G(\tau)\int_{-\infty}^{+\infty}S(\omega)\frac{e^{i\lambda v(u-\tau)}-1}{i\lambda}\,d\lambda d\tau\Big)||\longrightarrow 0\ ,$$

as $n\to\infty$ through integral values, from which the equation (5) follows for $v(u)$.

The right side of (5) satisfies the conditions of Tihonov's theorem. This follows by Lemma 1 where we substitute $t_0=\infty$, $E(mT+u-\tau)=0$ and $\Delta(mT+u-\tau)=v(u-\tau)$. If the right side of (5) is denoted by $c(v)$, then, by Lemma 1, $v\in M[0,\ T]$ implies $c(v)\in M[0,\ T]$ and $||c(v_1)-c(v_2)||\leqq\beta||v_1-v_2||$ for v_1 and v_2 from $M[0,\ T]$. By Tihonov's theorem it follows then that the equation (5) has a unique solution $v\in M[0,\ T]$ which may be obtained by iteration, beginning with an arbitrary function from $M[0],T$.

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A THEOREM ON EQUIDISTRIBUTION IN COMPACT GROUPS

GILBERT HELMBERG

1. Preliminaries. Throughout the discussions in the following sections, we shall assume that G is a compact topological group whose space is T_1 with an identity element e and with Haar-measure μ normalized in such a way that $\mu(G)=1$. G has a complete system of inequivalent irreducible unitary representations[1] $R^{(\lambda)}(\lambda \in \Lambda)$ where $R^{(1)}$ is the identity-representation and r_λ is the degree of $R^{(\lambda)}$. $R^{(\lambda)}(e)$ will then denote the identity matrix of degree r_λ.

The concept of equidistribution of a sequence of points was introduced first by H. Weyl [6] for the direct product of circle groups. It has been transferred to compact groups by B. Eckmann [1] and highly generalized by E. Hlawka [4][2]. We shall use it in the following from:

DEFINITION 1. Let[1] $\{x_\nu : \nu \in \omega\}$ be a sequence of elements in G and let, for any closed subset M of G, $N(M)$ be the number of elements in the set $\{x_\nu : x_\nu \in M,\ \nu \leqq N\}$.

The sequence $\{x_\nu : \nu \in \omega\}$ is said to be *equidistributed in* G if

$$(1) \qquad \lim_{N\to\infty} \frac{N(M)}{N} = \mu(M)$$

for all closed subsets M of G, whose boundaries have measured 0.

It is easy to see that a sequence which is equidistributed in G is also dense in G. As Eckmann has shown for compact groups with a countable base, and E. Hlawka for compact groups in general, the equidistribution of a sequence in G can be stated by means of the system $\{R^{(\lambda)} : \lambda \in \Lambda\}$ of representations of G.

LEMMA 1. *The sequence* $\{x_\nu : \nu \in \omega\}$ *is equidistributed in* G *if and only if*

$$(1') \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} R^{(\lambda)}(x_\nu) = 0 \qquad \textit{for all } \lambda \neq 1 .$$

Using this lemma, Eckmann arrives at the following theorem.

Received May 27, 1957. Presented at the 535th meeting of the Amer. Math. Soc. in Berkeley, Calif., April 20, 1957.

[1] In the following Λ and Λ' always denote any index-set, finite, countable, or uncountable, and ω denotes the set of positive integers $1, 2, \cdots$.

[2] Professor Hlawka has also noted in a letter to me that in order to secure the validity of Theorem 7 in [1] the lemma and footnote preceding it, it is necessary to change the definition of equidistribution into the form given below.

THEOREM 1. *Let g be an element of G such that*

$$|R^{(\lambda)}(g) - R^{(\lambda)}(e)| \neq 0 \qquad \text{for all } \lambda \neq 1 .$$

Then the sequence $\{g^{\nu} : \nu \in \omega\}$ is equidistributed in G.

It follows immediately that a group containing an element g with the above property is abelian and generated by a single element in the sense that the powers of g are dense in G. A group with the last property is called monothetic.

It is possible to extend this concept of generation of a group by one element to generation by a finite number of elements, that is, to ask for the smallest closed subgroup of G which contains a given finite set of elements of G (i. e., in which the set of all finite products of finite powers of these elements is dense).

DEFINITION 2. The finite set $\{g_k : k=1, 2, \cdots, n\}$ of elements of G is said to *generate the subgroup* H of G, if H is the smallest closed subgroup of G containing all g_k $(k=1, 2, \cdots, n)$.

Our subject in the following discussion will be a generalization of Eckmann's results in two directions indicated by that definition. First we shall try to find equidistributed sequences produced by finite set of elements in not necessarily commutative groups. In fact, the corresponding Theorem 2 will turn out to contain Theorem 1 as a special case. Furthermore we shall extend the definition of equidistribution in G to equidistribution in a subgroup of G.

DEFINITION 3. The sequence $\{x_{\nu} : \nu \in \omega\}$ of elements of a subgroup H is said to be *equidistributed in H* if it is equidistributed in the topological group H with respect to the relativized topology and with respect to the Haar-measure on the topological group H.

This definition is legitimate since H in the relativized topology is again a compact and T_1. Theorem 3 permits us to find sequences equidistributed in a subgroup of G and contains Theorem 2 as a special case.

In § 4 we compare our results with the results already known for finite groups which can be considered as compact groups in the discrete topology. Finally, we apply our results to abelian groups.

Before taking up this program, we state two rather obvious lemmas which will be helpful for deriving new equidistributed sequence from given ones. Clearly changing a finite number of elements of an equidistributed sequence has no influence on the property of being equidistributed.

LEMMA 2. *If the sequence $\{a_{\nu} : \nu \in \omega\}$ is equidistributed in G, then the sequence $\{a^{-1}_{\nu} : \nu \in \omega\}$ is also equidistributed in G.*

Proof. If M is an arbitrary closed subset of G whose boundary has measure 0, then M^{-1} is also closed its boundary has measure 0 and $\mu(M) = \mu(M^{-1})$ because of the fundamental properties of the Haar-measure μ. Let $N'(M)$ be the number of elements in the set $\{a_\nu : a_\nu \in M, r \leqq N\}$ and correspondingly $N''(M)$ the number of elements in $\{a_\nu^{-1} : a_\nu^{-1} \in M, \nu \leqq N\}$. Then $N''(M) = N'(M^{-1})$ and

$$\lim_{N\to\infty} \frac{N'(M^{-1})}{N} = \mu(M^{-1})$$

which holds because of our assumption for $\{a_\nu : \nu \in \omega\}$ is equivalent with

$$\lim_{N\to\infty} \frac{N''(M)}{N} = \mu(M) .$$

Therefore $\{a_\nu^{-1} : \nu \in \omega\}$ is equidistributed in G.

LEMMA 3. *If the sequences $\{a_\nu : \nu \in \omega\}$ and $\{b_\nu : \nu \in \omega\}$ are equidistributed in G, then the sequence $\{c_\nu : c_{2\nu-1} = a_\nu, c_{2\nu} = b_\nu, \nu \in \omega\}$ is also equidistributed in G.*

Proof. Define $N'(M)$, $N''(M)$ and $N(M)$ respectively for the sequences $\{a_\nu : \nu \in \omega\}$, $\{b_\nu : \nu \in \omega\}$ and $\{c_\nu : \nu \in \omega\}$ as above. For any positive integer N, let N_1 be the greatest integer in $(N+1)/2$ and let $N_2 = N - N_1$ (N_1 and N_2 are just the numbers of a's and b's among the first N c's). Then

$$\frac{N(M)}{N} = \frac{N_1'(M) + N_2''(M)}{N_1 + N_2} = \frac{N_1'(M)}{N} + \frac{N_2''(M)}{N} .$$

Starting from our assumption about $\{a_\nu : \nu \in \omega\}$ and $\{b_\nu : \nu \in \omega\}$ it is easy to show that

$$\lim_{N\to\infty} \frac{N_1'(M)}{N} = \lim_{N\to\infty} \frac{N_2''(M)}{N} = \frac{\mu(M)}{2}$$

for any closed subset M of G whose boundary has measure 0. But this implies (1) and the equidistribution of $\{c_\nu : \nu \in \omega\}$.

2. Non-commutative groups. A first generalization of Eckmann's Theorem 1 is given by the following.

THEOREM 2. *Let g_k ($k = 1, 2, \cdots, n$) be n elements of G such that for each $\lambda \neq 1$ there is at least one $g_{\bar{k}}$ for which*

$$|R^{(\lambda)}(g_{\bar{k}}) - R^{(\lambda)}(e)| \neq 0 .$$

Then the set of elements[3]

$$G' = \{g' : g' = g_1^{i_1} g_2^{i_2} \cdots g_n^{i_n},\ 0 \leqq i_k < \infty,\ k = 1, 2, \cdots, n\}$$

can be arranged in a sequence which is equidistributed in G.

Proof. We shall use Lemma 1. In order to simplify the notation of the proof, let us agree on the following. If A is the matrix (a_{ij}) then $\|A\|$ shall stand for the matrix $(|a_{ij}|)$, and if B is a matrix (b_{ij}) of the same degree r_λ as A then we shall write $\|A\| \leqq \|B\|$ for the simultaneous inequalities $|a_{ij}| \leqq |b_{ij}|$ for all i, j with $1 \leqq i, j \leqq r_\lambda$. The symbol $F^{(\lambda)}$ shall stand for the matrix of degree r_λ with the entries $f_{ij} = 1$ for all i, j with $1 \leqq i, j \leqq r_\lambda$. We can regard $\|A\|$ as matrix-norm for A for which the following relations hold (all matrices are of same degree r_λ).

$$
\begin{array}{lll}
 & \|kA\| = |k| \cdot \|A\| & (k \in K = \text{field of complex numbers}) \\
 & \|A+B\| \leqq \|A\| + \|B\| & \\
(2) & \|AB\| \leqq \|A\| \cdot \|B\| & \\
 & \|A\| \cdot \|B\| \leqq \|C\| \cdot \|D\| & \text{if } \|A\| \leqq \|C\| \text{ and } \|B\| \leqq \|D\| \\
 & \|A\| \leqq aF^{(\lambda)} & \text{if } a \geqq \max\{|a_{ij}| : 1 \leqq i, j \leqq r_\lambda\} \\
 & (F^{(\lambda)})^m = r_\lambda^{m-1} F^{(\lambda)} &
\end{array}
$$

Furthermore we shall write $\Pi_{l=1}^m h_l$ for the ordered product $h_1 h_2, \cdots, h_m$ and $\sum_{\underline{j}_l \leqq j_l \leqq \bar{j}_l}$ for

$$\sum_{j_1=\underline{j}_1}^{\bar{j}_1} \sum_{j_2=\underline{j}_2}^{\bar{j}_2}, \cdots, \sum_{j_m=\underline{j}_m}^{\bar{j}_m}$$

if it is clear that l goes from 1 to m.

In order to prove the theorem we first arrange the countable set G' in a sequence $\{g'_\nu : \nu \in \omega\}$ as follows, let g'_1 be e ($i_k = 0$ for $k = 1, 2, \cdots, n$); as the next $2^n - 1$ elements we take the products $\Pi_{k=1}^n g_k^{i_k}$ with $0 \leqq i_k \leqq 1$ ($k = 1, 2, \cdots, n$) and $\max\{i_k : k = 1, 2, \cdots, n\} = 1$ in any order. Then we take the $3^n - 2^n$ products $\Pi_{k=1}^n g_k^{i_k}$ with $0 \leqq i_k \leqq 2$ ($k = 1, 2, \cdots, n$) and $\max\{i_k : k = 1, 2, \cdots, n\} = 2$ in any order and so on.

The sequence so constructed $\{g'_\nu : \nu \in \omega\}$ contains all elements of G' and has the property that the first $(i+1)^n$ elements g'_ν ($\nu = 1, \cdots (i+1)^n$) are precisely all elements $\Pi_{k=1}^n g_k^{i_k}$ with $0 \leqq i_k \leqq i$, ($k = 1, 2, \cdots, n$). In order to show the equidistribution of this sequence in G we have to show that

[3] We allow any element of G to occur an arbitrary number of times in the set G' and similar sets formed below.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\nu=1}^{N}R^{(\lambda)}(g'_\nu)=0 \qquad \text{for all } \lambda\neq 1 \tag{3}$$

Let us assume that for a fixed $\lambda\neq 1$ the element $g_{\bar{k}}$ satisfies the condition

$$|R^{(\lambda)}(g_{\bar{k}})-R^{(\lambda)}(e)|\neq 0 \tag{4}$$

which means exactly that the matrix $R^{(\lambda)}(g_{\bar{k}})$ does not have the eigenvalue 1. For a given N let i be the greatest integer in $N^{1/n}-1$, such that

$$(i+1)^n \leqq N<(i+2)^n . \tag{5}$$

Then

$$\frac{1}{N}\sum_{\nu=1}^{N}R^{(\lambda)}(g'_\nu)=\frac{1}{N}\left[\sum_{\nu=1}^{(i+1)^n}R^{(\lambda)}(g'_\nu)+\sum_{\nu=(i+1)^n+1}^{N}R^{(\lambda)}(g'_\nu)\right]$$

where the second term in the square brackets vanishes if $N=(i+1)^n$; with the same qualification we have

$$\begin{aligned}\frac{1}{N}\sum_{\nu=1}^{N}R^{(\lambda)}(g'_\nu)&=\frac{1}{N}\left[\sum_{0\leqq i_k\leqq i}R^{(\lambda)}\left(\prod_{k=1}^{n}g_k^{i_k}\right)+\sum_{\nu=(i+1)^n+1}^{N}R^{(\lambda)}(g'_\nu)\right]\\&=\frac{(i+1)^n}{N}\left[\frac{1}{(i+1)^n}\sum_{0\leqq i_k\leqq i}R^{(\lambda)}\left(\prod_{k=1}^{n}g_k^{i_k}\right)+\frac{1}{(i+1)^n}\sum_{\nu=(i+1)^n+1}^{N}R^{(\lambda)}(g'_\nu)\right].\end{aligned} \tag{6}$$

Let us now consider separately the terms in the square brackets.

(a) Because of well-known properties of matrices and group-representations we can write

$$\begin{aligned}\frac{1}{(i+1)^n}\sum_{0\leqq i_k\leqq i}R^{(\lambda)}\left(\prod_{k=1}^{n}g_k^{i_k}\right)&=\frac{1}{(i+1)^n}\sum_{0\leqq i_k\leqq i}\prod_{k=1}^{n}[R^{(\lambda)}(g_k)]^{i_k}\\&=\prod_{k=1}^{n}\left\{\frac{1}{i+1}\sum_{i_k=0}^{i}[R^{(\lambda)}(g_k)]^{i_k}\right\}\\&=\prod_{k=1}^{\bar{k}-1}\left\{\frac{1}{i+1}\sum_{i_k=0}^{i}[R^{(\lambda)}(g_k)]^{i_k}\right\}\cdot\frac{1}{i+1}\sum_{i_{\bar{k}}=0}^{i}[R^{(\lambda)}(g_{\bar{k}})]^{i_{\bar{k}}}\\&\quad\times\prod_{k=\bar{k}+1}^{n}\left\{\frac{1}{i+1}\sum_{i_k=0}^{n}[R^{(\lambda)}(g_k)]^{i_k}\right\}.\end{aligned} \tag{7}$$

Again the first or last of the three factors vanishes if $\bar{k}=1$ or $\bar{k}=n$.

Since $R^{(\lambda)}(e)$ is the identity matrix, the following identity holds.

$$[R^{(\lambda)}(g_{\bar{k}})-R^{(\lambda)}(e)]\sum_{i_{\bar{k}}=0}^{i}[R^{(\lambda)}(g_{\bar{k}})]^{i_{\bar{k}}}=[R^{(\lambda)}(g_{\bar{k}})]^{i+1}-R^{(\lambda)}(e) .$$

Because of our assumption (4), we can solve this equation to obtain.

$$\sum_{i_{\bar{k}}=0}^{i} [R^{(\lambda)}(g_{\bar{k}})]^{i_{\bar{k}}} = [R^{(\lambda)}(g_{\bar{k}}) - R^{(\lambda)}(e)]^{-1} \cdot \{[R^{(\lambda)}(g_{\bar{k}})]^{i+1} - R^{(\lambda)}(e)\}$$

Since $R^{(\lambda)}$ is a unitary representation of G, we have for any $g \in G$ and any integral exponent j

$$(8) \qquad \| [R^{(\lambda)}(g)]^j \| = \| R^{(\lambda)}(g^j) \| \leqq F^{(\lambda)} .$$

According to our rules (2), the following inequalities then hold

$$\| [R^{(\lambda)}(g_{\bar{k}})]^{i+1} - R^{(\lambda)}(e) \| \leqq 2F^{(\lambda)} ,$$

$$\| [R^{(\lambda)}(g_{\bar{k}}) - R^{(\lambda)}(e)]^{-1} \| \leqq m_\lambda F^{(\lambda)} ,$$

where m_λ is a positive constant independent of i. This gives an upper bound for the matrix norm of the middle factor in (7)

$$\left\| \frac{1}{i+1} \sum_{i_{\bar{k}}=0}^{i} [R^{(\lambda)}(g_{\bar{k}})]^{i_{\bar{k}}} \right\| \leqq \frac{2m_\lambda}{i+1} [F^{(\lambda)}]^2 = \frac{2m_\lambda r_\lambda}{i+1} F^{(\lambda)} .$$

Furthermore by (8)

$$\left\| \frac{1}{i+1} \sum_{i_k=0}^{i} [R^{(\lambda)}(g_k)]^{i_k} \right\| \leqq \frac{1}{i+1} (i+1) F^{(\lambda)} = F^{(\lambda)} .$$

Replacing each factor in (7) by the stated bound for its matrix-norm, we get

$$(9) \qquad \begin{aligned} \left\| \frac{1}{(i+1)^n} \sum_{0 \leqq i_k \leqq i} R^{(\lambda)}\left(\prod_{k=1}^{n} g_k^{i_k} \right) \right\| &\leqq \prod_{k=1}^{\bar{k}-1} \{F^{(\lambda)}\} \cdot \frac{1}{i+1} \cdot 2m_\lambda r_\lambda F^{(\lambda)} \cdot \prod_{k=\bar{k}+1}^{n} \{F^{(\lambda)}\} \\ &= \frac{1}{i+1} \cdot 2m_\lambda r_\lambda^n F^{(\lambda)} . \end{aligned}$$

(b) Using (5) we get an upper bound for matrix norm of the second term in (6)

$$(10) \qquad \begin{aligned} \left\| \frac{1}{(i+1)^n} \sum_{\nu=(i+1)^n+1}^{N} R^{(\lambda)}(g'_\nu) \right\| &\leqq \frac{1}{(i+1)^n} \sum_{\nu=(i+1)^n+1}^{N} \| R^{(\lambda)}(g'_\nu) \| \\ &\leqq \frac{N-(i+1)^n}{(i+1)^n} F^{(\lambda)} && \text{(by (8))} \\ &< \frac{(i+2)^n-(i+1)^n}{(i+1)^n} F^{(\lambda)} && \text{(by (5))} \\ &= \left[\left(\frac{i+2}{i+1} \right)^n - 1 \right] F^{(\lambda)} \\ &= \left[\left(1 + \frac{1}{i+1} \right)^n - 1 \right] F^{(\lambda)} . \end{aligned}$$

(c) Let now $\varepsilon > 0$. Because of (9) we can find a number I_1 such that for all $i \geqq I_1$

$$\left\| \frac{1}{(i+1)^n} \sum_{0 \leqq i_k \leqq i} R^{(\lambda)}\left(\prod_{k=1}^{n} g_k^{i_k} \right) \right\| < \frac{\varepsilon}{2} F^{(\lambda)} . \tag{11}$$

Let

$$I_2 = \frac{2 - \left(\frac{\varepsilon}{2}+1\right)^{1/n}}{\left(\frac{\varepsilon}{2}+1\right)^{1/n} - 1} ;$$

then for $i \geqq I_2$ we have

$$\left(1 + \frac{1}{i+1}\right)^n - 1 \leqq \frac{\varepsilon}{2} \tag{12}$$

Now let $I = \max(I_1, I_2)$ and take $M = (I+2)^n$. Then from (6), (10) and the last two relations (11) and (12) it follows that

$$\left\| \frac{1}{N} \sum_{\nu=1}^{N} R^{(\lambda)}(g'_\nu) \right\| < \frac{(i+1)^n}{N} \left[\frac{\varepsilon}{2} F^{(\lambda)} + \frac{\varepsilon}{2} F^{(\lambda)} \right]$$

$$\leqq \varepsilon F^{(\lambda)} \qquad \text{for all } N \geqq M .$$

This shows the validity of (3) and the application of Lemma 1 completes the proof.

E. Hlawka, [4, §6] has shown that any sequence which is dense in G can be rearranged so as to be a sequence which is equidistributed in G. In view of that fact it should be emphasized that Theorem 2 (as well as any of the following ones) does not merely state that the set G' is dense in G; it states also the existence of a generally valid formula, as shown in the proof, for actually arranging the elements of G' in a sequence which is equidistributed in G.

Theorem 2 implies two more facts which are worth noting. First if we have n elements of G which satisfy the required condition, then we can, before actually producing the set G', arrange them in an entirely arbitrary order without affecting the equidistribution of the corresponding sequence in G. In the non-commutative case we shall therefore in general get different sequences containg different elements of G which are equidistributed in G. Second, we can add an arbitrary finite number of arbitrary elements $g_{n+1}, \cdots, g_m$ to our set of n elements $\{g_k : k = 1, 2, \cdots, n\}$ of G. The new set of $m(>n)$ elements of G still satisfies the condition of the theorem and, taken in any order, produces a set which can be arranged so as to be an equidistributed sequence in G.

The first remark together with Lemma 2 and Lemma 3 leads to the following.

COROLLARY 2.1. *If the elements* g_k $(k=1, 2, \cdots, n)$ *satisfy the condition of Theorem* 2, *then the sets*

$$G'' = \{g'' : g'' = g_1^{i_1} g_2^{i_2}, \cdots, g_n^{i_n}, \ -\infty < i_k \leqq 0, \ k = 1, 2, \cdots, n\}$$

and

$$G''' = \{g''' : g''' = g_1^{i_1} g_2^{i_2}, \cdots, g_n^{i_n}, \ -\infty < i_k < +\infty, \ k = 1, 2, \cdots, n\}$$

can be arranged in sequences which are equidistributed in G.

Proof. Let

$$\overline{G} = \{\overline{g} : \overline{g} = g_n^{i_n} g_{n-1}^{i_{n-1}}, \cdots, g_1^{i_1}, \ 0 \leqq i_k < +\infty, \ k = 1, 2, \cdots, n\} .$$

Then $\overline{G}$ can be arranged in a sequence equidistributed in G and $G''=\overline{G}^{-1}$. According to Lemma 2, G'' can also be arranged in a sequence which is equidistributed in G.

G''' is the union[3] of G' and $G''-e$ and according to Lemma 3 can be arranged in a sequence equidistributed in G.

COROLLARY 2.2. *If the elements* g_k $(k=1, 2, \cdots, n)$ *satisfy the condition of Theorem* 2, *then G is generated by* $\{g_k : k=1, 2, \cdots, n\}$.

Proof. We notice that G' is not an abstract subgroup of G. However, the subgroup H generated by $\{g_k : k=1, 2, \cdots, n\}$ must contain any finite product of finite powers of the g_k's. Therefore, it must contain all elements of the set G'. Since G' is dense in G, we have $H=G$.

3. **Subgroups.** If H is a subgroup of G, then any $R^{(\lambda)}(\lambda \in \Lambda)$ restricted to the elements of H, gives a representation $R^{*(\lambda)}$ of H. Each $R^{*(\lambda)}$ can be completely reduced into a direct sum of irreducible unitary representations of H which, as remarked before, is again a compact group. Let[1] $R'^{(\tau)}(\tau \in \Lambda')$ be the system of inequivalent irreducible unitary representations of H, so obtained. $R'^{(1)}$ again denotes the identity representation of H, obtained e. g. by restricting $R^{(1)}$ to H.

It can be shown without difficulty that $\{R'^{(\tau)} : \tau \in \Lambda'\}$ is a complete system of inequivalent irreducible representations of H. In order to do that we have by the Stone-Weinstrass-theorem to show that the entries of the system $\{R'^{(\tau)} : \tau \in \Lambda'\}$ span a linear space which is an algebra closed under pointwise multiplication and under conjugation and which separates points in H. But all these properties hold for the system $\{R^{(\lambda)} : \lambda \in \Lambda\}$, and from this we have obtained $\{R'^{(\tau)} : \tau \in \Lambda'\}$ only by changing the base in each $R^{(\lambda)}$, restricting it to H and selecting a system of linearly independent entries.

We can apply Theorem 2 to a subgroup H in the following form.

THEOREM 3. *Let H be a subgroup of G and let h_k $(k=1, 2, \cdots, n)$ be elements of H with the property that for each $\lambda \in \Lambda$ there is at least one element $h_{\bar{k}}$ such that the multiplicity of the eigenvalue 1 in $R^{(\lambda)}(h_{\bar{k}})$ is exactly the multiplicity with which the identity-representation $R'^{(1)}$ of H is contained in $R^{*(\lambda)}$.*

Then the set

$$H' = \{h' : h' = h_1^{i_1} h_2^{i_2}, \cdots , h_n^{i_n},\ 0 \leqq i_k < +\infty,\ k = 1, 2, \cdots , n\}$$

can be arranged in a sequence which is equidistributed in H.

Proof. From the above remarks we can conclude that any irreducible representation of H is contained in some $R^{*(\lambda)}$. Suppose that for a certain $\bar{\tau} \neq 1$ and for each $k=1, 2, \cdots , n$ we have

$$|R'^{(\bar{\tau})}(h_k) - R'^{(\bar{\tau})}(e)| = 0 .$$

This implies that $R'^{(\bar{\tau})}(h_k)$ has the eigenvalue 1 for each $k=1, 2, \cdots , n$. The representation $R'^{(\bar{\tau})}$ is contained in some $R^{*(\lambda)}$ which may contain also $R'^{(1)}$ with multiplicity m. But then each $R^{(\lambda)}(h_k)$ $(k=1, 2, \cdots , n)$ would have the eigenvalue 1 at least with multiplicity $m+1$ which contradicts our assumption.

Therefore for each $\tau \neq 1$ there has to be at least one $h_{\bar{k}}$ such that

$$|R'^{(\tau)}(h_{\bar{k}}) - R'^{(\tau)}(e)| \neq 0$$

and the conclusion of Theorem 2 applies to the topological group H.

Again we notice that the order, in which the elements h_k are used to produce the set H' is insignificant. By exactly the same reasoning as in §2, we obtain the following.

COROLLARY 3.1. *If the elements h_k $(k=1, 2, \cdots , n)$ satisfy the condition of Theorem 3, then the sets*

$$H'' = \{h'' : h'' = h_1^{i_1} h_2^{i_2}, \cdots , h_n^{i_n},\ -\infty < i_k \leqq 0,\ k = 1, 2, \cdots , n\}$$

and

$$H''' = \{h''' : h''' = h_1^{i_1} h_2^{i_2}, \cdots , h_n^{i_n},\ -\infty < i_k < +\infty,\ k = 1, 2, \cdots , n\}$$

can be arranged in sequences which are equidistributed in H.

COROLLARY 3.2. *If the elements h_k $(k=1, 2, \cdots , n)$ satisfy the condition of Theorem 3, then H is generated by $\{h_k : k=1, 2, \cdots , n\}$.*

4. Finite groups. Let now G be a not necessarily commutative finite group of order o, considered as a finite compact group with the discrete topology. The Haar-measure of G is then defined by $\mu(g)=1/o$ for any element $g \in G$.

The theorems stated so far are valid in general and therefore also for finite groups, since G was nowhere required in the definitions, lemmas and proofs to have infinitely many different group elements or inequivalent, irreducible representations. However, it is of not much use to talk about infinite sequences in a finite group. Therefore it seems justified to modity the concept of equidistribution of a sequence to the situation in finite groups in the following way:

DEFINITION 4. Let $\{x_\nu : \nu=1, 2, \cdots, N\}$ be a finite sequence of elements of G and let $N(M)$ be the number of elements in the set $\{x_\nu : x_\nu \in M, \nu \leqq N\}$ for any subset M of G.

The sequence $\{x_\nu : \nu=1, 2, \cdots, N\}$ is said to be *equidistributed in* G if

$$\frac{N(M)}{N} = \mu(M) \tag{13}$$

for all subsets M of G.

The formal translation of Definition 1 to finite groups, however, admits a much less complicated statement of equidistribution of finite sequence in a finite group which in turn reflects the intuitive meaning of equidistribution in infinte groups. In contrast to the infinite case the order of the element in the finite sequence $\{x_\nu : \nu=1, 2, \cdots, N\}$ is completely irrelevant. Instead of talking about a finite sequence of elements of G, we might therefore just as well talk about a finite set of elements of G (which may contain any element of G arbitrarily often). If M contains m elements, then $\mu(M)=m/o$. Especially if $M=\{g\}$ (a single element of G) (13) gives $N(g)=N/o$ for any element $g \in G$ which means that $\{x_\nu : \nu=1, 2, \cdots, N\}$ contains each element of G equally often. Conversely, if the latter is true, then $N(M)=mN/o$ and (13) holds for any subset M of G. So we can give the following better definition.

DEFINITION 4′. The finite set $\{x_\nu : \nu=1, 2, \cdots, N\}$ of elements of G is said to be equidistributed in G if it contains every element of G equally often.

In the same way we modify Definition 3.

DEFINITION 5. The finite set $\{x_\nu : \nu=1, 2, \cdots, N\}$ of elements of a subgroup H is said to be *equidistributed in* H if it contains every element of H equally often.

The theorems obtained so far are then completely transferrable to the finite case [3 Theorems 8 and 9]. Let now $\{R^{(\lambda)} : \lambda=1, 2, \cdots, l\}$ be a complete system of inequivalent irreducible unitary representations of G and let $R^{(1)}$ again be the identity-representation of G. Furthermore let $o(g)$ be the order of any element $g \in G$. The results are as follows.

THEOREM 4. *Let g_k $(k=1, 2, \cdots, n)$ be n elements of the finite group G such that for each $\lambda \neq 1$ there is at least one $g_{\bar{k}}$ for which*

$$|R^{(\lambda)}(g_{\bar{k}}) - R^{(\lambda)}(e)| \neq 0 .$$

Then the set

$$G' = \{g' : g' = g_1^{i_1} g_2^{i_2}, \cdots, g_n^{i_n},\ 0 \leqq i_k < o(g_k),\ k = 1, 2, \cdots, n\}$$

is equidistributed in G and contains each element $g \in G$ exactly

$$\frac{1}{o} \prod_{k=1}^{n} o(g_k)$$

times.

COROLLARY 4.1. *If the elements g_k $(k=1, 2, \cdots, n)$ satisfy the condition of Theorem 4, then G is generated by $\{g_k : k=1, 2, \cdots, n\}$.*

THEOREM 5. *Let H be a subgroup of order $o(H)$ of the finite group G and let h_k $(k=1, 2, \cdots, n)$ be elements of H with the property that for each $\lambda=1, 2, \cdots, l$ there is at least one element $h_{\bar{k}}$ such that the multiplicity of the eigenvalue 1 in $R^{(\lambda)}(h_{\bar{k}})$ is exactly the multiplicity with which the identity representation $R'^{(1)}$ of H is contained in $R^{*(\lambda)}$.*

Then the set

$$H' = \{h' : h' = h_1^{i_1} h_2^{i_2}, \cdots, h_n^{i_n},\ 0 \leqq i_k < o(h_k),\ k = 1, 2, \cdots, n\}$$

is equidistributed in H and contains each element $h \in H$ exactly $o(H)^{-1} \prod_{k=1}^{n} o(h_k)$ times.

COROLLARY 5.1. *If the elements h_k $(k=1, 2, \cdots, n)$ satisfy the condition of Theorem 5, then H is generated by $\{h_k : k=1, 2, \cdots, n\}$.*

The Corollaries 2.1 and 3.1, transferred to the finite case, coincide with Theorems 4 and 5. There is a last case which might be of interest, where H is a finite discrete subgroup of the infinite compact group G. Take the notation as defined in the corresponding cases. We get the following.

THEOREM 6. *Let H be a finite discrete subgroup of order $o(H)$ of the infinite compact group G and let h_k $(k=1, 2, \cdots, n)$ be elements of*

H with the property that for each $\lambda \in A$ there is at least one element $h_{\bar{k}}$ such that the multiplicity of the eigenvalue 1 in $R^{(\lambda)}(h_{\bar{k}})$ is exactly the multiplicity with which the identity-representation $R'^{(1)}$ of H is contained in $R^{*(\lambda)}$.

Then the set

$$H' = \{h' : h' = h_1^{i_1} h_2^{i_2}, \cdots, h_n^{i_n},\ 0 \leqq i_k < o(h_k),\ k = 1, 2, \cdots, n\}$$

is equidistributed in H and contains each element $h \in H$ exactly $o(H)^{-1} \prod_{k=1}^{n} o(h_k)$ times.

Proof. By the same reasoning as in the proof of Theorem 3 we assert that each irreducible representation of H is contained in some $R^{(\lambda)}$, restricted to H. Then as there was done with Theorem 2 we apply Theorem 4 to the finite group H.

COROLLARY 6.1. *If the elements h_k ($k=1, 2, \cdots, n$) satisfy the conditions of Theorem 6, then H is generated by $\{h_k : k=1, 2, \cdots, n\}$.*

It may be remarked that Theorems 4 to 6 can be deduced also from Theorems 2 and 3 without going back to the corresponding condition imposed on the generating elements, by means of the following lemma.

LEMMA 4. *Let g_k ($k=1, 2, \cdots, n$) be elements of finite order $o(g_k)$ ($k=1, 2, \cdots, n$) of an arbitrary compact group G.*

Then (i) the finite set

$$\overline{G} = \{\bar{g} : \bar{g} = g_1^{i_1} g_2^{i_2}, \cdots, g_n^{i_n},\ 0 \leqq i_k < o(g_k),\ k = 1, 2, \cdots, n\}$$

is equidistributed in G if and only if (ii) the set

$$G' = \{g' : g' = g_1^{i_1} g_2^{i_2}, \cdots, g_n^{i_n},\ 0 \leqq i_k < +\infty,\ k = 1, 2, \cdots, n\}$$

can be arranged as in the proof of Theorem 2 in a sequence which is equidistributed in G.

Proof. (i) $\to$ (ii). From (i) it follows immediately that G is finite. Let m be the least common multiple of the numbers $o(g_k)$ ($k=1, 2, \cdots, n$). If we arrange the elements of G' in a sequence $\{g'_\nu : \nu \in \omega\}$ as in the proof of Theorem 2, then we observe that

$$\{g'_\nu : \nu \leqq (pm)^n\} = \{g' : g' = g_1^{i_1} g_2^{i_2}, \cdots, g_n^{i_n},\ 0 \leqq i_k < pm,\ k = 1, 2, \cdots, n\}$$

(p=positive integer) is just composed of $(pm)^n / \prod_{k=1}^{n} o(g_k)$ times[3] the set $\overline{G}$. So for $N=(pm)^n$ in (1′) we get

$$(14) \qquad \frac{1}{(pm)^n} \sum_{\nu=1}^{(pm)^n} R^{(\lambda)}(g'_\nu) = \frac{1}{\prod_{k=1}^{n} o(g_k)} \sum_{0 \leqq i_k < o(g_k)} R^{(\lambda)}\Big(\prod_{k=1}^{n} g_k^{i_k}\Big) .$$

If (i) holds it can be shown that the right-side sum in (14) is the 0-matrix for $\lambda\neq 1$. If N is an integer between $(pm)^n$ and $[(p+1)m]^n$ then the left-side term in (1′) can be split up into $(pm)^n/N$ times the left-side term of (14) and $1/N$ times a sum of $N-(pm)^n$ unitary matrices. But

$$\frac{N-(pm)^n}{N} \leqq \frac{[(p+1)m]^n-(pm)^n}{(pm)^n} = \left(1+\frac{1}{p}\right)^n - 1$$

can be made arbitrarily small as in the proof of Theorem 2 and by the method used there we arrive at (1′).

$(ii)\rightarrow(i)$. We first observe that G' contains only finitely many different elements, namely those contained in $\bar{G}$, and G is finite. Again (14) holds. Since (14) gives just the value of the left-side term of (1′) for $N=(pm)^n$ $(p\in\omega)$, we can conclude from the validity of (1′) that

$$\sum_{0\leqq i_k<o(g_k)} R^{(\lambda)}\left(\prod_{k=1}^{n} g_k^{i_k}\right) = \sum_{\bar{g}\in\bar{G}} R^{(\lambda)}(\bar{g})$$

has to be the 0-matrix for $\lambda\neq 1$. But from this follows (i) by the well-known properties of irreducible representations of a finite group.

5. Abelian groups. Let now G be a (finite or infinite) compact abelian group. The irreducible representations are of degree 1 and instead of talking about a complete system of inequivalent irreducible unitary representations $R^{(\lambda)}$ $(\lambda\in\Lambda)$ we may talk about a complete system of inequivalent characters $\chi^{(\lambda)}$ $(\lambda\in\Lambda)$, where $\chi^{(1)}$ denotes the identity-character. As can be seen easily, the conditions of Theorem 2 and 3 take the specially simple form "$\chi^{(\lambda)}(g_{\bar{k}})\neq 1$ for $\lambda\neq 1$" and "for each λ for which $\chi^{(\lambda)}(h)\neq 1$ for some element $h\in H$ there is at least one element $h_{\bar{k}}$ such that $\chi^{(\lambda)}(h_{\bar{k}})\neq 1$" respectively. However, here we can make a stronger statement than in the preceding theorems.

THEOREM 7. *Let g_k $(k=1,2,\cdots,n)$ be elements of the abelian group G. Furthermore, let*

$$G' = \{g' : g' = g_1^{i_1} g_2^{i_2}, \cdots, g_n^{i_n},\ 0\leqq i_k < +\infty,\ k=1,2,\cdots,n\}$$

and let $\{g'_\nu : \nu\in\omega\}$ be the sequence in which the elements of G' have been arranged as in the proof of Theorem 2.

A necessary and sufficient condition for (i) $\{g_k : k=1,2,\cdots,n\}$ to generate G and (ii) $\{g'_\nu : \nu\in\omega\}$ to be equidistributed in G is that for each $\lambda\neq 1$ there is at least one $g_{\bar{k}}$ such that

$$\chi^{(\lambda)}(g_{\bar{k}}) \neq 1\ .$$

Proof. The statement about sufficiency is exactly Theorem 2 together with Corollary 2.2. Let us now assume that $\{g'_\nu : \nu \in \omega\}$ is equidistributed in G. Then $\{g_k : k=1, 2, \cdots, n\}$ generates G. Suppose that for a given $\lambda \neq 1$ we have $\chi^{(\lambda)}(g_k)=1$ for all $k=1, 2, \cdots, n$. Take a fixed element $g \in G$ with $\chi^{(\lambda)}(g) \neq 1$ and an arbitrary small positive number $\varepsilon < |\chi^{(\lambda)}(g)-1|$. Since finite products of finite powers of the elements g_k $(k=1, 2, \cdots, n)$ are dense in G and since $\chi^{(\lambda)}$ is a continuous character on G there has to be an element $g' = g_1^{j_1} g_2^{j_2} \cdots g_n^{j_n}$ such that

$$|\chi^{(\lambda)}(g) - \chi^{(\lambda)}(g')| < \varepsilon .$$

Since $\chi^{(\lambda)}(g')=1$ this implies $|\chi^{(\lambda)}(g)-1| < \varepsilon$. But this contradicts our assumption about ε.

THEOREM 8. *Let h_k $(k=1, 2, \cdots, n)$ be elements of a subgroup H of the abelian group G. Furthermore let*

$$H' = \{h' : h' = h_1^{i_1} h_2^{i_2}, \cdots, h_n^{i_n},\ 0 \leqq i_k < +\infty,\ k = 1, 2, \cdots, n\}$$

and let $\{h'_\nu : \nu \in \omega\}$ be the sequence in which the elements of H' have been arranged as in the proof of Theorem 3.

A necessary and sufficient condition for (i) *$\{h_k : k=1, 2, \cdots, n\}$ to generate H and* (ii) *$\{h'_\nu : \nu \in \omega\}$ to be equidistributed in H is that for each λ for which $\chi^{(\lambda)}(h) \neq 1$ for some element $h \in H$ there is at least on element $h_{\bar{k}}$ such that $\chi^{(\lambda)}(h_{\bar{k}}) \neq 1$.*

Proof. Again the sufficiency of the above condition is stated in Theorem 3 and Corollary 3.2. On the other had, if $\{h'_\nu : \nu \in \omega\}$ is equidistributed in H, then $\{h_k : k=1, 2, \cdots, n\}$ generates H and we can prove our claim exactly as in the proof of the preceding theorem.

Naturally there hold similar statements as Corollaries 2.1 and 3.1. For finite abelian groups we can, by obvious modifications, arrive at conclusions about equidistribution of finite sets as in §4, see [3, Theorems 10 and 11].

If we take as our abelian group G the direct product of p circle groups, the p-dimensional toroidal group, then Theorems 7 and 8 give us well-known theorems of Kronecker [5, p. 83 Theorem 4] and Weyl [6, Theorem 4]. It has been shown by Halmos and Samelson and again by Eckmann (see [1, Theorems 2 and 5] and [2, Theorem II* and Corollary]) that the p-dimensional toroidal group as well as any separable connected compact abelian group is monothetic.

In contrast to the situation in abelian groups the condition of Theorem 2 is not necessary for the existence of an equidistributed sequence of the form $\{g'_\nu : \nu \in \omega\}$ (as constructed there) in a non-commutative group. A simple counter example is given by the tetrahedral group A_4 (the alternating group of 4 variables). Let g_1 and g_2 be two different elements of order 2 and g_3 an arbitrary element of order 3. If we denote by $R^{(4)}$ the irreducible representation of A_4 of degree 3 it can be easily checked[4] that $|R^{(4)}(g_k)-R^{(4)}(e)|=0$ for $k=1, 2, 3$. However, the set of 12 element $\overline{G}=\{\overline{g}:\overline{g}=g_1^{i_1}g_2^{i_2}g_3^{i_3},\ 0\leqq i_1<2,\ 0\leqq i_2<2,\ 0\leqq i_3<3\}$ is equidistributed in G. By Lemma 4 it follows that the set

$$G' = \{g' : g' = g_1^{i_1} g_2^{i_2} g_3^{i_3},\ 0 \leqq i < \infty,\ k = 1, 2, 3\}$$

can be arranged in a sequence which is equidistributed in G. A counter-example disproving the necessity of the condition of Theorem 3 is given by any group containing A_4 as a subgroup, for example A_4 itself or the symmetric group of 4 variables.

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[4] The characters of g_1, g_2 and g_3 in $R^{(4)}$ are -1, -1 and 0 respectively.

SUBFUNCTIONS AND THE DIRICHLET PROBLEM

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1. Introduction. In previous papers [1 ; 6] the notion of subharmonic functions was generalized by replacing the dominating family of harmonic functions by a more general family of functions. The object was to require of the dominating functions the minimum properties necessary to study the boundary value problem by subfunction techniques. In a natural way these properties were separated into two parts: first, those properties sufficient to obtain functions which are solutions in the interior of a domain and, second, those properties sufficient to obtain agreement of the solution with the prescribed boundary values on the boundary of the domain. In particular the aim was to choose properties which would be sufficient to insure that a solution would take on prescribed boundary values at any boundary point p at which an exterior circle could be drawn intersecting the closed domain only in the point p. In a recent paper Inoue [5] points out an error in this second aspect of [1]. Inoue then lists properties of the dominating functions which are sufficient to insure the regularity of boundary points at which exterior triangles can be drawn. In his paper these properties are embodied in six postulates the first four of which are essentially the same as the first four postulates of [1]. Postulates 5 and 6 given by Inoue are used in studying the behavior at the boundary and are naturally more restrictive but they are such that the theory can be applied to elliptic partial differential equations which have the property that the difference between two solutions is subharmonic when positive.

In the present paper we use only the portion of the theory of subfunctions which is based on the first four postulates of [1] to obtain some results concerning the Dirichlet problem for certain types of elliptic equations. We shall give some results concerning the linear equation

$$\Delta z+a(x, y)z_x+b(x, y)z_y+c(x, y)z=f(x, y) \ , \tag{1}$$

where $\Delta z=\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}$, and the quasi-linear equation

$$a(p, q)r+2b(p, q)s+c(p, q)t=0 \ , \tag{2}$$

where $p=\dfrac{\partial z}{\partial x}$, $q=\dfrac{\partial z}{\partial y}$, $r=\dfrac{\partial^2 z}{\partial x^2}$, $s=\dfrac{\partial^2 z}{\partial x \partial y}$, and $t=\dfrac{\partial^2 z}{\partial x^2}$. In particular we

Received September 9, 1957. In revised form January 6, 1958.

shall give a theorem concerning the Dirichlet problem for the minimal surface equation

$$(3)\qquad (1+q^2)r-2pqs+(1+p^2)t=0$$

for non-convex regions. The result is quite weak but is perhaps of some interest since results ef this type are very meagre indeed.

2. $\{F\}$-functions and sub-$\{F\}$ functions. In this section we shall list for convenience the postulates satisfied by the $\{F\}$-functions and some theorems given in [1]. For simplicity our language will be in terms of the plane, however, our statements in this section could be phrased in terms of Euclidean space of any number of dimensions.

Let D be a given plane domain and let $\{\kappa\}$ be the family of all circles with radii less than some fixed number and such that $\bar{K}=K+\kappa\subset D$ where K is the open circle bounded by κ and $\bar{K}$ its closure. Throughout the paper we shall use Ω to indicate an arbitrary bounded domain such that $\bar{\Omega}\subset D$ and the boundary of Ω will be represented by ω. We shall use single small italic letters in this section to represent points in the plane.

Let there be given a family of functions $\{F(x)\}$, which we shall call $\{F\}$-functions, satisfying the postulates that follow.

POSTULATE 1. For any $\kappa\in\{\kappa\}$ and any continuous boundary value function $h(x)$ defined on κ, there is a unique $F(x\,;h\,;\kappa)\in\{F(x)\}$ such that

(a) $F(x\,;h\,;\kappa)=h(x)$ on κ,

and (b) $F(x\,;h\,;\kappa)$ is continuous on $\bar{K}$.

POSTULATE 2. If $h_1(x)$ and $h_2(x)$ are continuous on $\kappa\in\{\kappa\}$ and if $h_1(x)-h_2(x)\leqq M$ on κ, $M\geqq 0$, then

$$F(x\,;h_1\,;\kappa)-F(x\,;h_2\,;\kappa)\leqq M$$

in K; further, if the strict inequality holds at a point of κ, then the strict inequality holds throughout K.

POSTULATE 3. For any $\kappa\in\{\kappa\}$ and any collection $\{h_\nu(x)\}$ of functions $h_\nu(x)$ which are continuous and uniformly bounded on κ, the functions $F(x\,;h_\nu\,;\kappa)$ are equicontinuous in K.

DEFINITION 1. The function $s(x)$ is defined to be a sub-$\{F\}$ function, or simply a subfunction, in D provided

(a) $s(x)$ is bounded on every closed subset of D,

(b) $s(x)$ is upper semicontinuous in D, and (c) $s(x) \leqq F(x) \in \{F(x)\}$ on $\kappa \in \{\kappa\}$ implies $s(x) \leqq F(x)$ in K.

DEFINITION 2. The function $S(x)$ is defined to be a super-$\{F\}$ function or a superfunction in D provided—$S(x)$ is a sub-$\{-F\}$ function in D.

Let $g(x)$ be a bounded function defined on ω, the boundary of Ω, and define

$$g_*(x_0) = \liminf_{x \in \omega \to x_0} g(x) \ ,$$

and

$$g^*(x_0) = \limsup_{x \in \omega \to x_0} g(x) \ .$$

DEFINITION 3. The function $\phi(x)$ is an *under-function* (relative to $g(x)$) in Ω if $\phi(x)$ is continuous in $\bar{\Omega}$, is sub-$\{F\}$ in Ω, and $\phi(x) \leqq g(x)$ on ω.

DEFINITION 4. The function $\psi(x)$ is an *over-function* (relative to $g(x)$) in Ω if $\psi(x)$ is continuous in $\bar{\Omega}$, is a super-$\{F\}$ function in Ω, and $\psi(x) \geqq g(x)$ on ω.

POSTULATE 4. If Ω is any bounded domain comprised together with its boundary ω in D and if $g(x)$ is any bounded function defined on ω, then the associated families of over-functions and under-functions are both non-null.

DEFINITION 5. By a *solution of the Dirichlet Problem* for Ω relative to $\{F(x)\}$ and relative to a given bounded boundary value function $g(x)$ on ω, we shall mean a function $H(x)$ which is continuous in Ω, satisfies

$$(4) \qquad g_*(x_0) \leqq \liminf_{x \in \bar{\Omega} \to x_0} H(x) \leqq \limsup_{x \in \bar{\Omega} \to x_0} H(x) \leqq g^*(x_0)$$

at each $x_0 \in \omega$, and is such that for each $\kappa \in \{\kappa\}$ with $\bar{K} \subset \Omega$ we have

$$(5) \qquad H(x) \equiv F(x\,;H\,;\kappa) \text{ in } \bar{K} \,.$$

DEFINITION 6. We shall say that a function $H(x)$ which is continuous in Ω, and which satisfies (5) for each $\kappa \in \{\kappa\}$ with $\bar{K} \subset \Omega$, is an $\{F\}$-function in Ω.

DEFINITION 7. Given a bounded domain Ω such that $\bar{\Omega} \subset D$ and a bounded function $g(x)$ defined on ω. We denote by $H_*(x)$ and $H^*(x)$ the functions defined by

$$H_*(x) = \sup_{\phi \in \{\phi\}} \phi(x) ,$$

and

$$H^*(x) = \inf_{\varphi \in \{\psi\}} \psi(x) ,$$

where $\{\phi\}$ and $\{\psi\}$ are the associated families of under-functions and over-functions respectively.

THEOREM 1. *Given any bounded domain Ω with $\bar{\Omega} \subset D$ and any bounded function $g(x)$ defined on ω, then the associated functions $H_*(x)$ and $H^*(x)$ are $\{F\}$-functions in Ω* [1 ; p. 303].

DEFINITION 8. The point $x_0 \in \omega$ is a *regular* boundary point of Ω relative to $\{F(x)\}$ provided that for every bounded function $g(x)$ defined on ω the associated functions $H_*(x)$ and $H^*(x)$ satisfy (4) at x_0.

THEOREM 2. *If all points of ω are regular boundary points of Ω, and $g(x)$ is continuous on ω, then the Dirichlet problem for Ω, relative to $\{F(x)\}$ and $g(x)$, has a unique solution* [1 ; p. 304].

The next theorem shows that regularity of a boundary point "in the small" implies regularity "in the large".

DEFINITION 9. For a point $x_0 \in \omega$, a circle κ with center at x_0 and with $\bar{K} \subset D$, and constants $\varepsilon > 0$, M, and N, a function

$$s(x) \equiv s(x ; \kappa ; \varepsilon, M, N)$$

is a *barrier subfunction* provided:

(a) $s(x)$ is continuous in $\bar{\Omega} \cap \bar{K}$,
(b) $s(x)$ is a sub-$\{F\}$ function in $\Omega \cap K$,
(c) $s(x_0) \geqq N - \varepsilon$,
(d) $s(x) \leqq N + 2\varepsilon$ on $\omega \cap \bar{K}$,

and (e) $s(x) \leqq M$ on $\bar{\Omega} \cap \kappa$.

DEFINITION 10. With the notation of Definition 9, a function $S(x) \equiv S(x ; \kappa ; \varepsilon, M, N)$ is a *barrier superfunction* provided:

(a) $S(x)$ is continuous in $\bar{K} \cap \bar{\Omega}$,
(b) $S(x)$ is a super-$\{F\}$ function in $K \cap \Omega$,
(c) $S(x_0) \leqq N + \varepsilon$,
(d) $S(x) \geqq N - 2\varepsilon$ on $\omega \cap \bar{K}$,

and (e) $S(x) \geqq M$ on $\bar{\Omega} \cap \kappa$.

THEOREM 3. *If for $x_0 \in \omega$ and for each set of constants $\varepsilon > 0$, M, and N, there exists a sequence of circles $\kappa_n = \kappa_n(x_0)$ with centers at x_0 and* *radii* $r_n(x_0) \to 0$ *for which barrier subfunctions* $s(x\,;\kappa_n\,;\varepsilon, M, N)$ *and barrier superfunctions* $S(x\,;\kappa_n\,;\varepsilon, M, N)$ *exist, then* x_0 *is a regular boundary point of* Ω *relative to* $\{F(x)\}$ [1 ; p. 305].

3. **Equicontinuity at the boundary.** In this section, before turning our attention to differential equations, we shall show that a property of $\{F\}$-functions given as Postulate 8 in [1] is a consequence of Postulates 1 and 2.

THEOREM 4. *For any circle* $\kappa \in \{\kappa\}$, *if the functions* $\{h_\nu(x)\}$, *uniforuly bounded and continuous on* κ, *are equicontinuous at* $x_0 \in \kappa$, *then the functions* $F(x\,;h_\nu\,;\kappa)$, *defined in* $\overline{K}$, *are equicontinuous at* x_0.

Proof. Assume that $|h_\nu(x)| \leqq M$ on κ for all $h_\nu(x) \in \{h_\nu(x)\}$. Since the functions $\{h_\nu(x)\}$ are equicontinuous at x_0, it follows that given $\varepsilon > 0$ there exists an arc σ of κ with midpoint at x_0 such that

$$|h_\nu(x) - h_\nu(x_0)| < \varepsilon \qquad \text{on } \sigma$$

for all $h_\nu(x) \in \{h_\nu(x)\}$. Now let the function $g(x)$ be continuous on κ, $g(x) > M$ on $\kappa - \sigma$, $g(x) \geqq -M + \varepsilon$ on σ, and $g(x_0) \leqq -M + 2\varepsilon$. For any $h_\nu(x) \in \{h_\nu(x)\}$ set

$$c_\nu = h_\nu(x_0) + M \geqq 0\ ,$$

then

$$F(x\,;h_\nu\,;\kappa) - c_\nu < F(x\,;g\,;\kappa)$$

on κ. Therefore, by Postulate 2

$$F(x\,;h_\nu\,;\kappa) - c_\nu < F(x\,;g\,;\kappa) \qquad \text{in } \overline{K}$$

for each $h_\nu(x) \in \{h_\nu(x)\}$. Since $F(x\,;g\,;\kappa)$ is continuous in $\overline{K}$, there exists a circle κ_1 with center at x_0 such that

$$F(x\,;g\,;\kappa) - F(x_0\,;g\,;\kappa) < \varepsilon \qquad \text{in } \overline{K} \cap \overline{K}_1\ .$$

Then

$$F(x\,;h_\nu\,;\kappa) - c_\nu < F(x\,;g\,;\kappa) < \varepsilon + F(x_0\,;g\,;\kappa) \leqq 3\varepsilon + F(x_0\,;h_\nu\,;\kappa) - c_\nu$$

in $\overline{K} \cap \overline{K}_1$, hence, for any $h_\nu(x) \in \{h_\nu(x)\}$

$$F(x\,;h_\nu\,;\kappa) - F(x_0\,;h_\nu\,;\kappa) < 3\varepsilon \qquad \text{in } \overline{K} \cap \overline{K}_1\ .$$

By a similar argument there exists a circle κ_2 with center x_0 such that

$$F(x\,;h_\nu\,;\kappa) - F(x_0\,;h_\nu\,;\kappa) > -3\varepsilon \qquad \text{in } \overline{K} \cap \overline{K}_2\ .$$

Hence, if κ_3 is the smaller of κ_1 and κ_2, then

$$|F(x;h_\nu;\kappa)-F(x_0;h_\nu;\kappa)|<3\varepsilon \qquad \text{in } \overline{K}\cap\overline{K}_3$$

and the functions $F(x;h_\nu;\kappa)$ are equicontinuous at x_0.

Theorem 4 obviously remains valid under weaker conditions. For example, the theorem remains valid if Postulate 1 is weakened by assuming that the boundary value problem is solvable for some class of continuous boundary value functions defined on κ which under the uniform topology is dense in the set of all continuous functions defined on κ. Also, Theorem 4 remains valid if instead of dealing with a circle $\kappa\in\{\kappa\}$ we state the theorem in terms of a bounded demain Ω with $\overline{\Omega}\subset D$ and assume x_0 is a regular boundary point of Ω. However, in this case the proof draws on Postulates 3 and 4 as well as Postulates 1 and 2.

4. Applications to elliptic partial differential equations. In this section we shall show that the solutions of certain types of elliptic partial differential equations satisfy Postulates 1 to 4. We shall also consider some regularity criteria for boundary points with respect to these equations. It will be more convenient in this section to return to the customary (x, y) representation of points in the plane.

First we shall consider Postulate 2 since it states a characteristic property of the solutions of a wide class of elliptic differential equations. We consider the function $E(x, y, z, p, q, r, s, t)$ and make the following assumptions:

(1) E is continuous in all 8 variables in the region T defined by

$$T:\begin{cases}(x,y)\in D\\ -\infty<z,p,q,r,s,t<+\infty\end{cases}$$

where D is a domain in the xy-plane.

(2) The first partial derivatives E_z, E_p, E_q, E_r, E_s, and E_t are continuous in T, $E_s^2-4E_rE_t<0$, $E_r>0$, and $E_z\leqq 0$ in T.

THEOREM 5. *The solutions of the elliptic partial differential equation*

$$E(x,y,z,p,q,r,s,t)=0 \tag{6}$$

where $p=\dfrac{\partial z}{\partial x}$, $q=\dfrac{\partial z}{\partial y}$, $r=\dfrac{\partial^2 z}{\partial x^2}$, $s=\dfrac{\partial^2 z}{\partial x\partial y}$, *and* $t=\dfrac{\partial^2 z}{\partial y^2}$ *satisfy Postulate* 2.

THEOREM 6. *The functions $s(x, y)$ and $S(x, y)$ of class $C^{(2)}$ in the subdomain $\Omega\subset D$, are respectively a subfunction and a superfunction in Ω with respect to solutions of* (6) *if and only if*

$$E(x,y,s,s_x,s_y,s_{xx},s_{xy},s_{yy})\geqq 0 \tag{7}$$

and

(8) $$E(x, y, S, S_x, S_y, S_{xx}, S_{xy}, S_{yy}) \leqq 0$$

in Ω.

The proofs of Theorems 5 and 6 follow immediately from the maximum principle for solutions of elliptic partial differential equations which has been discussed by Hopf [4].

We consider now the linear elliptic equation

(1) $$L(z) \equiv \Delta z + a(x, y) z_x + b(x, y) z_y + c(x, y) z = f(x, y) .$$

We assume that D is a bounded plane domain such that $a(x, y)$, $b(x, y)$, $c(x, y)$ and $f(x, y)$ are Hölder continuous in $\overline{D}$ and $c(x, y) \leqq 0$ in $\overline{D}$.

THEOREM 7. *The solutions of* (1) *satisfy Postulates* 1, 2, 3, *and* 4.

Proof. It follows from Theorem 5 that the condition $c(x, y) \leqq 0$ in $\overline{D}$ insures that Postulate 2 is satisfied.

It is known [9] that there is an $r_0 > 0$, depending on $\max[|a|, |b|, |c|, |f|]$ in $\overline{D}$, such that Postulate 1 is satisfied for the family $\{\kappa\}$ of circles with radii less than or equal to r_0 and with $\overline{K} = K + \kappa \subset D$. The uniqueness part of Postulate 1 follows since Postulate 2 is satisfied.

If $\kappa \in \{\kappa\}$, if (x_0, y_0) is an interior point of K, and if $z(x, y)$ is continuous in $\overline{K}$, is of class $C^{(2)}$ in $\overline{K}$, and is a solution of (1) in K, then $|z_x(x_0, y_0)| \leqq M$ and $|z_y(x_0, y_0)| \leqq M$, where M depends on $\max[|a|, |b|, |c|, |f|]$ in $\overline{K}$, $\max|z(x, y)|$ on κ, the radius of κ, and the distance from (x_0, y_0) to κ [9]. This implies that Postulate 3 is satisfied.

Let Ω be any domain such that $\overline{\Omega} \subset D$ and let $g(x, y)$ be any bounded function defined on ω. Then, if $u(x, y) = \gamma[\alpha - e^{\beta x}]$ where α, β, and γ are constants,

$$L[u] = \gamma(\alpha - e^{\beta x}) \left[c(x, y) - \frac{(\beta^2 + \beta a(x, y) e^{\beta x}}{\alpha - e^{\beta x}} \right] .$$

Choose β so that $\beta > \max|a(x, y)|$ in $\overline{D}$, then choose α so that $\alpha - e^{\beta x} > 1$ in $\overline{\Omega}$. It is then clear that $\gamma_1 > 0$ can be chosen large enough that the function $\psi(x, y) = \gamma_1[\alpha - e^{\beta x}]$ will simultaneously satisfy the conditions:

$$L[\psi] \leqq f(x, y) \text{ in } \Omega \text{ and } \psi(x, y) \geqq g(x, y) \text{ on } \omega.$$

Hence, it follows from Definition 4 and Theorem 6 that $\psi(x, y)$ is an over-function. Similarly, if $\gamma_2 > 0$ is taken large enough,

$$\phi(x, y) = -\gamma_2[\alpha - e^{\beta x}]$$

will be an under-function. Postulate 4 is satisfied.

Since Postulates 1 to 4 are satisfied it follows from Theorem 1 that, given any domain Ω with $\bar{\Omega}\subset D$ and any bounded function $g(x, y)$ defined on ω, the associated functions $H^*(x, y)$ and $H_*(x, y)$ exists and are solutions of (1) in Ω.

THEOREM 8. *Let Ω be a domain with $\bar{\Omega}\subset D$ and let $(x_0, y_0)\in\omega$ be such that a circle κ_0 can be drawn with $\bar{K}_0\subset\bar{D}$ and $\bar{K}_0\cap\bar{\Omega}=(x_0, y_0)$. Then (x_0, y_0) is a regular boundary point of Ω relative to solutions of* (1)

Proof. Making use of Theorem 3 we see that to establish the regularity of (x_0, y_0) it is sufficient to show that barrier subfunctions and barrier superfunctions can be constructed for all sufficiently small circles with centers at (x_0, y_0). We shall consider only the barrier superfunctions since the barrier subfunctions can be dealt with in an exactly parallel way.

By the method used in Theorem 7 we can select a functions $S_0(x, y)$ which is continuous in $\bar{\Omega}$, is of class $C^{(2)}$ in Ω, and satisfies

$$L[S_0]\leqq f(x, y) \qquad \text{in } \Omega\,. \tag{9}$$

Now assume that constants $\varepsilon>0$, M, and N are given. Let (x_1, y_1) be the center of κ_0 and r_0 its radius. Let κ_1 be a circle with center at (x_0, y_0) and radius $r_1<r_0$ taken small enough that

$$S_0(x, y)\geqq S_0(x_0, y_0)-\varepsilon \qquad \text{on } \omega\cap\bar{K}_1\,. \tag{10}$$

Let

$$r=\sqrt{(x-x_1)^2+(y-y_1)^2}$$

and

$$w(x, y)=r_0^{-n}-r^{-n}\,.$$

One can easily verify that, if n is chosen large enough, then

$$L[w]\leqq 0 \qquad \text{in } \Omega\,, \tag{11}$$

furthermore, $w(x, y)$ is continuous in $\bar{\Omega}$, $w(x_0, y_0)=0$, and $w(x, y)>0$ elsewhere in $\bar{\Omega}$.

Now we consider two cases: $N-S_0(x_0, y_0)\geqq 0$ and $N-S_0(x_0, y_0)<0$. First we assume $N-S_0(x_0, y_0)\geqq 0$, then we can choose $h_1>0$ such that

$$h_1 w(x, y)\geqq M+\max_{(x,y)\in\bar{\Omega}}|S_0(x, y)| \qquad \text{on } \kappa_1\cap\bar{\Omega}\,. \tag{12}$$

The function

$$S(x, y\,;\kappa_1\,;\varepsilon, M, N)=h_1 w(x, y)+S_0(x, y)+N-S_0(x_0, y_0)$$

is then a barrier superfunction at (x_0, y_0) for the circle κ_1. This follows immediately from (9), (10), (11), (12) and the definition of a barrier superfunction.

Now assume $N-S_0(x_0, y_0)<0$. Again it is easily verified that, for $\delta>0$ chosen sufficiently large, the function

(13) $$v(x, y)=[N-S_0(x_0, y_0)]e^{\delta(x-x_0)}$$

satisfies

(14) $$L[v]\leqq 0 \qquad \text{in } \Omega \ .$$

Let the circle κ_2 with center at (x_0, y_0) and radius $r_2<r_1$ be chosen small enough that

(15) $$v(x, y)\geqq N-S_0(x_0, y_0)- \qquad \text{on } \omega\cap\overline{K}_2 \ .$$

Then let $h_2>0$ be taken large enough that

(16) $$h_2 w(x, y)\geqq M+\max_{(x,y)\in\overline{\Omega}}[|S_0(x, y)|+|v(x, y)|] \qquad \text{on } \kappa_2\cap\overline{\Omega} \ .$$

It follows from (13), (14), (15), and (16) that

$$S(x, y\,;\,\kappa_2\,;\,\varepsilon, M, N)=h_2 w(x, y)+v(x, y)+S_0(x, y)$$

is a barrier superfunction at (x_0, y_0) with respect to the circle κ_2.

THEOREM 9. *Let D be a domain in which the coefficient functions in* (1) *are Hölder continuous. Then, if Ω is any bounded domain with $\overline{\Omega}\subset D$ and is such that corresponding to each $(x_0, y_0)\in\omega$ there is a circle κ with $\overline{\Omega}\cap\overline{K}=(x_0, y_0)$ and if $g(x, y)$ is any continuous function defined on ω, there is a unique function $z(x, y)$ which is continuous in $\overline{\Omega}$, is of class $C^{(2)}$ and satisfies* (1) *in Ω, and is equal to $g(x, y)$ on ω.*

Proof. This is an immediate consequence of Theorems 2 and 8.

In our consideration of the quasi-linear equation

(2) $$a(p, q)r+2b(p, q)s+c(p, q)t=0$$

we are going to employ two sets of conditions on the coefficient functions, first, conditions (A): $a(p, q)$, $b(p, q)$, and $c(p, q)$ have Hölder continuous first partial derivatives, $ac-b^2=1$, and $a>0$ for all (p, q).

Bers [2] has proved that, if a, b, and c satisfy conditions (A), then there exist functions $k(p, q)$, $\theta(p, q)$, and $\Lambda(p, q)$ with $k(p, q)>0$, $\theta(0, 0)=\Lambda(0, 0)=0$, and which are such that

$$\frac{\partial\theta}{\partial p}=ka \qquad \frac{\partial\Lambda}{\partial p}+\frac{\partial\theta}{\partial q}=2kb \qquad \frac{\partial\Lambda}{\partial q}=kc \ .$$

We now state conditions (B) on the coefficients of (2): There exists an $\varepsilon > 0$ such that

$$a\left(\frac{1+p^2}{w}\right)+c\left(\frac{1+q^2}{w}\right)+2b\left(\frac{pq}{w}\right)\leqq 2\varepsilon \tag{17}$$

for all (p, q) where $w=\sqrt{1+p^2+q^2}$, further θ and Λ can be chosen so that

$$\frac{\theta^2+\Lambda^2}{p\theta+q\Lambda}\leqq 1 \tag{18}$$

for all (p, q). Conditions (A) and (B) are satisfied by the minimal surface equation (3) if it is normalized so that $ac-b^2=1$. For this reason Finn [3] calls equations (2) which satisfy conditions (A) and (B) equations of "minimal surface type".

In our application of the theory of §2 to equation (2), we let D be the xy-plane and $\{\kappa\}$ the family of all circles in the plane.

THEOREM 10. *If equation* (2) *satisfies conditions* (*A*) *and* (*B*), *then its solutions satisfy Postulates* 1 *to* 4.

Proof. Nirenberg [7; p. 138] has proved that if Γ is any convex domain in the plane with boundary γ which is of finite length, which can be represented parametrically by

$$\gamma:\begin{cases}x=x(s)\\ y=y(s)\end{cases}$$

in terms of arc length s where $x(s)$ and $y(s)$ are of class $C^{(3)}$, and which has positive curvature everywhere, and if $g(s)$ has a Hölder continuous second derivative on γ, then there is a function $z(x, y)$ continuous in $\bar{\Gamma}$, of class $C^{(2)}$ and a solution of (2) in Γ, and such that $z(x(s), y(s)) = g(s)$ on γ.

Finn [3] has shown that if (2) satisfies conditions (A) and (B) and if $z(x, y)$ is continuous in $\bar{K}$, is of class $C^{(2)}$ in K, and is a solution of (2) in K, then at any point $(x_0, y_0)\in K$ $|z_x(x_0, y_0)|\leqq M$ and $|z_y(x_0, y_0)|\leqq M$ where M depends on $\max|z(x, y)|$ on κ, the radius of κ, the distance from (x_0, y_0) to κ, and other quantities which are fixed for any particular equation (2). Using standard arguments [3; p. 411], one can then use Nirenberg's result to prove that Postulate 1 is satisfied. The bounds on the first partial derivatives of solutions imply that Postulate 3 is satisfied. That Postulate 2 is satisfied follows from Theorem 5 and since planes are solutions of (2) Postulate 4 is obviously satisfied.

Thus, we can conclude that, if Ω is any bounded domain and $g(x, y)$

is any bounded function defined on ω, the functions $H^*(x, y)$ and $H_*(x, y)$ of Theorem 1 exist and are solutions of (2) in Ω. In particular this is true of the minimal surface equation.

THEOREM 11. *Let equation* (2) *satisfy conditions* (*A*) *and* (*B*) *and let* Ω *be any bounded plane domain with boundary* ω. *If* $(x_0, y_0) \in \omega$ *is such that there is a circle* κ *with center at* (x_0, y_0) *and a straight line* π *such that* $\pi \cap (\bar{K} \cap \bar{\Omega}) = (x_0, y_0)$, *then* (x_0, y_0) *is a regular boundary point of* Ω *relative to solutions of* (2).

Proof. Since planes are solutions of (2), barrier subfunctions and superfunctions can obviously be constructed at (x_0, y_0) for all sufficiently small circles with centers at (x_0, y_0).

It follows that if equation (2) satisfies conditions (A) and (B), then in order that the Dirichlet problem have a solution for any convex domain whose boundary contains no straight line segments, it is sufficient that the Dirichlet problem have a solution for circles. Of course it is well known that the Dirichlet problem for the minimal surface equation always has a solution for convex domains whether or not their boundaries contain straight line segments. It is known that the Dirichlet problem for equation (2) is not always solvable for non-convex domains. In particular an example of a boundary value problem for a non-convex region which is not solvable for the minimal surface equation was given by H. A. Schwarz [8; p. 42]. For a given domain Ω with boundary ω, those points of ω which satisfy the criterion of Theorem 11 are regular with respect to equation (2), those points which are interior points of straight line segments of ω are possibly regular, but it seems likely that all other points of ω are not regular relative to solutions of (2). The possibility remains that the Dirichlet problem for (2) for certain types of non-convex domains may be solvable if the boundary values are suitably restricted. Our last theorem contains a weak result in this direction.

Let ω_c be the set of points of ω which satisfy the regularity criterion of Theorem 11. Let $\omega_n = \omega - \omega_c$ and for $\delta > 0$ let ω_δ be the set of points of ω which belong to ω_n or are within a distance δ of points of ω_n.

THEOREM 12. *Let* Ω *be a bounded plane domain with boundary* ω *for which there is an* $R > 0$ *such that for every* $(x, y) \in \omega$ *a circle* κ *of radius* R *may be drawn with* $\bar{\Omega} \cap \bar{K} = (x, y)$. *If for a given* $\delta > 0$ *the boundary value function* $g(x, y)$ *is continuous on* ω, *is constant on each component of* ω_δ, *and is such that*

$$V_g \equiv \max_{(x,y)\in\omega} g(x,y) - \min_{(x,y)\in\omega} g(x,y) \leqq \begin{cases} \dfrac{\delta}{\sqrt{2}} & \text{for } \delta \leqq R \\ R\sqrt{\dfrac{\delta}{\delta+R}} & \text{for } \delta > R\,, \end{cases} \tag{19}$$

then the Dirichlet problem for Ω with boundary values $g(x, y)$ has a unique solution for the minimal surface equation (3).

Proof. As we have already observed the functions $H_*(x, y)$ and $H^*(x, y)$ both exist and are solutions of (3) in Ω. Since the function $g(x, y)$ is continuous on ω, it is sufficient to show that inequality (4) is satisfied at each point of ω. This implies that the functions $H_*(x, y)$ and $H^*(x, y)$ are both continuous in $\bar{\Omega}$, agree on ω, and consequently coincide in $\bar{\Omega}$ to give the unique solution of the Dirichlet problem. Since by Theorem 11 the points of ω_c are regular, it will be sufficient to show that at each point of ω_n we can construct an over-function and an under-function which take on the given boundary value at the point.

Let $(x_0, y_0) \in \omega_n$ and let κ_0 be a circle of radius R such that $\bar{K}_0 \cap \bar{\Omega} = (x_0, y_0)$. Translate the origin to the center of κ_0 and rotate the axes so that (x_0, y_0) becomes the point $(R, 0)$. Draw the circle κ_1 with center at $(R, 0)$ and radius δ. Then the function

$$S_1(x, y) = \frac{R\sqrt{x^2+y^2} - Rx}{\sqrt{x^2+y^2}} \tag{20}$$

is of class $C^{(2)}$ on comp $\bar{K}_0$, $S_1(x, y) \geqq 0$ on comp K_0, and $S_1(R, 0) = 0$. Furthermore, by substituting $S_1(x, y)$ in the left-hand member of equation (3) one can verify that inequality (8) is satisfied in Ω. It follows from Theorem 6 that $S_1(x, y)$ is a superfunction in Ω with respect to solutions of (3). Finally, we also have that on $\kappa_1 \cap \text{comp}\, K_0$

$$\min S_1(x, y) = \begin{cases} \dfrac{\delta}{\sqrt{2}} & \text{if } \delta \leqq R \\ R\sqrt{\dfrac{\delta}{\delta+R}} & \text{if } \delta > R\,. \end{cases} \tag{21}$$

We define the function $S_2(x, y)$ by

$$S_2(x, y) = S_1(x, y) + g(R, 0)\,. \tag{22}$$

The function $S_2(x, y)$ is clearly also a superfunction in Ω because of the form of equation (3). Now let $M = \max g(x, y)$ on ω, then the function

$$\psi(x, y) = \begin{cases} M & \text{in } \bar{\Omega} \cap \text{comp}\, \bar{K}_1 \\ \min\,[M, S_2(x, y)] & \text{in } \bar{\Omega} \cap \bar{K}_1 \end{cases} \tag{23}$$

is the desired over-function. To see this we first observe that $\psi(x, y)$ is continuous in $\bar{\Omega}$ since by (19), (21), and (22) $S_2(x, y) \geqq M$ on $\bar{\Omega} \cap \kappa_1$. The argument that $\psi(x, y)$ is a superfunction is the same as that given in [1 ; p. 306]. From the definitions of $S_2(x, y)$ and ω_δ it follows that $\psi(R, 0) = g(R, 0)$ and $\psi(x, y) \geqq g(x, y)$ on ω.

Similarly

$$s_1(x, y) = -S_1(x, y) ,$$

and

$$s_2(x, y) = s_1(x, y) + g(R, 0)$$

are subfunctions in Ω. The function $\phi(x, y)$ defined by

$$\phi(x, y) = \begin{cases} m & \text{in } \bar{\Omega} \cap \operatorname{comp} \bar{K}_1 \\ \max\,[m, s_2(x, y)] & \text{in } \bar{\Omega} \cap \bar{K}_1 \end{cases}$$

where $m = \min g(x, y)$ on ω is an under-function with $\phi(R, 0) = g(R, 0)$. Thus inequality (4) holds at every point of ω and

$$H^*(x, y) \equiv H_*(x, y) \qquad \text{in } \bar{\Omega}$$

constituting the unique solution of the Dirichlet problem.

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THE STRUCTURE OF IDEMPOTENT SEMIGROUPS (I)

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1. Introduction. The first step in the study of idempotent semigroups has been made by David McLean [3] and is stated as follows.

THEOREM 1. *Let S be an idempotent semigroup. Then there exist a semilattice Γ and a disjoint family of rectangular subsemigroups of S indexed by Γ, $\{S_\gamma : \gamma \in \Gamma\}$, such that*

(i) $$S = \cup \{S_\gamma : \gamma \in \Gamma\}$$

and

(ii) $$S_\gamma S_\delta \subset S_{\gamma\delta} \qquad \textit{for} \quad \gamma, \delta \in \Gamma.$$

However, the structure of S is not determined, in general, by knowing only the structures of Γ and of all S_γ.

In this paper and the subsequent papers we shall study some special idempotent semigroups which are defined by some identities, where the decomposition theorem above plays an important role. This paper will be chiefly concerned with the study of regular idempotent semigroups (for the definition see below), which can be considered as a quite general class of idempotent semigroups. Also characterizations of identities for some special idempotent semigroups are obtained.

2. Rectangular bands. An *idempotent semigroup* or *band* [1] is a semigroup which satisfies the identity $a^2=a$.

A semigroup satisfying the identity

$$aba = a \quad (ab = a, ba = a)$$

is called *rectangular* (*left singular, right singular*). These semigroups are all idempotent. And a left (right) singular semigroup is rectangular. Conversely we have the following

LEMMA 1. *A rectangular semigroup is the direct product of a left singular semigroup and a right singular semigroup. Moreover this factorization is unique up to isomorphism.*

Proof. Let S be a rectangular semigroup. Then since

Received December 20, 1957, and in revised form February 26, 1958. This work was partially supported by the National Science foundation, U. S. A. An abstract of this paper has appeared in Proc. Jap. Acad., **33** (1957) p. 642.

$$xS \supset x(yS) = (xy)S \supset (xy)(xS) = (xyx)S = xS ,$$

we have

(1) $$\begin{cases} xyS = xS. \text{ Dually ,} \\ Sxy = Sy. \end{cases}$$

Therefore we have also

(2) $$\begin{cases} (xS)(yS) = (xS)(yxS) = (xSyx)S = xS. \quad \text{Dually ,} \\ (Sx)(Sy) = Sy. \end{cases}$$

Let A (B) be the set of all subsets of S of the form xS (Sx). Then A (B) forms a left (right) singular semigroup, with respect to the usual multiplication induced by that of S, on account of (2).

Let $p: S \rightarrow A$ $(q: S \rightarrow B)$ be the mapping defined by

$$p(x) = xS \quad (q(x) = Sx) .$$

Then by (1) and (2), p and q are onto homomorphisms.

Let $r: S \rightarrow A \times B$ be the mapping defined by

$$r(x) = (p(x), q(x)) .$$

Then r is a homomorphism. Take any element of $A \times B$, say (xS, Sy). Then $r(xy) = (xyS, Sxy) = (xS, Sy)$, by (1). Thus r is onto. On the other hand, if $r(z) = (xS, Sy)$, then $zS = xS$ and $Sz = Sy$. Therefore by rectangularity we have

$$xy = (xSx)(ySy) = (zSx)(ySz) = z(SxyS)z = z .$$

Thus r is an isomorphism between S and $A \times B$, where A (B) is left (right) singular.

Let $r': S \rightarrow A' \times B'$ be an isomorphism, where A' (B') is left (right) singular. Define $p': S \rightarrow A'$ and $q': S \rightarrow B'$ by $r'(x) = (p'(x), q'(x))$, then they are onto homomorphisms.

If $p(x) = p(y)$, that is $xS = yS$, then

$$p'(xS) = p'(x)p'(S) = p'(x) \quad \text{and} \quad p'(yS) = p'(y) .$$

Therefore $p'(x) = p'(y)$. Thus we have an onto homomorphism $f: A \rightarrow A'$ $(g: B \rightarrow B')$ such that $p' = fp$ $(q' = gq)$.

Now f (g) must be one-to-one. For, let $xS \neq yS$, $f(xS) = f(yS)$. Then $xyS = xS \neq yS$, therefore $xy \neq y$. But

$$p'(xy) = fp(xy) = f(xyS) = f(xS) = f(yS) = fp(y) = p'(y) ,$$
$$q'(xy) = gq(xy) = g(Sxy) = g(Sy) = gq(y) = q'(y) .$$

Therefore $r'(xy) = r'(y)$, which contradicts the assumption that r' is an isomorphism. Thus f and g must be isomorphisms.

This ends the proof of Lemma 1.

REMARK 1. The above defined A (B) is the set of all minimal right (left) ideals of S.

LEMMA 2. *A band is rectangular if and only if it satisfies the identity* $abc=ac$.

Proof. (1) *Sufficiency.* If a band S satisfies the above identity, then simply put $c=a$, which proves that S is rectangular.

(2) *Necessity.* Assume that S is a rectangular band, then $a(bc)a=a$. Therefore $abc=ab(cac)=(abca)c=ac$, which proves the above identity.

REMARK 2. Now we have established the equivalence between two identities, $aba=a$ and $abc=ac$, on idempotent semigroups. Thus either one of them can define rectangularity.

Also each one of the following identities on bands is equivalent to rectangularity :

(1) $$ax_1x_2\cdots x_na = a \qquad (n \geqq 1),$$

(2) $$ax_1\cdots x_{i-1}ax_i\cdots x_{j-1}ax_j\cdots\cdots\cdots x_na = a \qquad (1 < i < j < \cdots < n),$$

(3) $$ax_1x_2\cdots x_nb = ab \qquad (n \geqq 1),$$

(4) $$ax_1\cdots x_{i-1}c_1x_i\cdots x_{j-1}c_2x_j\cdots\cdots\cdots x_nb = ab\ ,$$

where c_k is either a or b for each k $(1<i<j<\cdots<n)$,

(5) $$ax_1x_2\cdots x_nb = ax_{i_1}x_{i_2}\cdots x_{i_r}b \qquad (1 \leqq i_1 < i_2 < \cdots < i_r \leqq n,\ r < n).$$

These facts raise the problem of determining the conditions for identities to be equivalent to rectangularity. It will be discussed in § 5 below, and there we will find that the equivalence of the above identities with rectangularity is merely a special case of Theorem 6.

REMARK 3. If we consider the two identities, $aba=a$ and $abc=ac$, for general semigroups, then they are not equivalent. The former defines a rectangular band, but the latter defines a little wider class of semigroups which contains rectangular bands.

However we have the following Lemma 3.

A semigroup S is called *total* if every element of S can be written as the product of two elements of S, that is $S^2=S$.

LEMMA 3. *A total semigroup is rectangular if and only if it satisfies the identity $abc=ac$.*

Proof. (1) *Sufficiency* Let S be total. Assume the identity $abc=ac$. Pick $a \in S$, then $a=xy$ for some elements x, y. Then

$$a^2 = (xy)^2 = (xy)(xy) = x(yx)y = xy = a .$$

Thus S is a band. Therefore by Lemma 2, S must be rectangular.

(2) *Necessity.* Obvious, because any rectangular semigroup satisfies the given identity by Lemma 2. This ends the proof of Lemma 3.

Let S be a semigroup which satisfies the identity $abc=ac$. Consider the mapping $f\colon S \to S$ defined by $f(x)=x^2$. Then f is a homomorphism of S into S, because

$$f(xy) = (xy)^2 = x(yx)y = xy = x(xy)y = x^2y^2 = f(x)f(y) .$$

Let R be the image of S under f:

$$R = f(s) = \{x^2 : x \in S\} .$$

Then obviously $R^2 \subset R \subset S^2$. Conversely, every element of S^2 is idempotent, because $(xy)^2=x(yx)y=xy$. Therefore we have $R^2=R=S^2$. Now since R is total, R is rectangular by Lemma 3.

Hence, defining S_r by $S_r=\{x : x \in S, x^2=r\}$, S is decomposed in the following way :

$$S = \cup \{S_r : r \in R\} , \quad \text{where} \quad S_rS_t = \{rt\} .$$

For, if $x \in S_r, y \in S_t,$ then $x^2=r, y^2=t$ and so $xy=x^2y^2=rt$. Thus we have the following

THEOREM 2. *Let S be a semigroup satisfying the identity $abc=ac$. Then there exists a rectangular subsemigroup R of S and a partition of S with R as its index set, such that*

$$S = \cup \{S_r : r \in R\} ,$$

where

$$S_r \cap S_t = \Box , \quad \textit{the null set,} \qquad \textit{if } r \neq t,$$
$$r \in S_r$$

and

$$S_rS_t = \{rt\} .$$

Thus the "if" part of Lemma 3 is a special case of this Theorem.

3. **The structure of one-sided regular bands.** A band is called (1) *left regular*, (2) *right regular* or (3) *regular* if it satisfies the identity:

$$(1) \qquad aba = ab\ ,$$

$$(2) \qquad aba = ba$$

or

$$(3) \qquad abaca = abca\ ,$$

respectively.

Then the following lemmas are obvious by these definitions.

LEMMA 4. *A left (right) regular band is regular.*

LEMMA 5. *The direct product of (left, right) regular bands is also (left, right) regular.*

LEMMA 6. *Any subsemigroup of a (left, right) regular band is also (left, right) regular.*

LEMMA 7. *A left (right) singular band is left (right) regular.*

LEMMA 8. *A rectangular band is regular.*

LEMMA 9. *A band is left (right) singular if and only if it is both left (right) regular and rectangular.*

LEMMA 10. *A band is commutative if and only if it is both left and right regular.*

For a total semigroup we have the following.

LEMMA 11. *A total semigroup is a left (right) regular band if and only if it satisfies the identity $aba=ab$ ($aba=ba$).*

Proof. The necessity is trivial. So it is sufficient to prove the idempotence from the above identity

Let S be a total semigroup, that is $S^2=S$. Then any element $x \in S$ can be written as the product of two elements of S, say, $x=ab$ for some $a, b \in S$. Therefore

$$x^2 = (ab)^2 = a(bab) = a(ba) = aba = ab = x\ .$$

Thus we have $x^2=x$, or S is idempotent.

Let S be a band. Then by Theorem 1 there exist a semilattice Γ and a disjoint family of rectangular subsemigroups of S indexed by Γ, $\{S_\gamma : \gamma \in \Gamma\}$, such that

$$\text{(i)} \qquad S = \cup \{S_\gamma : \gamma \in \Gamma\}$$

and

$$\text{(ii)} \qquad S_\gamma S_\delta \subset S_{\gamma\delta} \quad \text{for} \quad \gamma, \delta \in \Gamma$$

(See McLean [3, p. 111], also see A. H. Clifford [1, p. 501]).

Furthermore Γ is determined uniquely up to isomorphism, and accordingly so is S_γ.

We call Γ the *structure semilattice*, and S_γ the (γ-)*kernel*. A homomorphism $p: S \to \Gamma$ defined by $p(S_\gamma) = \gamma$ is called *natural*. Also in this case we write $S \sim \sum \{S_\gamma : \gamma \in \Gamma\}$, and call it the *structure decomposition* of S.

Then we have the following corollaries to Theorem 1.

COROLLARY 1. *Each kernel S_γ is a maximal rectangular subsemigroup of S. Moreover any rectangular subsemigroup of S is contained in one and only one kernel.*

Proof. Let $S \sim \sum \{S_\gamma : \gamma \in \Gamma\}$ be the structure decomposition of S and let $p: S \to \Gamma$ be natural. If R is a rectangular subsemigroup of S, then $p(R)$ is also a rectangular subsemigroup of Γ. Since Γ is a semilattice, $p(R)$ is reduced to a single element, say $\gamma = p(R)$, and according $R \subset p^{-1}(\gamma) = S_\gamma$. Namely R is contained in one and only one S_γ since the S_γ's are disjoint. On the other hand S_γ is rectangular for each $\gamma \in \Gamma$. Therefore each kernel S_γ is a maximal rectangular subsemigroup of S.

COROLLARY 2. *For any (onto) homomorphism $q: S \to \Delta$, where Δ is a semilattice, there exists a unique (onto) homomorphism $f: \Gamma \to \Delta$, such that $q = fp$, where $p: S \to \Gamma$ is natural.*

Proof. Since $q(S_\gamma)$ is rectangular, it must be a single element in Δ. Now we have a mapping f: $\Gamma \to \Delta$ defined by $f(\gamma) = q(S_\gamma)$. Then it is easy to see that $q = fp$.

COROLLARY 3. *Let $q: S \to \Delta$ be an onto homomorphism, where Δ is a semilattice. If $q^{-1}(\delta)$ is rectangular for all $\delta \in \Delta$, then the mapping f defined above is an isomorphism. More precisely, we can consider Δ as the structure semilattice of S, $q^{-1}(\delta)$ as the δ-kernel and q as the natural homomorphism, that is $S \sim \sum \{q^{-1}(\delta) : \delta \in \Delta\}$.*

Proof. Since $q^{-1}(\delta)$ is rectangular, it is contained in S_γ for some γ by Corollary 1 above. Now we have

$$\gamma = p(S_\gamma) \supset pq^{-1}(\delta) = p(fp)^{-1}(\delta) = pp^{-1}f^{-1}(\delta) = f^{-1}(\delta) .$$

Therefore f must be one-to-one.

THEOREM 3. *A band is left (right) regular, if and only if its kernels are all left (right) singular* (Naoki Kimura [2, p. 117]).

Proof. Let $S \sim \sum\{S_\gamma : \gamma \in \Gamma\}$ be the structure decomposition of a band S.

(1) Let S be left regular. Then each γ-kernel S_γ of S is rectangular. Also it is left regular by Lemma 6.

Therefore S_γ must be left singular by Lemma 9.

(2) Let every kernel of S be left singular. Let $a \in S_\alpha, b \in S_\beta$. Then $ab, ba \in S_{\alpha\beta} = S_{\beta\alpha}$. Thus, by the left singularity of $S_{\alpha\beta}$, we have $aba = ab^2a = (ab)(ba) = ab$, which proves that S is left regular.

4. The structure of regular bands. Let Γ be a semilattice. Let A and B be bands having Γ as their structure semilattice. Let $A \sim \sum\{A_\gamma : \gamma \in \Gamma\}$, $B \sim \sum\{B_\gamma : \gamma \in \Gamma\}$ be their structure decompositions.

Form the direct product $D = A \times B$. Then $C_\gamma = A_\gamma \times B_\gamma$ can be considered as a rectangular subsemigroup of D. Also $C = \cup\{C_\gamma : \gamma \in \Gamma\}$ is a subsemigroup of D. Moreover the structure decomposition of C is $C \sim \sum\{C_\gamma : \gamma \in \Gamma\}$.

Let $p : A \rightarrow \Gamma$, $q : B \rightarrow \Gamma$ be the natural homomorphisms. Then

$$C = \{(x, y) : x \in A, y \in B, p(x) = q(y)\} ,$$

and $r : C \rightarrow \Gamma$ defined by $r(x, y) = p(x) = q(y)$ is the natural homomorphism. We call C the *spined product* of A and B with respect to Γ. Note that this product depends not only on A, B and Γ but also on the natural homomorphism p and q [2, p. 28].

LEMMA 12. *The spined product of a left regular band and a right regular band is regular.*

Proof. Since the spined product of A and B is a subsemigroup of the direct product of A and B, we have the lemma by Lemmas 4, 5 and 6.

Now we shall prove the converse of this lemma which plays an essential part in the structure theorem of regular bands.

LEMMA 13. *Let $S \sim \sum\{S_\gamma : \gamma \in \Gamma\}$ be a regular band. Then there exist a left regular band $A \sim \sum\{A_\gamma : \gamma \in \Gamma\}$ and a right regular band $B \sim \sum\{B_\gamma : \gamma \in \Gamma\}$, both of which have the same structure semilattice Γ, such* *that S is isomorphic to the spined product of A and B with respect to Γ.*

Proof. Let $S \sim \sum\{S_\gamma : \gamma \in \Gamma\}$ be a regular band. Since each γ-kernel S_γ is rectangular we can assume that $S_\gamma = A_\gamma \times B_\gamma$, where A_γ is left singular and B_γ is right singular. Let

$$A = \cup\{A_\gamma : \gamma \in \Gamma\}, B = \cup\{B_\gamma : \gamma \in \Gamma\}, T = A \times B .$$

Then S can be identified as a subset of T. We shall prove that A and B can be considered as idempotent semigroups. Let

$$a \in A_\alpha, c \in A_\beta, b, b' \in B_\alpha, d, d' \in B_\beta .$$

Then $(a, b), (a, b') \in S_\alpha, (c, d), (c, d') \in S_\beta$. Put $(e, f) = (a, b)(c, d), (e', f') = (a, b')(c, d')$. Then both (e, f) and (e', f') belong to $S_{\alpha\beta}$.

Since $A_{\alpha\beta}$ is left singular and $B_{\alpha\beta}$ is right singular, we have

$$(e, f)(e', f') = (e, f') .$$

On the other hand we have

$$\begin{aligned}
(e, f)(e', f') &= (a, b)(c, d)(a, b')(c, d') \\
&= (a, b'b)(c, d'd)(a, bb')(c, d) \quad \text{(by right singularity of } B_\alpha \text{ and } B_\beta) \\
&= (a, b')(a, b)(c, d')(c, d)(a, b)(a, b')(c, d') \\
&= (a, b')(a, b)(a, b')(c, d')(a, b)(c, d)(a, b)(a, b')(c, d') \\
&\qquad \text{(by repeated use of regularity)} \\
&= (a, b'bb')(c, d')(a, b)(c, d)(a, bb')(c, d') \\
&= (a, b')(c, d')(a, b)(c, d)(a, b')(c, d') \\
&= (e', f')(e, f)(e', f') \qquad \text{(by definition)} \\
&= (e', f'). \qquad \text{(by rectangularity of } S_{\alpha\beta})
\end{aligned}$$

Hence

$$(e, f') = (e', f') \quad \text{or} \quad e = e' .$$

Thus e is determined by a and c only, and does not depend on b or d. Similarly, f is also determined by b and d only.

Now we can define $m : A \times A \to A$, $n : B \times B \to B$ by

$$(m(a, c), n(b, d)) = (a, b)(c, d) = (e, f) .$$

Thus A and B become multiplicative systems where m and n are multiplications on them, and A_γ and B_γ are subsystems which are a left singular band and a right singular band, respectively. Also $T = A \times B$ is a multiplicative system.

Consider the projections $p: T \to A$ defined by $p(a, b)=a$ and $q: T \to B$ defined by $q(a, b)=b$. They are homomorphisms. Therefore the mappings p and q with their domain restricted to $S \subset T$ are also homomorphisms, and their images are $A=p(S)$ and $B=q(S)$. Since homomorphisms preserve any relation defined by identities, as a result, associativity and idempotency hold in both A and B, because S is a band. Thus both A and B are bands.

Since A_γ is left singular, and B_γ is right singular, they are rectangular, and since Γ is a semilattice,

$$A \sim \sum \{A_\gamma : \gamma \in \Gamma\}, \quad B \sim \sum \{B_\gamma : \gamma \in \Gamma\}$$

become the structure decompositions of A and B, by Corollary 3 to Theorem 1.

Thus there exist a left regular band A and a right regular band B such that S is the spined product of A and B with respect to Γ.

Lemmas 12 and 13 prove the following

THEOREM 4. *A band is regular if and only if it is the spined product of a left regular band and a right regular band.*[1]

COROLLARY 1. *Any regular band is imbedded into the direct product of a left regular band and a right regular band.*

Proof. Immediately from Theorem 4.

COROLLARY 2. *Let S be the spined product of A and B with respect to Γ and let T be the spined product of C and D with respect to Δ, where $A \sim \sum \{A_\gamma : \gamma \in \Gamma\}$ and $C \sim \sum \{C_\delta : \delta \in \Delta\}$ are left regular, and $B \sim \sum \{B_\gamma : \gamma \in \Gamma\}$ and $D \sim \sum \{D_\delta : \delta \in \Delta\}$ are right regular.*

Let $k: S \to T$ be a homomorphism, then there exist a homomorphism $h: \Gamma \to \Delta$ and homomorphisms $f: A \to C$ and $g: B \to D$ satisfying (1) *$k(a, b)=(f(a), g(b))$ and* (2) *$hp=rf$ and $hq=sg$, that is the diagram*

$$\begin{array}{ccccc} A & \xrightarrow{p} & \Gamma & \xleftarrow{q} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{r} & \Delta & \xleftarrow{s} & D \end{array}$$

is analytic, where p, q, r and s are the natural homomorphisms.

Proof. Let $u: S \to \Gamma$, $v: T \to \Delta$ be the natural homomorphisms. Then since $vk: S \to \Delta$ is a homomorphism, by Corollary 2 to Theorem 1, there exists a unique homomorphism $h: \Gamma \to \Delta$ such that $vk=hu$.

Therefore $v(k(S_\gamma))=hu(S_\gamma)=h(\gamma)$, and so $k(S_\gamma) \subset v^{-1}(\delta)=T_\delta$, where $\delta=h(\gamma)$. Now the homomorphism $k_\gamma: S_\gamma \to T_\delta$ defines uniquely homomorphisms $f_\gamma: A_\gamma \to C_\delta$ and $g_\gamma: B_\gamma \to D_\delta$ such that $k_\gamma(a, b)=(f_\gamma(a), g_\gamma(b))$,

[1] Miyuki Yamada has obtained this theorem also, according to a recent communication from him to the author.

where k_γ is the homomorphism k with its domain restricted to S_γ.

Since A and B are the union of A_γ and B_γ for $\gamma \in \Gamma$, f_γ and g_γ determined uniquely mappings $f: A \to C$ and $g: B \to D$, such that

$$f(a) = f_\gamma(a) \quad \text{if} \quad a \in A_\gamma ,$$

and

$$g(b) = g_\gamma(b) \quad \text{if} \quad b \in B_\gamma .$$

Then it is obvious that $k(a, b) = (f(a), g(b))$. Therefore if $(a, b) \in S$, $(a', b') \in S$, then we have

$$\begin{aligned}(f(aa'), g(bb')) &= k(aa', bb') = k((a, b)(a', b')) = k(a, b)k(a', b') \\ &= (f(a), g(b))(f(a'), g(b')) = (f(a)f(a'), g(b)g(b')) ,\end{aligned}$$

which proves that f and g are homomorphisms.

Since $(a, b) \in S_\gamma$ implies $(f(a), g(b)) = k(a, b) \in T_\delta$, where $\delta = h(\gamma)$, we have $rf(a) = \delta = h(\gamma) = hp(a)$, namely, $rf = hp$. Similarly, $sg = hq$.

COROLLARY 3. *In Corollary* 2, *k is* (1) *one-to-one into,* (2) *onto or* (3) *one-to-one onto, respectively, if and only if there exists h, f and g, all of which are* (1) *one-to-one into,* (2) *onto or* (3) *one-to-one onto, satisfying all the conditions in Corollary* 2.

Proof. Sufficiency. It is easily proved by considering the mapping k defined by $k(a, b) = (f(a), g(b))$, in each case.

Necessity (1) Let k be one-to-one into. Then $k^{-1}(T_\delta)$ is rectangular if it is not empty, and so it is contained in only one S_δ by Corollary 1 to Theorem 1. Therefore h is one-to-one into. Now it is easy to see that f and g are one-to-one.

(2) Let k be onto. Then

$$h(\Gamma) = h(u(S)) = vk(S) = v(T) = \Delta ,$$

which shows that h is onto. Now it is obvious that f and g are onto.

(3) Obvious by (1) and (2).

The case (3) of this corollary can be restated as follows.

COROLLARY 4. *The decomposition of a regular band into the spined product of a left regular band and a right regular band is unique up to idomorphism.*

5. **Characterizations by identities.** Let $X = \{x, y, \cdots\}$ be a set whose elements we will call *variables*. A *word* is an element of the free semigroup $F = F(X)$. A pair of words (P, Q) is called an *identity* and is written $P = Q$.

Let S be a band. Then we say S *satisfies* an identity $P=Q$, if $f(P)=f(Q)$ for every homomorphism $f\colon F\to S$.

An identity $P=Q$ is said to *imply* an identity $P'=Q'$ if any band satisfying $P=Q$ also satisfies $P'=Q'$. Thus any identity implies the identity $x^2=x$, that is idempotence. If $P=Q$ implies $P'=Q'$ and $P'=Q'$ implies $P=Q$, then the identities are *equivalent*.

Let X' also be a set of variables. Let $t_0\colon X\to X'$ be any transformation, then it induces a homomorphism $t\colon F(X)\to F(X')$ which coincides with t_0 on X.

It is easy to see that $P=Q$ implies $t(P)=t(Q)$.

LEMMA 14. *$P=Q$ implies $t(P)=t(Q)$ for any transformation of variables t_0.*

The following lemmas are also straightforward.

LEMMA 15. *If $P=Q$ implies $P=P'$, $Q=Q'$, then $P=Q$ implies $P'=Q'$.*

LEMMA 16. *If $P=Q$ implies $P'=Q'$, then $P=Q$ implies both $PP'=QQ'$ and $P'P=Q'Q$.*

REMARK 4. We can take the free idempotent semigroup generated by X instead of the free semigroup generated by X. It makes no essential difference in the argument.

An identity $P=Q$ is said to be *homotypical* if both P and Q contain the same variables explicitly, otherwise it is said to be *heterotypical*. Thus an identity $xy=x$ is heterotypical, but an identity $xy=yx$ is homotypical.

If P is a word $x_1x_2\cdots x_n$, then we call x_1 the *head* of P and x_n the *tail* of P.

THEOREM 5. *An identity $P=Q$ is equivalent to left (right) singularity if and only if*[2]

(1) *$P=Q$ is heterotypical,*

(2) *P and Q have the same (different) heads,*

(3) *P and Q have different (the same) tails.*

Proof. Sufficiency. Let $P=Q$ satisfy (1), (2) and (3) above. Then the words P and Q are expressed by $x\cdots x_1$ and $x\cdots x_2$, respectively, where x_1 is different from x_2, and either x_1 or x_2, but not both, may be the same as x. By assumption (1) either P or Q contains a variable y, which the other does not. Assume that P contains y.

A transformation $X\to X$ defined by $y\to y$, all other variables $\to x$ sends the words P, Q to P', Q' where P' is $x\cdots y\cdots x$ or $x\cdots y$ $(\cdots$

[2] The "only if" part will be proved right after the proof of Lemma 17 below.

stands for x's, y's or nothing) and Q' is x^n for some positive integer n. Now any band satisfies the identities $P'=xyx$ or $P'=xy$, according as P' *is* $x\cdots y\cdots x$ or $x\cdots y$, and $Q'=x$. Thus by using Lemmas 14 and 15 we have that $P=Q$ implies either (i) $xyx=x$ or (ii) $xy=x$.

Since (i) is rectangularity, and rectangularity implies both $P=xx_1$ and $Q=xx_2$ by Lemma 2, the identity $P=Q$ implies $xx_1=xx_2$ by Lemma 15. It is now easy to see that the identity $xx_1=xx_2$ implies left singularity, by a suitable transformation. (ii) itself shows left singularity.

Thus $P=Q$ implies left singularity.

Conversely, the identity $xy=x$ implies any identity of the form

$$x\cdots y = x\cdots x \quad \text{or} \quad x\cdots y = x\cdots z\ ,$$

where x, y, z are all different and $\cdots$ stand for any sequence of variables. Thus $xy=x$ implies any identity satisfying the conditions in the theorem.

THEOREM 6. *An identity $P=Q$ is equivalent to rectangularity if and only if*[2]

(1) *$P=Q$ is heterotypical,*

(2) *P and Q have the same heads,*

(3) *P and Q have the same tails.*

Proof. Let $P=Q$ be an identity satisfying (1), (2) and (3) above. Then we can assume that the word P is $x\cdots y\cdots z$ and Q is $x\cdots z$, where Q does not contain the variable y, while z can be the same as x. Now the transformation $y\rightarrow y$, all other variables $\rightarrow x$, implies the identity $xyx=x$, which is equivalent to rectangularity.

Conversely, rectangularity $xyx=x$ implies any identity of the form $x\cdots z=x\cdots z$ by Lemma 2. Thus it implies any identity satisfying the above conditions.

REMARK 5. It is easily verified that all identities mentioned in Remark 2 satisfy the above three conditions.

THEOREM 7. *An identity $P=Q$ is equivalent to triviality, that is $x=y$, if and only if*[2]

(1) *$P=Q$ is heterotypical,*

(2) *P and Q have different heads,*

(3) *P and Q have different tails.*

Proof. Let $P=Q$ be an identity satisfying the above condition. Then it implies both identities $zP=zQ$ and $Pz=Qz$, where z is a variable which is not contained in both P and Q, by Lemma 16. The former is equivalent to left singularity, while the latter is equivalent to right singularity by Theorem 5. Thus $P=Q$ implies both left and right singularity. Hence it implies triviality.

Conversely, triviality implies any identity.

LEMMA 17. *Any semilattice satisfies any homotypical identity.*

Proof. Let $P=Q$ be any homotypical identity whose variables are $x_1, \cdots, x_n$. Let S be any semilattice. Then it is clear that S satisfies the both identities, $P=x_1x_2\cdots x_n$ and $x_1x_2\cdots x_n=Q$. Thus S satisfies the identity $P=Q$.

Proof of the necessity in Theorem 5, 6 and 7.

Let $P=Q$ be an identity which is equivalent to triviality, left (right) singularity or rectangularity. Let S be the two-element semilattice. If $P=Q$ is homotypical, then S satisfies this identity by the preceding lemma. But S is not rectangular nor left (right) singular nor trivial. So this identity can not be homotypical. Thus it must be heterotypical. This takes care of the part (1) of the theorems.

Let $A(B)$ be the two-element left (right) singular band. Then $A(B)$ is neither right (left) singular nor trivial. Also $A(B)$ satisfies any identity $P=Q$ if the heads (tails) of P and Q are the same.

(i) Assume that $P=Q$ is equivalent to triviality. Then the heads (tails) of P and Q must be different. For, if not, $A(B)$ which is not trivial satisfies this identity. This proves the necessity of (2) and (3) in Theorem 7.

(ii) Assume that $P=Q$ is equivalent to left (right) singularity. Then the tails (heads) of P and Q must be different. For, if not, then $B(A)$, which is not left (right) singular, satisfies this identity. This takes care of (3) of Theorem 5.

Now the heads (tails) of P and Q must be the same. For, if not, this identity is equivalent to triviality by Theorem 7, which has already been proved completely. But there exists a left (right) singular band which is not trivial, for example, $A(B)$.

This takes care of (2) of Theorem 5.

(iii) Assume that $P=Q$ is equivalent to rectangularity.

Then the heads of P and Q are the same and so are their tails. For, if not, the identity is equivalent to triviality or left singularity or right singularity by the preceding argument. Also there exists a band which is rectangular but neither left nor right singular nor trivial, for example, $A\times B$. This ends the proof of (2) and (3) in Theorem 6.

Thus the classification of all heterotypical identities into four distinct cases is now completed.

THEOREM 8. *An identity $P=Q$ is equivalent to commutativity if it satisfies the following conditions:*

(1) *$P=Q$ is homotypical,*
(2) *P and Q have different heads,*
(3) *P and Q have different tails.*

Proof. Let $P=Q$ be an identity satisfying the above conditions (1), (2) and (3). Then we can assume the word P is $x\cdots$, and Q is $y\cdots$, $x\neq y$. Thus $P=Q$ implies $Pxy=Qxy$. Now the transformation: $y\to y$, all other variables $\to x$ on the latter identity implies the identity $xy=yxy$, which is equivalent to right regularity. Similarly, $P=Q$ implies left regularity. Thus by Lemma 10 the identity $P=Q$ implies commutativity.

Conversely, commutativity implies any homotypical identity.

Before stating the conditions for an identity to be equivalent to left or right regularity, we shall introduce the concept of initial and final parts, by which we can reduce both members of an identity to simpler forms.

If the word P', say $x_{i_1}x_{i_2}\cdots x_{i_k}$, is the word which is obtained by writing down all the distinct variables of the word P, say $x_1x_2\cdots x_n$, from the left, we call P' the *initial part* of P and denote it by $q(P)$. Similarly, we can define the *final part* of P, $r(P)$, dually with respect to left and right. Thus if the word P is $xyxzx$, then the initial part and the final part of P are xyz and yzx, respectively, that is $q(P)$ is xyz and $r(P)$ is yzx.

When P and Q have the same initial (final) parts, we say that the identity $P=Q$ is *coinitial* (*cofinal*). Note that if an identity is coinitial or cofinal then it must be homotypical.

THEOREM 9. *An identity $P=Q$ is equivalent to left (right) regularity, if it satisfies the following two conditions:*
(1) *$P=Q$ is coinitial (cofinal),*
(2) *P and Q have different tails (heads).*

Proof. Let $P=Q$ satisfy the above two conditions. Then by (1) P and Q must have the same head, say x, By (2) one of P or Q has a tail which is different from x, say P is $x\cdots y$, where $y\neq x$.

Let t_0 be the transformation defined by $y\to y$, all other variables $\to x$. Then we have two identities $t_0(P)=xy$ and $t_0(Q)=xyx$. Thus we have left regularity, $xy=xyx$.

Conversely it is obvious that left regularity implies $P=P'$ for any word P, where P' is the initial part of P. Thus left regularity implies any coinitial identity. Hence it implies any identity satisfying the above conditions.

The problem of finding the characteristic conditions for an identity on bands to be equivalent to regularity still remains open.

6. **Free regular bands.** By the *free* (*left, right*) *regular band* generated by a non-empty set X, we mean a band S such that

(1) there exists a mapping $i: X \to S$, which is called the imbedding mapping,

(2) $i(X)$ generates S,

(3) S is (left, right) regular,

(4) for any (left, right) regular band T and for any mapping j: $X \to T$, there exists a homomorphism $h: S \to T$ such that $j = hi$.

REMARK 6. In this definition, the imbedding mapping is not assumed to be one-to-one, but this property is proved easily in this case. Also it is easy to see that if there are two such free regular bands for a given set X, then they are isomorphic fixing every point of X pointwise under the imbedding mappings. So if there exists a free (left, right) regular band, it is unique up to isomorphism. The homomorphism h in (4) is also unique.

The free commutative band, i. e., the free semilattice generated by X is defined similarly.

In this section we shall construct the free (left, right) regular band from a given set X.

Let X be a non-empty set. Let S be the set of all non-empty subsets of X consisting of a finite number of points. Then we have the following

LEMMA 18. *The above defined S is the free semilattice generated by X under the multiplication defined by $yz = y \cup z$, where $\cup$ denotes the union operation.*

Proof. It is obvious that S forms a semilattice generated by $\{\{x\}: x \in X\}$. Let $i: X \to S$ be defined by $i(x) = \{x\}$. Let T be a semilattice and j: $X \to T$ any mapping. Then the mapping $h: S \to T$ defined by $h(y) = j(x_1)j(x_2)\cdots j(x_n)$ where $y = \{x_1, x_2, \cdots, x_n\}$, is a homomorphism by commutativity and by idempotence, satisfying $j(x) = h(\{x\}) = h(i(x))$, that is $j = hi$. Thus S is the free semilattice generated by X.

Let X be a non-empty set. Let $F = F(X)$ be a free semigroup generated by X. Then F is the set of all finite sequences of points of X with juxtaposition multiplication. We imbed X in F in the natural way under $k: X \to F$.

Consider the two mappings, the initial part $q: F \to F$ and the final part $r: F \to F$, defined in the preceding section. Let $A_0 = q(F) \subset F$, $B_0 = r(F) \subset F$ be the images of F under q, r. Note that not only are q and

r not homomorphisms but also A_0 and B_0 can never be subsemigroups of F. To make them form bands we define other multiplications in A_0 and in B_0 as follows:

$$m(a, a') = q(aa'), \text{ for } a, a' \in A_0 ,$$
$$n(b, b') = r(bb'), \quad \text{for } b, b' \in B_0 .$$

Let $a, a' a'' \in A_0$. Then

$$m(m(a, a'), a'') = m(q(aa'), a'') = q(q(aa')a'') = q(aa'a'') .$$

Similarly $m(a, m(a',a''))=q(aa'a'')$. Therefore m is an associative multiplication on A_0.

Moreover $m(a, a) = q(aa) = q(a) = a$ and

$$m(m(a, a'), a) = q(aa'a) = q(aa') = m(a, a') .$$

Therefore A_0 forms a left regular band under the multiplication m. Similarly, B_0 is a right regular band under the multiplication n. We shall denote these bands by A and B instead of A_0 and B_0, because of the difference of multiplications.

It is now simple to see that $q: F \to A$ and $r: F \to B$ are both onto homomorphisms. Since F is generated by $k(X)$, A and B are generated by $i(X)$ and $j(X)$, respectively, where $i=qk$ and $j=rk$.

Let A' be any left regular band and $i': X \to A'$ any mapping. Since F is the free semigroup generated by X, there exists a homomorphism $f: F \to A'$ such that $i'=fk$. For any $w \in F$ we have $f(w)=f(q(w))$, because A' is left regular. Thus there exists a homomorphism $h: A \to A'$ such that $f=hq$. Therefore

$$i' = fk = (hq)k = h(qk) = hi .$$

Hence A is the free left regular band generated by X. Similarly B is the free right regular band generated by X.

Consider the free semilattice Γ generated by X with its imbedding $c: X \to \Gamma$ (Lemma 18). Then since Γ is both left and right regular, there exist homomorphisms $s: A \to \Gamma$ and $t: B \to \Gamma$ such that $si=c=tj$. It is obvious that

$$s(a) = \{x_1, x_2, \cdots, x_n\} \text{ , if } a = i(x_1)i(x_2)\cdots i(x_n) .$$

Let $A_\gamma=s^{-1}(\gamma)$ and $B_\gamma=t^{-1}(\gamma)$ for $\gamma \in \Gamma$. Then it is easy to see that $A_\gamma(B_\gamma)$ is left (right) singular. Thus by Corollary 3 to Theorem 1, $A\sim\sum\{A_\gamma : \gamma \in \Gamma\}$ and $B\sim\sum\{B_\gamma : \gamma \in \Gamma\}$ are the structure decomposition of A and B. Thus we have the following

THEOREM 10. *Let X be a non-empty set. Let Γ be the free semilattice obtained in Lemma 18. Let A (B) be the set of all linearly ordered* *non-empty finite subsets of* X *together with the multiplication defined by juxtaposition deleting all second letters which appear in the expression of the juxtaposition product reading from the left (right). Let* $s: A \to \Gamma$ (t: $B \to \Gamma$) *be the mapping defined by* $s(a)$ ($t(b)$)*=the set of all distinct points contained in* a (b). *Let* $A_\gamma = s^{-1}(\gamma)$ *and* $B_\gamma = t^{-1}(\gamma)$ *for* $\gamma \in \Gamma$. *Then* A (B) *is the free left (right) regular band generated by* X *and* $A \sim \sum\{A_\gamma : \gamma \in \Gamma\}$ ($B \sim \sum\{B_\gamma : \gamma \in \Gamma\}$) *is its structure decomposition.*

COROLLARY. *The free left (right) regular band generated by* n *elements consists of* $\sum_{i=1}^{n}\binom{n}{i} i! = n!\sum_{i=0}^{n-1} 1/i!$ *elements.*

Proof. Each A_γ consists of $i!$ elements when γ contains i elements, since A_γ consists of all permutations of points of γ.

Let A (B) and Γ be the free left (right) regular band and the free semilattice generated by X with the imbedding mappings $i: X \to A$ (j: $X \to B$) and $c: X \to \Gamma$, respectively. Since Γ is both left and right regular, there exist homomorphisms $s: A \to \Gamma$ and $t: B \to \Gamma$ such that $si = c = tj$.

Let C be the spined product of A and B with respect to Γ with s and t as spine homomorphisms. Then C is the subset of $A \times B$ consisting of elements (a, b) such that $s(a) = t(b)$. Now since $s(i(x)) = (si)(x) = c(x) = (tj)(x) = t(j(x))$ the element $(i(x), j(x))$ is in C. Define $k: X \to C$ by $k(x) = (i(x), j(x))$.

Now we shall prove that $k(X)$ generates C.

Pick any element $(a, b) \in C$. Then $s(a) = t(b) \in \Gamma$. Since A and B are generated by X, we have

$$a = i(x_1)i(x_2)\cdots i(x_m)\ , \quad b = j(y_1)j(y_2)\cdots j(y_n)\ .$$

Thus

$$c(x_1)c(x_2)\cdots c(x_m) = s(a) = t(b) = c(y_1)c(y_2)\cdots c(y_n)\ .$$

Therefore the subset consisting of the points $x_1, x_2, \cdots, x_m$ coincides with that consisting of the points $y_1, y_2, \cdots, y_n$.

Since A is left regular we have

$$i(x_1)i(x_2)\cdots i(x_m)i(y_1)i(y_2)\cdots i(y_n) = i(x_1)i(x_2)\cdots i(x_m) = a\ .$$

Similarly,

$$j(x_1)j(x_2)\cdots j(x_m)j(y_1)j(y_2)\cdots j(y_n) = j(y_1)j(y_2)\cdots j(y_n) = b\ .$$

Thus we have $(a, b) = k(x_1)k(x_2)\cdots k(x_m)k(y_1)k(y_2)\cdots k(y_n)$, which proves that $k(X)$ generates C.

Next, we shall prove that C is the free regular band generated by X.

Let C' be any regular band and $k': X \to C'$ any mapping. Since C' is regular, it is the spined product of a left regular band A' and a right regular band B' with respect to a semilattice Γ', where $s': A' \to \Gamma'$ and $t': B' \to \Gamma'$ are the spine homomorphisms. Now the freeness of A, B and Γ implies the existence of homomorphisms $f: A \to A'$, $g: B \to B'$ and $h: \Gamma \to \Gamma'$ such that

$$(1) \qquad fi = u'k', \quad gj = v'k', \quad hc = d'k' ,$$

where $u': C' \to A'$, $v': C' \to B'$ are natural and $d': C' \to \Gamma'$ are such that

$$(2) \qquad S'u' = d' = t'v' .$$

Let $u: C \to A$, $v: C \to B$ be natural and $d: C \to \Gamma$ be such that $su=d=tv$. Take $(a, b) \in C$. Then $s(a)=t(b)$ by definition. Since C is generated by $k(X)$, there exist $x_1, x_2, \cdots, x_n \in X$. such that

$$a = i(x_1)i(x_2)\cdots i(x_n), \; b = j(x_1)j(x_2)\cdots j(x_n) .$$

Put $a'=f(a)$ and $b'=g(b)$. Then by (1) and (2) we have

$$s'(a') = s'f(a) = \prod_{\nu=1}^{n} s'fi(x_\nu) = \prod_{\nu=1}^{n} s'u'k'(x_\nu) = \prod_{\nu=1}^{n} d'k'(x_\nu) ,$$
$$t'(b') = t'g(b) = \prod_{\nu=1}^{n} t'gj(x_\nu) = \prod_{\nu=1}^{n} t'v'k'(x_\nu) = \prod_{\nu=1}^{n} d'k'(x_\nu) .$$

Thus $s'(a')=t'(b')$. Therefore $(a', b') \in C'$, that is $(f(a), g(b)) \in C'$. Hence there exists a mapping $p: C \to C'$ defined by $p(a, b)=(f(a), g(b))$.

It is now easy to see that p is a homomorphism. Moreover for $x \in X$ we have by (1)

$$k'(x) = (u'k'(x), v'k'(x)) = (fi(x), gj(x)) = pk(x) ,$$

because $k(x)=(i(x), j(x))$, and accordingly $k'=pk$. This completes the proof that C is the free regular band generated by X. Thus we have the following

THEOREM 11. *Let X be a non-empty set. Let A, B and Γ be the free left regular, the free right regular and the free commutative band generated by X, respectively, so that Γ is regarded as the structure semilattice of both A and B. Then the free regular band generated by X is the spined product of A and B with respect to Γ.*

COROLLARY. *The regular band generated by n distinct elements consists of*

$$\sum_{i=1}^{n} \binom{n}{i} (i\,!)^2 = n\,! \sum_{i=1}^{n} \frac{i\,!}{(n-i)\,!}$$

elements.

Proof. Each $A_\gamma \times B_\gamma$ consists of $(i\,!)^2$ elements when contains i elements.

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A METHOD OF APPROXIMATING THE COMPLEX ROOTS OF EQUATIONS

STEPHEN KULIK

1. The method described in this paper presents an algorithm by which at least two roots of an equation can be approximated starting with the same first approximation. This is achieved by introducing a parameter and choosing its numerical value appropriately. In particular, in case of real roots, two adjacent or the largest and the smallest roots are approximated by the use of two different values of the parameter. This is discussed in §3. In case of conjugate imaginary roots the real and imaginary parts of the approximations are easily separated. This is discussed in §4.

2. Let $f(z)$ be an analytic function within and upon a circle C, and let the roots of the equation $f(z)=0$ within and upon the circle be denoted by a_j, $j=1, 2, \cdots$, and their multiplicities by m_j respectively.

We consider the expansion into the partial fractions of $(u-z)^k f'(z)/f(z)$, where k is a positive integer and $u \neq a_j$ or z but otherwise arbitrary,

$$(u-z)^k f'(z)/f(z)=\sum_{j=1} m_j(u-a_j)^k/(z-a_j)+\psi_1(z) \ , \tag{1}$$

where $\psi_1(z)$ is analytic within and upon the circle, and the sum is taken over all the roots a_j starting with $j=1$.

By differentiating (1) $n-1$ times and dividing by $(-1)^{n-1}(n-1)!$ we derive

$$Q_{n,k}/[f(z)]^n=\sum_{j=1} m_j(u-a_j)^k/(z-a_j)^n+\psi_n(z) \ , \qquad n \geqq k, \tag{2}$$

where $\psi_n(z)$ is analytic within and upon C. The function $Q_{n,k} \equiv Q_{n,k}(z, u)$ can be evaluated by the formula

$$Q_{n,k}=\sum_{j=0}^{n-1}\binom{k}{j}(u-z)^{k-j}[f(z)]^j D_{n-j} \tag{3}$$

with D_n evaluated recursively

$$\begin{aligned} D_n&=\sum_{j=0}^{n-2} f^{(j+1)}(z)[-f(z)]^j D_{n-j-1}/(j+1)!+f^{(n)}(z)[-f(z)]^{n-1}/(n-1)! \ , \\ D_0&=1, \ D_1=f'(z) \end{aligned} \tag{4}$$

The function $Q_{n,k}$ can also be evaluated recursively, and both $Q_{n,k}$ and

Received November 14, 1957. Presented at the meeting of the American Mathematical Society at Pennsylvania State University, August 27-30, 1957. Sponsored by the OOR., U.S. Army, 1957.

D_n can be expressed in the form of determinants [1, 2].

We rewrite (2) as follows

$$Q_{n,k}/[f(z)]^n = m_1(u-a_1)^k/(z-a_1)^n \times\left\{1+(z-a_1)^n/m_1(u-a_1)^k\left[\sum_{j=2} m_j(u-a_j)^k/(z-a_j)^n+\psi_n(z)\right]\right\}, \tag{5}$$

where the summation starts with $j=2$. Now, if we assume that u and z are given such values that

$$|(u-a_1)/(z-a_1)| > |(u-a_j)/(z-a_j)|, \qquad j=2, 3, \cdots, \tag{6}$$

and

$$|(u-a_1)/(z-a_1)| > |(u-\zeta)/(z-\zeta)|,$$

for any ζ on C, the following result follows:

$$(u-a_1)^{c-b}/(z-a_1)^a = \lim_{n\to\infty} Q_{n,n-b}/[f(z)]^a Q_{n-a,n-c}, \tag{7}$$

where a, b, and c are constants satisfying the conditions imposed on the subscripts of $Q_{n,k}$ in (2). An approximation to a_1 is obtained with a finite n.

Of particular practical value are the cases when the left hand side of the equation has only $u-a_1$, or $z-a_1$, or both of the first or second degree.

3. The reason for introducing the parameter u into the problem is that more than one root can be approximated with the same D_1, D_2, $\cdots D_n$ by using different appropriately chosen values of u. This will be illustrated when the left hand side of (7) is either $u-a_1$, $(u-a_1)/(z-a_1)$, or $1/(z-a_1)$, namely:

$$(z-a_1)/(u-a_1) = \lim_{n\to\infty} f(z)Q_{n-1,n-1}/Q_{n,n} \tag{8}$$

$$z-a_1 = \lim_{n\to\infty} f(z)Q_{n-1,n-1}/Q_{n,n-1} \tag{9}$$

$$u-a_1 = \lim_{n\to\infty} Q_{n,n}/Q_{n,n-1} \tag{10}$$

$$(z-a_1)/(u-a_1) = \lim_{n\to\infty} f(z)Q_{n-1,n-2}/Q_{n,n-1} \tag{11}$$

Let us assume that $z=x$ is real and that the two roots closest to x are also real, $a_1<x<a_2$, $x-a_1<a_2-x$. Then, as it can easily be verified, an approximation to a_1 can be obtained with any $u_1>x$, and $\infty<u_1<[(a_1+a_2)x-2a_1a_2]/(2x-a_1-a_2)$; an approximation to a_2 can be obtained with any $[(a_1+a_2)x-2a_1a_2]/(2x-a_1-a_2)<u_2<x$ (Diagram 1). The

above inequalities defining u_1 and u_2 should also be used when $a_1 < x$ is the largest real root of an equation and a_2 the smallest (Diagram 2).

Before applying any of (8)—(11), an approximation to a_1 can be obtained by using a more particular case of (7) [1, 2]:

$$z - a_1 = \lim_{n\to\infty} f(z) D_{n-1}/D_n = \lim_{n\to\infty} [f^{(n)}(z) D_n]^{1/n} \tag{12}$$

This gives an idea as to the location of the root closest to x.

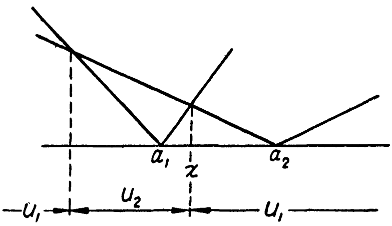

Diagram 1

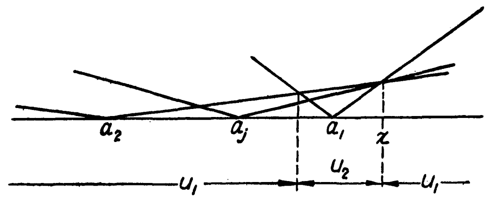

Diagram 2

4. Let now $z = x$ be real equidistant from two conjugate imaginary roots $a+bi$ and $a-bi$. Then u can be taken in the form $x+ti$ and the real and imaginary parts in the equations (8)—(11) can easily be separated. In this case, if x is closer to $a+bi$ and $a-bi$ than to any other root of the equation, and if the equation has no more imaginary roots, any positive t can be taken to approximate $a-bi$ (Diagram 3). If the equation has another imaginary root not much more distant from x than $a-bi$, and with real part closer to x than a, a large value of t would be required (Diagram 4). The imaginary root $a-bi$ can be approximated with some positive t (Diagram 5) even if there is a real root which is closer to x than to $a-bi$, but not very close, and if the real part, a, is close to x.

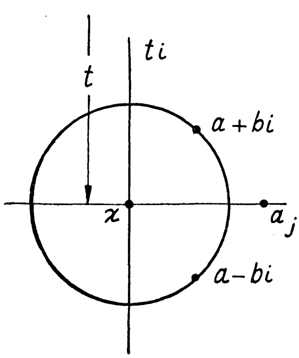

Diagram 3

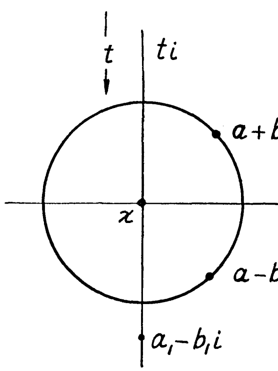

Diagram 4

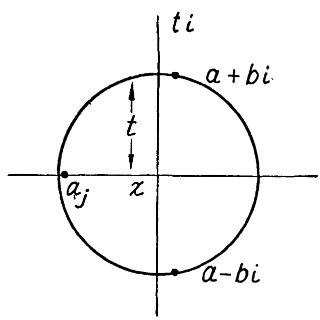

Diagram 5

We shall now give the explicit formulas for the real and imaginary parts of a root a_1 in the four cases given by (8)-(11).

We designate $z=x+ti$, as before, where x and t are real, $t>0$, $a_1=a-bi$, $Q_{n,n}=A_{n,n}+iB_{n,n}$, $Q_{n,n-1}=A_{n,n-1}+iB_{n,n-1}$, where

$$A_{n,n}=-t^2\sum_{j=1}(-1)^{j-1}\binom{n}{2j}t^{2j-2}[f(x)]^{n-2j}D_{2j}\ ,\qquad\qquad (13)$$
$$B_{n,n}=t\sum_{j=1}(-1)^{j-1}\binom{n}{2j-1}t^{2j-2}[f(x)]^{n-2j+1}D_{2j-1}\ ;$$

$$A_{n,n-1}=\sum_{j=1}(-1)^{j-1}(2j^{n-1}-2)t^{2j-2}[f(x)]^{n-2j+1}D_{2j-1}\ ,\qquad\qquad (14)$$
$$B_{n,n-1}=t\sum_{j=1}(-1)^{j-1}(2j^{n-1}-1)t^{2j-2}[f(x)]^{n-2j}D_{2j}\ .$$

The sums being taken over all j, for which the binomial coefficients do not vanish, starting with $j=1$.

Now by using (8)-(11) we get respectively

$$x-a=\lim_{n\to\infty} t\,f(x)[B_{n-1,n-1}(f(x)A_{n-1,n-1}-A_{n,n}) -A_{n-1,n-1}(f(x)B_{n-1,n-1}-B_{n,n})]/\Delta\ ,\qquad (15)$$

$$b=-\lim_{n\to\infty} t\,f(x)[A_{n-1,n-1}(f(x)A_{n-1,n-1}-A_{n,n}) -B_{n-1,n-1}(f(x)B_{n-1,n-1}-B_{n,n})]/\Delta\ ,\qquad (15_1)$$

where

$$\Delta=[f(x)A_{n-1,n-1}-A_{n,n}]^2+[f(x)B_{n-1,n-1}-\mathrm{B}_{n,n}]^2\ ;$$

$$x-a=\lim_{n\to\infty} f(x)(A_{n,n-1}A_{n-1,n-1}+B_{n,n-1}B_{n-1,n-1})/(A^2_{n,n-1}+B^2_{n,n-1})\ ,\qquad (16)$$

$$b=\lim_{n\to\infty} f(x)(A_{n,n-1}B_{n-1,n-1}-B_{n,n-1}A_{n-1,n-1})/(A^2_{n,n-1}+B^2_{n,n-1})\ ;\qquad (16_1)$$

$$x-a=\lim_{n\to\infty}(A_{n,n}A_{n,n-1}+B_{n,n}B_{n,n-1})/(A^2_{n,n-1}+B^2_{n,n-1})\ ,\qquad (17)$$

$$t-b=\lim_{n\to\infty}(A_{n,n}B_{n,n-1}-B_{n,n}A_{n,n-1})/(A^2_{n,n-1}+B^2_{n,n-1})\ ;\qquad (17_1)$$

$$x-a=\lim_{n\to\infty} t\,f(x)[B_{n-1,n-2}(f(x)A_{n-1,n-2}-A_{n,n-1}) -A_{n-1,n-2}(f(x)B_{n-1,n-2}-B_{n,n-1})]/\Delta_1\ ,\qquad (18)$$

$$b=\lim_{n\to\infty} t\,f(x)[A_{n-1,n-2}(f(x)A_{n-1,n-2}-A_{n,n-1}) -B_{n-1,n-2}(f(x)B_{n-1,n-2}-B_{n,n-1})/\Delta_1\ ,\qquad (18_1)$$

where

$$\Delta_1=[f(x)A_{n-1,n-2}-A_{n,n-1}]^2+[f(x)B_{n-1,n-2}-B_{n,n-1}]^2\ .$$

5. Results analogous to those presented above can be obtained by considering other expansions similar to those given by (2). We mention

here one such result assuming that $f(z)$ has only simple zeros. We consider then $(u-z)^k/f(z)$ instead of $(u-z)^k f'(z)/f(z)$ and derive the equation

$$Q^1_{n,k}/[f(z)]^n = \sum_{j=1} A_j(u-a_j)^k/(z-a_j)^n + \psi^1_n(z) , \tag{19}$$

where A_j are constants.

$$Q^1_{n,k} = \sum_{j=0}^{n-1} \binom{k}{j} (u-z)^{k-j} [f(z)]^j P_{n-j-1} \tag{20}$$

$$P_n = \sum_{j=0}^{n-1} f(z)^{(j+1)} [-f(z)]^j P_{n-j-1} , \qquad P_0 = 1. \tag{21}$$

It would suffice now to replace $Q_{n,k}$ by $Q^1_{n\,k}$ in all the previous formulas.

6. If $f(z)$ is a polynomial of degree N and $k=n$, then the last member on the right hand side of (2) equals $-N$. If taken to the left hand side, it would contribute the term $N[f(z)]^n$ to $Q_{n,n}$, consequently, $N[f(z)]^n$ and $N[f(z)]^{n-1}$ will be contributed to $A_{n,n}$ and $A_{n-1,n-1}$ respectively in (8), (9), and (10). In case of equation (19), however, the last member would be reduced to $-A_1 - A_2 - \cdots - A_N$.

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A NOTE ON A PAPER OF L. GUTTMAN

A. C. MEWBORN

In a recent paper L. Guttman [2] obtained, using a result of von Neumann on the theory of games, lower bounds for the largest characteristic root of the matrix AA' where A is a real matrix of order $m \times n$. As Guttman points out his bounds are non-trivial only if some row or column of A has only positive or only negative elements. I wish to show that Guttman's results, and even a better result, are an immediate corollary of a well known theorem on Hermitian matrices: that each diagonal element lies between the smallest and largest characteristic roots (see e.g. [1]). Moreover, if AA' be replaced by AA^* then A can be real or complex and a non-trivial result is always obtained.

THEOREM 1. *Let $A=(a_{ij})$ be an $m \times n$ matrix with real or complex elements. Let λ be the largest characteristic root of the $m \times m$ non-negative definite Hermitian matrix $B=AA^*=(b_{ij})$. Then*

$$(1) \qquad \lambda \geqq \max_i \sum_{j=1}^{n} |a_{ij}|^2$$

$$(2) \qquad \lambda \geqq \max_j \sum_{i=1}^{m} |a_{ij}|^2$$

Proof. Let b_{rr} be the largest diagonal element of B. Then

$$\lambda \geqq b_{rr} = \sum_{j=1}^{n} |a_{rj}|^2 = \max_i \sum_{j=1}^{n} |a_{ij}|^2 ,$$

and (1) is proved. Now the non-zero characteristic roots of AA^* are the same as those of A^*A. Then (2) follows as above if we consider A^*A instead of AA^*.

The bounds in (1) and (2) can be replaced by the weaker bounds

$$(3) \qquad \lambda \geqq n \cdot \max_i \left(\min_j |a_{ij}|^2 \right)$$

$$(4) \qquad \lambda \geqq m \cdot \max_j \left(\min_i |a_{ij}|^2 \right)$$

respectively, and even these bounds are obviously better than Guttman's.

Theorem 1 can be improved further.

THEOREM 2. *Under the hypotheses of Theorem* 1 *we have*

Received October 25, 1957. The author is a National Science Foundation fellow.

$$(5)\quad 2\lambda \geqq \max_{i,j}\left[\sum_{\nu=1}^{n}(|a_{i\nu}|^2+|a_{j\nu}|^2)+\left\{\left[\sum_{\nu=1}^{n}(|a_{i\nu}|^2-|a_{j\nu}|^2)\right]^2+4\left|\sum_{\nu=1}^{n}a_{i\nu}\bar{a}_{j\nu}\right|^2\right\}^{1/2}\right]$$

$$(6)\quad 2\lambda \geqq \max_{i,j}\left[\sum_{\nu=1}^{m}(|a_{\nu i}|^2+|a_{\nu j}|^2)+\left\{\left[\sum_{\nu=1}^{m}(|a_{\nu i}|^2-|a_{\nu j}|^2)\right]^2+4\left|\sum_{\nu=1}^{m}\bar{a}_{\nu i}a_{\nu j}\right|^2\right\}^{1/2}\right]$$

Proof. It was shown in [1] that the largest root of an Hermitian matrix is greater than or equal to the larger of the two roots of any principal minor of order two of the matrix. Suppose the principal minor or order two of B having the largest root lies in the r, s rows and columns of B. Then

$$\begin{aligned}2\lambda &\geqq b_{rr}+b_{ss}+[(b_{rr}-b_{ss})^2+4|b_{rs}|^2]^{1/2}\\ &=\sum_{\nu=1}^{n}(|a_{r\nu}|^2+|a_{s\nu}|^2)+\left\{\left[\sum_{\nu=1}^{n}(|a_{r\nu}|^2-|a_{s\nu}|^2)\right]^2+4\left|\sum_{\nu=1}^{n}a_{r\nu}\bar{a}_{s\nu}\right|^2\right\}^{1/2}\end{aligned}$$

and (3) follows. (4) is proved similarly by considering A^*A instead of B.

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ON THE PRINCIPAL FREQUENCY OF A MEMBRANE

ZEEV NEHARI

1. **Let D denote a simply-connected region** in the xy-plane whose boundary consists of a finite number of piecewise smooth arcs. If λ is the principal frequency of a homogeneous membrane which covers D and is kept fixed at its boundary C, then, according to a well-known theorem of Rayleigh [3], λ is not smaller than the principal frequency of a circular membrane of equal area and density. This may also be expressed by saying that the homogeneous circular membrane has the lowest principal frequency among all homogeneous membranes of the same mass.

In this paper we shall be concerned with the possible generalizations of Rayleigh's theorem to the case of non-homogeneous membranes. It is clear that no general result of this type is to be expected unless certain restrictions are imposed on the density distribution of the membrane. Indeed, it is easily shown that the principal frequency of a membrane of given mass can be made arbitrarily small if enough of the mass is concentrated in a small area interior to D. It is therefore necessary to add conditions which prevent the excessive accumulation of mass at interior points of the membrane. As the following theorem shows, a sufficient condition of this type is the requirement that the density distribution $p(x, y)$ be such that $\log p(x, y)$ is subharmonic, i.e., that the mean value of $\log p(x, y)$ on any circular circumference inside D is not smaller than the value of $\log p(x, y)$ at the center.

THEOREM I. *If λ is the principal frequency of a membrane of given mass whose density distribution $p(x, y)$ is such that $\log p(x, y)$ is subharmonic, then*

$$\lambda \geqq \lambda_0 , \tag{1}$$

where λ_0 is the principal frequency of a homogeneous circular membrane of the same mass.

The conclusion of Theorem I will in general not hold if the restriction on $p(x, y)$ is replaced by the somewhat weaker condition that $p(x, y)$ be subharmonic. The following theorem shows, moreover, that — at least in the case of a circular membrane — inequality (1) is reversed if $p(x, y)$ is assumed to be superharmonic.

THEOREM II. *If λ is the principal frequency of a circular membrane*

Received January 3, 1958. This research was supported by the United States Air Force Office of Scientific Research.

of given mass whose density distribution $p(x, y)$ is superharmonic, then

$$\lambda \leqq \lambda_0 ,$$

where λ_0 is the principal frequency of a homogeneous circular membrane of the same mass.

Theorems I and II will be proved in §§ 2 and 3, respectively. In § 4, Theorem I will be applied to the proof of the following result on homogeneous membranes.

THEOREM III. *Let α be an analytic subarc of C which is concave with respect to D. If Λ denotes the principal frequency of a homogeneous membrane whose boundary is free along α and fixed along $C-\alpha$, then*

$$\Lambda \geqq \Lambda_0 ,$$

where Λ_0 is the principal frequency of a homogeneous semi-circular membrane of equal mass whose boundary is free along the diameter and fixed along the semi-circle.

2. The principal frequency of the membrane with the continuous density distribution $p(x, y)$ is the lowest eigenvalue λ of the differential equation

$$u_{xx} + u_{yy} + \lambda p(x, y)u = 0 \tag{3}$$

with the boundary condition $u=0$. λ may also be defined as the minimum of the Rayleigh quotient

$$J(U) = \frac{\iint_D (U_x^2 + U_y^2)dx\,dy}{\iint_D pU^2 dx\,dy} \tag{4}$$

if $U(x, y)$ ranges over the class of functions which vanish on C and for which U, U_x, U_y are continuous in $D+C$. To prove Theorem I, we have to show that, under the assumptions concerning $p(x, y)$ the integral $\lambda\iint_D p\,dx\,dy$ attains its minimum in case D is a circular disk and $p(x, y)$ is constant, i.e., we have to demonstrate the inequality

$$\lambda\iint_D p(x, y)\,dx\,dy \geqq \pi j_0^2 , \tag{5}$$

where j_0 is the smallest zero of the Bessel function $J_0(r)$.

We denote by u the first eigenfunction of (3). The function u is not zero in D, and may be normalized in such a way that $0 \leqq u \leqq 1$ in D. We use the symbol C_ρ for the level curve, or curves, $u=\rho$ and we set $A(\rho)=\iint_{D_\rho} p\,dx\,dy$, where D_ρ is the subset of D at which $u \geqq \rho$. If C_ρ consists of n closed Jordan curves (sections of which may coincide), these will be denoted by $C_{\rho,1}, C_{\rho,2}, \cdots, C_{\rho,n}$. The proof of (5) will use a symmetrization procedure [3] in which the curve, or curves, C_ρ is replaced by a circle about the origin of radius r, where $\pi r^2 = A(\rho)$. If v is the function which takes the value ρ at all points of this circle, and R is defined by $\pi R^2 = A(0) = \iint_D p\,dx\,dy$, we shall show that

$$\iint_D p\,u^2\,dx\,dy = \int_0^{2\pi}\int_0^R v^2\,r\,dr\,d\theta \tag{6}$$

and

$$\iint_D (u_x^2+u_y^2)\,dx\,dy \geqq \int_0^{2\pi}\int_0^R (v_x^2+v_y^2)\,r\,dr\,d\theta\ . \tag{7}$$

If $J(u)$ and $J(v)$ denote the Rayleigh quotients (3) of u and v for their respective domains of definition, it will follow from (6) and (7) that

$$\lambda\iint_D p\,dx\,dy = \pi R^2 J(u) \geqq \pi R^2 J(v)\ .$$

Since $v(R, \theta)=0$, $J(v)$ is not smaller than the principal frequency $j_0^2 R^{-2}$ of a homogeneous circular membrane of radius R and density 1. Theorem I will therefore be proved if (6) and (7) are established.

We denote by C_ρ^* the level curve $u=\rho-d\rho$, where $d\rho=\varepsilon>0$ and ε is small. If dn is the length of the piece of the normal to C between C_ρ and C_ρ^*, the area between C_ρ and C_ρ^* will be, except for a correction term of order ε^2,

$$dA = \int_{C_\rho} p\,dn\,ds = \sum_{\nu=1}^{n}\int_{C_{\rho,\nu}} p\,dn\,ds\ . \tag{8}$$

where s is the length parameter on C_ρ. Since $A(\rho)=\pi r^2$, we thus have

$$2\pi r\,dr = \int_{C_\rho} p\,dn\,ds\ . \tag{9}$$

By the Schwarz inequality, we have

$$(d\rho)^2\Big(\int_{C_{\rho,\nu}} \sqrt{p}\,ds\Big)^2 = \Big(\int_{C_{\rho,\nu}} \frac{d\rho}{dn}\sqrt{p}\,dn\,ds\Big)^2 \leqq \int_{C_{\rho,\nu}} p\,dn\,ds \int_{C_{\rho,\nu}} \Big(\frac{d\rho}{dn}\Big)^2 dn\,ds\ .$$

It follows therefore that

$$(d\rho)^2 \sum_{\nu=1}^{n}\Big(\int_{C_{\rho,\nu}} \sqrt{p}\,ds\Big)^2 \leqq \int_{C_\rho} p\,dn\,ds \int_{C_\rho}\Big(\frac{d\rho}{dn}\Big)^2 dn\,ds\ .$$

Since, up to an ε^2-correction, $\int_{C\rho}\left(\frac{d\rho}{dn}\right)^2 dn\,ds$ is the contribution $D_\varepsilon(u)$ of the area between C_ρ and C_ρ^* to the Dirichlet integral on the left-hand side of (7), we thus have, in view of (9).

$$(d\rho)^2 \sum_{\nu=1}^{n}\left(\int_{C_{\rho,\nu}} \sqrt{p}\,ds\right)^2 \leqq 2\pi r\, D_{d\rho}(u)\ .$$

On the other hand, the contridution $D_{dr}(v)$ of the circular ring between r and $r+dr$ to the dirichlet integral $D(v)$ is (again with an ε^2-correction) $2\pi r\left(\frac{d\rho}{dr}\right)^2 dr$. Hence,

$$D_{dr}(v) \sum_{\nu=1}^{n}\left(\int_{C_{\rho,\nu}} \sqrt{p}\,ds\right)^2 \leqq 4\pi^2 r^2\, D_{d\rho}(u)\ .$$

Since $\pi r^2 = \iint_{D\rho} p\,dx\,dy$, this may also be written

$$D_{dr}(v) \sum_{\nu=1}^{n}\left(\int_{C_{\rho,\nu}} \sqrt{p}\,ds\right)^2 \leqq 4\pi\, D_{d\rho}(u) \iint_{D_\rho} p\,dx\,dy\ . \tag{10}$$

We shall prove presently that, under our assumptions regarding the function $p(x, y)$, the inequality

$$4\pi \iint_G p\,dx\,dy \leqq \left(\int_\Gamma \sqrt{p}\,ds\right)^2 \tag{11}$$

holds for any rectifiable Jordan curve Γ and the region G bounded by it. If the simply-connected region enclosed by $C_{\rho,\nu}$ is denoted by $D_{\rho,\nu}$, (11) implies that

$$4\pi \iint_{D_\rho} p\,dx\,dy \leqq 4\pi \sum_{\nu=1}^{n} \iint_{D_{\rho,\nu}} p\,dx\,dy \leqq \sum_{\nu=1}^{n}\left(\int_{C_{\rho,\nu}} \sqrt{p}\,ds\right)^2 .$$

Combining this with (10), we obtain

$$D_{dr}(v) \leqq D_{d\rho}(u)$$

and this entails (7). (6) follows from the fact that, by (9),

$$\iint_{D^*-D} p\,u^2 dx\,dy = \rho^2 \int_{C_\rho} p\,dn\,ds + O(\varepsilon^2) = 2\pi\, r\, v^2 dr + O(\varepsilon^2)\ .$$

To complete the proof of Theorem I, we have to show that (11) holds for a function $p(x, y)$ which is positive and continuons in a simply-connected region G and on its boundary Γ, and which is such that $\log p(x, y)$ is subharmonic. Because of the latter property, we have

$\log p(x, y) \leqq \sigma(x, y)$ in G if $\sigma(x, y)$ is the harmonic function in G whose boundary values on Γ coincide with those of $\log p(x, y)$. Hence, (11) will be proved if we can show that

$$4\pi \iint_G e^{2\sigma} \, dx \, dy \leqq \left(\int_\Gamma e^\sigma \, ds \right)^2 ,$$

where $\sigma(x, y)$ is any harmonic function in G which is continuous in $G+\Gamma$. Now $e^\sigma = |g(z)|$, where $g(z)$ is a regular analytic function in G which is continuous and does not vanish in $G+\Gamma$. If we set $g(z)=f'(z)$, we thus have to show that

$$4\pi \iint_G |f'(z)|^2 \, dx \, dy \leqq \left(\int_\Gamma |f'(z)| \, ds \right)^2 , \tag{12}$$

where $f(z)$ is regular in G, and $f'(z)$ is continuous and does not vanish in $G+\Gamma$.

If $f(z)$ is univalent in G, (12) reduces to the isoperimetric inequality

$$4\pi \iint_{G^*} d\xi \, d\eta \leqq \left(\int_{\Gamma^*} |dw| \right)^2 \qquad (w=\xi+i\eta)$$

for the region G^* (bounded by Γ^*) onto which G is mapped by the transformation $w=f(z)$. In the general case we have, by Green's formula,

$$\iint_G |f'(z)|^2 \, dx \, dy = \frac{1}{2i} \int_\Gamma \bar{f} \, f' dz = \frac{1}{2} \int_{\Gamma^*} (\xi \, d\eta - \eta \, d\xi) ,$$

and (12) is seen to be equivalent to the general isoperimetric inequality

$$2\pi \int_{\Gamma^*} (\xi d\,\eta - \eta \, d\xi) \leqq \left(\int_{\Gamma^*} |dw| \right)^2$$

proved by Hurwitz [1, p. 97] for arbitrary piecewise smooth closed curves Γ^* which may be self-intersecting. This completes the proof of Theorem I.

3. **We now turn to the proof of Theorem II.** Since $p(x, y)$ is superharmonic in the circle $x^2+y^2 \leqq R^2$, it follows from a well-known result [2] that

$$q(r) = \int_0^{2\pi} p(x, y) d\theta \qquad (x+iy=re^{i\theta}) \tag{13}$$

is a non-increasing function of r in the interval $[0, R]$. The same is evidently true of its mean value

$$\frac{2}{r^2}\int_0^r t\,q(t)dt\,.$$

If we set $\tau(r)=\int_0^r t\,q(t)dt$, we may therefore conclude that

(14) $$\tau(r) \geqq \frac{r^2}{R^2}\tau(R)\,.$$

If λ denotes the lowest eigenvalue of the problem $\Delta v+\lambda p\,v=0$ with the boundary condition $v=0$ on the circumference $r=R$, we bave

$$\frac{1}{\lambda} \geqq \frac{\int_0^{2\pi}\int_0^R p\,u^2\,r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R (u_x^2+u_y^2)r\,dr\,d\theta}$$

where u is any function which satisfies the boundary and admissibility conditions. In particular, we may take for u the lowest eigenfunction of the problem

(15) $$[r\,u'(r)]' + \lambda_0\,r\,u(r) = 0,\ u(R) = u'(0) = 0\,.$$

This yields

$$\frac{1}{\lambda} \geqq \frac{\int_0^R r\,q(r)\,u^2dr}{2\pi\int_0^R r\,u'^2\,dr}\,.$$

In view of the definition of $\tau(r)$, we have

(16) $$\int_0^R r\,q(r)u^2dr = \int_0^R \tau'(r)u^2dr = -2\int_0^R \tau(r)\,u\,u'dr\,.$$

Since $u(r)\geqq 0$ in $[0, R]$, it follows from (15) that $r\,u'(r)$ is a non-increasing function of r. Because of $u'(0)=0$, we must therefore have $u'(r)\leqq 0$ throughout the interval. We thus conclude from (14) and (16) that

$$\int_0^R rq(r)\,u^2\,dr \leqq -2\frac{\tau(R)}{R^2}\int_0^R r^2\,u\,u'\,dr = 2\frac{\tau(R)}{R^2}\int_0^R r\,u^2\,dr\,.$$

Hence,

$$\frac{1}{\lambda} \geqq \frac{\tau(R)}{\pi R^2}\frac{\int_0^R r\,u^2\,dr}{\int_0^R r\,u'^2dr} = \frac{\tau(R)}{\pi R^2\,\lambda_0} = \frac{\tau(R)}{\pi j_0^2}\,,$$

where j_0 is the first zero of the Bessel function $J_0(x)$. Since

$$\tau(R) = \int_0^R r\, q(r)dr = \int_0^{2\pi}\int_0^R p\, r\, dr\, d\theta\ ,$$

we finally obtain

$$\lambda \iint_D p\, dx\, dy \leqq \pi j_0^2\ ,$$

and this is equivalent to the assertion of Theorem II.

4. In Theorem III, we are concerned with a boundary value problem of different type. If α is an analytic subarc of C, we are considering the problem

$$\Delta u + \Lambda u = 0,\ u = 0 \text{ on } C - \alpha, \frac{\partial u}{\partial n} = 0 \text{ on } \alpha\ . \tag{17}$$

We shall show that, under the assumption that α is concave with respect to the interior of the membrane, the smallest eigenvalue Λ of (17) takes its smallest possible value in the case of a semicircular membrane of the same area, where α coincides with the diameter bounding the membrane. It may be noted that for non-concave arcs α the assertion of Theorem III will in general not be true; as suitable examples show, Λ may in this case be made arbitrarily small.

We introduce the analytic function $f(z)$ which maps the semicircle $|z|<R$, $\Im\{z\}>0$ conformally onto the region D covered by the membrane, and transforms the segment $-R<z<R$ into the open arc α. The value of R may be chosen in such a war that the semicircle has the same area as D. Since α is analytic, $f(z)$ will be regular and the mapping will be conformal on the segment $-R<z<R$. Accordingly, the function $v(z)$ defined by $v(z)=u[f(z)]$ will satisfy the boundary condition $\partial v/\partial n=0$ on this linear segment, and (17) is transformed into the problem

$$\Delta v + \Lambda |f'(z)|^2 v = 0,\ v = 0 \text{ for } z = Re^{i\phi},\ 0 \leqq \phi \leqq \pi, \tag{18}$$
$$\frac{\partial v}{\partial n} = 0 \text{ for } -R < z < R\ .$$

We now define a function $p(z)$ by $p(z)=|f'(z)|^2$ for $|z|\leqq R$, $\Im\{z\}\geqq 0$, and $p(z)=|f'(\bar{z})|^2$ for $|z|\leqq R$, $\Im\{z\}<0$. This function is continuous in $|z|\leqq R$, and we may consider the eigenvalue problem

$$\Delta w + \Lambda^* p\, w = 0,\ w = 0 \text{ for } |z| = R\ . \tag{19}$$

It is easy to see that

$$\Lambda^* \leqq \Lambda\ , \tag{20}$$

where Λ and Λ^* are the lowest eigenvalues of (18) and (19), respectively. Indeed, we have

(21) $$\Lambda^* \leqq \frac{\iint_{D_R}(\eta_x^2+\eta_y^2)dx\,dy}{\iint_{D_R} p\eta^2\,dx\,dy}$$

where D_R denotes the disk $|z|<R$ and η satisfies the boundary and admissibility conditions. If η is identified with $v(z)$ in the upper half of D_R, and with $v(\bar{z})$ in the lower half, these conditions are satisfied and the right-hand side of (21) reduces to Λ.

The next step is to show that $\log p(z)$ is a subharmonic function in $|z|<R$. This is certainly true in both the upper and the lower open halves of $|z|<R$; indeed, in both these regions $\log p(z)$ is even harmonic. To show that $\log p(z)$ is subharmonic throughout $|z|<R$ it is therefore only necessary to derive the inequality

(22) $$\log\ p(x) \leqq \frac{1}{2\pi}\int_0^{2\pi} \log\ p(x+\varepsilon\, e^{i\theta})d\theta\ ,$$

where x is any value such that $-R<x<R$ and ε is a sufficiently small positive number. Since $p(z)$ is symmetric with respect to the horizontal axis, this is equivalent to

$$\log\ p(x) \leqq \frac{1}{\pi}\int_0^{\pi} \log\ p(x+\varepsilon\, e^{i\theta})d\theta\ ,$$

or, in view of the definition of $p(z)$ in the upper half of the disk $|z|<R$,

(23) $$\log\ |f'(x)| \leqq \frac{1}{\pi}\int_0^{\pi} \log\ |f'(x+\varepsilon\, e^{i\theta})|d\theta\ .$$

Since $f(z)$ is regular for $-R<z<R$, we have

$$\frac{1}{\pi}\int_0^{\pi} \log\ |f'(x+\varepsilon\, e^{i\theta})|\,d\theta = \Re\left\{\frac{1}{\pi}\int_0^{\pi} \log\ f'(x+\varepsilon\, e^{i\theta})d\theta\right\}$$

$$= \Re\left\{\frac{1}{\pi}\int_0^{\pi}\left[\log\ f'(x) + \varepsilon\, e^{i\theta}\frac{f''(x)}{f'(x)} + O(\varepsilon^2)\right]d\theta\right\}$$

$$= \log\ |f'(x)| - \frac{2\varepsilon}{\pi}\Re\left\{\frac{1}{i}\frac{f''(x)}{f'(x)}\right\} + O(\varepsilon^2)\ .$$

A comparison with (23) shows therefore that (22) will be satisfied for sufficiently small ε if, and only if, $\Im\{f''(x)/f'(x)\}<0$. If $\phi(x)$ is the angle between the tangent to the curve $w=f(x)$ and the positive

x-direction, this is equivalent to $\phi'(x)<0$. This condition will therefore be satisfied if, and only if, the curve $w=f(x)$ — that is the arc α — is concave with respect to the interior of D. We add that the points at which $\Im\{f''/f'\}=0$ are either isolated, or else this expression vanishes identically for $-R<x<R$ and α is a linear segment. Evidently, the subharmonicity of $p(z)$ in not destroyed by isolated points of this nature. If α is a linear segment, the assertion of Theorem III follows from Rayleigh's theorem and an elementary symmetry argument.

In accordance with the hypotheses of Theorem III, log $p(z)$ will thus be subharmonic in $|z|<R$ and we may apply Theorem I, i.e., inequality (5). In view of the definition of $p(z)$, we have

$$2\Lambda^*\iint_{D_R}|f'(z)|^2\,dx\,dy = \Lambda^*\iint_{|z|<R} p(z)\,dx\,dy \geqq \pi j_0^2 .$$

Taking account of (20) and the fact that $\iint_{D_R}|f'(z)|^2\,dx\,dy$ is the area A of D, we obtain

$$A\Lambda \geqq \frac{\pi}{2} j_0^2 = A j_0^2 R^{-2} ,$$

and this is equivalent to the assertion of Theorem III since $j_0^2 R^{-2}$ is the principal frequency of the membrane of density 1 which covers D_R and has the indicated boundary conditions.

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REMARKS ON DE LA VALLÉE POUSSIN MEANS AND CONVEX CONFORMAL MAPS OF THE CIRCLE

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Introduction. The aims of the present remarks are similar to those pursued by L. Fejér in several papers in the early nineteen thirties and well described by the title of one of his paper: *Gestaltliches über die Partialsummen und ihre Mittelwerte bei der Fourierreihe und der Potenzreihe.* However, the means which we use to realize these aims are different. Fejér discovered the remarkable behavior of certain Cesàro means, especially that of the third Cesàro means for even or odd functions of certain simple basic shapes. In what follows we show that the de la Vallée Poussin means possess such shape-preserving properties to a much higher degree thanks to their variation diminishing character.

Before stating our results, we have to explain a few concepts.

Variation diminishing Transformations on the Circle. If $a_1, a_2, \cdots, a_n$ s a finite sequence of real numbers we shall denote by $v(a)$ or $v(a_\nu)$ the number of variations of sign in the terms of this sequence. By the number $v_c(a)$ of cyclic variations of sign of our sequence we mean the following: If all $a_\nu=0$ we set $v_c(a)=0$. If $a_i\neq 0$ we set

$$v_c(a)=v(a_i, a_{i+1}, \cdots, a_n, a_1, a_2, \cdots, a_{i-1}, a_i) .$$

If we think of the a_ν as arranged clockwise in cyclic order, it becomes obvious that $v_c(a)$ does not depend on the particular non-vanishing term a_i we start with. Notice that $v_c(a)$ is always an even number. Let now $f(t)$ be a real-valued function of period 2π. Let $t_1, t_2, \cdots, t_n$ be such that

$$(1) \qquad t_1<t_2< \cdots <t_n<t_1+2\pi .$$

We may now define the number $v_c(f)$ of *cyclic variations of sign of* $f(t)$ by

$$(2) \qquad v_c(f)=\sup v_c(f(t_\nu)) ,$$

the supremum being taken for all finite sequences $\{t_\nu\}$ subject to (1).

Received October 21, 1957. This paper was prepared partly under the sponsorship of the Office of Naval Research, and partly under the sponsorship of the United States Air Force, Office of Scientific Research, Air Research and Development Command.

Also $v_c(f)$, if finite, is even. Thus $v_c(\sin t)=2$, $v_c(\sin 2t)=4$, $v_c(|\sin t|)=0$.

We now describe what is meant by a *variation diminishing transformation on the circle* (See [4]). Such a transformation is characterized by a non-negative weight-function, or kernel, $\Omega(t)$, of period 2π, of bounded variation and normalized by the conditions

(3) $$\frac{1}{2\pi}\int_0^{2\pi}\Omega(t)dt=1,\ \Omega(t)=\frac{1}{2}(\Omega(t+0)+\Omega(t-0))\ .$$

Let $f(t)$ be an arbitrary periodic function, with period 2π, real-valued and integrable (cf. §1.2); let us form its convolution transform

(4) $$g(t)=\frac{1}{2\pi}\int_0^{2\pi}\Omega(t-\tau)f(\tau)d\tau\ .$$

We say that this transformation is *variation diminishing* provided that the inequality

(5) $$v_c(g)\leqq v_c(f)$$

holds for each f. We mean the same thing if we say that $\Omega(t)$ is a *variation diminishing kernel.*

V-means. One of our aims is to show that the de la Vallée Poussin kernels

(6) $$\omega_n(t)=\frac{(n\,!)^2}{(2n)\,!}\left(2\cos\frac{t}{2}\right)^{2n},$$

the Fourier expansion of which has the simple form

(6′) $$\omega_n(t)=\frac{1}{\binom{2n}{n}}\sum_{-n}^{n}\binom{2n}{n+\nu}e^{i\nu t}=1+2\sum_{1}^{n}\frac{n\,!}{(n-\nu)\,!}\frac{n\,!}{(n+\nu)\,!}\cos\nu t\ ,$$

possess the property of being variation diminishing for $n=1, 2, 3, \cdots$. For $\Omega(t)=\omega_n(t)$ the transformation (4) becomes

(7) $$V_n(t)=\frac{(n\,!)^2}{(2n)\,!}\frac{1}{2\pi}\int_0^{2\pi}\left(2\cos\frac{t-\tau}{2}\right)^{2n}f(\tau)d\tau\ ,$$

and defines the de la Vallée Poussin means, or simply V-means, of the function $f(t)$. It is easily verified (See [14] and [5, p. 15]) that $V_n(t)$ is a trigonometric polynomial of an order not exceeding n, which is readily expressed in terms of the Fourier coefficients of $f(t)$. Indeed, if

(8) $$f(t)\sim\sum_{-\infty}^{\infty}c_\nu e^{i\nu t},\ (c_{-\nu}=\bar{c}_\nu)\ ,$$

we obtain by convoluting (6′) and (8)

$$V_n(t)=\frac{1}{\binom{2n}{n}}\sum_{-n}^{n}\binom{2n}{n+\nu}c_\nu e^{i\nu t}\,. \tag{9}$$

In terms of the real Fourier series ($2c_\nu=a_\nu-ib_\nu$)

$$f(t)\sim\frac{1}{2}a_0+\sum_1^\infty(a_\nu\cos\nu t+b_\nu\sin\nu t) \tag{10}$$

we find

$$V_n(t)=\frac{1}{2}a_0+\frac{1}{\binom{2n}{n}}\sum_1^n\binom{2n}{n+\nu}(a_\nu\cos\nu t+b_\nu\sin\nu t) \tag{11}$$

or

$$V_n(t)=\frac{1}{2}a_0+\sum_1^n\frac{n\,!}{(n-\nu)\,!}\,\frac{n\,!}{(n-\nu)\,!}(a_\nu\cos\nu t+b_\nu\sin\nu t)\,. \tag{12}$$

Main Results. Our principal result is the following

THEOREM 1. *The inequalities*

$$v_c(V_n)\leqq Z_c(V_n)\leqq v_c(f) \tag{13}$$

hold for an arbitrary integrable function $f(t)$. *(We let* $Z_c(V_n)$ *denote the number of real zeros of* $V_n(t)$ *within a period including multiplicities.)*

The first inequality $v_c(V_n)\leqq Z_c(V_n)$, which is obvious, shows that Theorem 1 states considerably more than the variation diminishing property of the kernel $\omega_n(t)$ which amounts to $v_c(V_n)\leqq v_c(f)$. In Part I we give two proofs of Theorem 1, both based on a theorem due to Sylvester [12]. The first proof uses the result of Sylvester's theorem, the second uses the method of one of its proofs.

In Part II we discuss applications of the variation diminishing property of V-means. Theorem 1 gives a useful lower bound for $v_c(f)$ if a certain number of Fourier coefficients of $f(t)$ are known. It is shown how this implies easily some results by Sturm, A. Hurwitz, Pólya and Wiener. In §5 we study the simplest classes of discontinuous periodic functions; the behavior of their V-means is described by Theorems 3 and 4. Fejér's Theorem III [1, p. 86] has an analogue for V-means which is our Theorem 5 below. All this refers to *real* periodic functions. However, the shape-preserving properties of V-means appear to best advantage if applied to *complex-valued* periodic functions.

Let us state here the main result of §6 concerning convex maps of the circle. Let K denote the class of those "schlicht" power series $\sum_1^\infty a_\nu z^\nu$ which map $|z|<1$ onto some convex domain. Let

(14) $$f(z)=\sum_1^\infty c_\nu z^\nu,\ (c_1=1)\ ,$$

(15) $$V_n(z)=\frac{1}{\binom{2n}{n}}\sum_{\nu=1}^{n}\binom{2n}{n+\nu}c_\nu z^\nu$$

be the de la Vallée Poussin mean, or V-mean, of the power series (14). It is known that the partial sums of the series (14) need not belong to K. G. Szegö has shown [13] that if $F(z)\in K$ then all partial sums of (14) are "schlicht" in the circle $|z|<1/4$ and map it onto convex domains, and that $1/4$ is here the largest constant. That the V-means belong to K is one part of the following

THEOREM 2. For

(16) $$f(z)\in K$$

it is necessary and sufficient that

(17) $$V_n(z)\in K \text{ for } n=1, 2, \cdots .$$

The sufficiency part does not even assume the regularity of (14) in the unit circle, as for any formal power series (14) the assumption (17) imply that (14) converges and defines an element of K.

Additional Results. Parts I and II are followed by two appendices which contain related materials, but are almost independent of the main text.

Appendix I brings out a certain analogy between approximations to two kinds of functions: periodic functions and functions defined in a finite interval. It will be shown that the shape-preserving properties of the V-means, which approximate functions of period 2π, are analogous to the shape-preserving properties of the so called Bernstein polynomials which approximate functions defined in $[0,1]$. For the definition of these polynomials see §7 where also their variation diminishing property (Theorem 6) is stated and proved.

Appendix II is devoted to a conjecture on power series which represent a conformal one to one mapping of the unit circle onto a convex domain. The conjecture is that the Hadamard composition, or convolution, of two such power series is again a power series of the same kind (see §9). We do not know whether this conjecture is true or not (it seems to us more likely that it is true) but at any rate, in view of the partial results which we have obtained (§§10 and 11), the problem to prove or to disprove the conjecture seems to us worth while.

PART I. THE DE LA VALLÉE POUSSIN SUMMATION METHOD IS VARIATION DIMINISHING

1.1 A theorem of Sylvester. In the course of his work on Newton's rule of signs J. J. Sylvester discovered a remarkable theorem concerning the real zeros of polynomials of the form

$$\sum_{\nu=1}^{m} c_\nu (x-\xi_\nu)^q$$

(see [**12**, p. 408], [**7**] and also [**9**, vol. 2, Problem 79, p. 50]). In Sylvester's theorem q may assume any positive integral value, a fact which is important for its proof which proceeds by induction in q. We need Sylvester's result only for $q=2n$ and state it as follows.

LEMMA 1. *Let* $\xi_1<\xi_2<\cdots<\xi_m$, $(m\geqq 2)$, *be given reals and consider the polynomial*

$$P(x)=\sum_{\nu=1}^{m} c_\nu (x-\xi_\nu)^{2n}$$

(*with real* $c_\nu\neq 0$ *for all* ν), *which we assume not to vanish identically. Then*

$$Z(P;\,-\infty<x<\infty)\leqq v(c_1, c_2, \cdots, c_m, c_1)\ ,$$

where the left side denotes the number of real zeros of $P(x)$ *while the right side is the number of variations of sign in the sequence displayed.*

The significance for us of Sylvester's results is that it easily yields the following

LEMMA 2. Let

$$-\pi<\tau_1<\tau_2<\cdots<\tau_m<\pi, \qquad m\geqq 2\ , \tag{1.1}$$

be given reals and consider the trigonometric polynomial

$$T(t)=\sum_{\nu=1}^{m} c_\nu\left(\sin\frac{t-\tau_\nu}{2}\right)^{2n}, \tag{1.2}$$

(*for real* $c_\nu\neq 0$ *for all* ν), *which we assume not to vanish identically. Then*

$$Z(T;-\pi<t<\pi)\leqq v(c_1, c_2, \cdots, c_m, c_1)\ .$$

Proof. We introduce the new variable

$$x = \tan\frac{t}{2} \qquad -\pi < t < \pi \tag{1.3}$$

whose range is $-\infty < x < \infty$. The images of the τ_ν we denote by

$$\xi_\nu = \tan\frac{\tau_\nu}{2}$$

and these give rise to the identities

$$\left(\sin\frac{t-\tau_\nu}{2}\right)^2 = \frac{(x-\xi_\nu)^2}{(1+x^2)(1+\xi_\nu^2)}\,, \qquad \nu = 1, \cdots, m\,,$$

Thus (1.2) may be expressed in terms of x by

$$T(t) = \frac{1}{(1+x^2)^n}\sum_{\nu=1}^{m} c_\nu \gamma_\nu (x-\xi_\nu)^{2n}\,,$$

where the γ_ν are positive and so Lemma 2 immediately follows from Sylvester's Lemma 1.

We now recast our result in the following more useful form;

LEMMA 3. *Let $\tau_1, \tau_2, \cdots, \tau_m$ $(m \geqq 2)$ be m points in counter-clockwise order on the circle such that τ_m should not overtake or even reach τ_1. We may express these requirements by assuming that*

$$\tau_1 < \tau_2 < \cdots < \tau_m < \tau_1 + 2\pi\,. \tag{1.4}$$

Let

$$T_n(t) = \sum_{\nu=1}^{m} c_\nu \omega_n(t-\tau_\nu)\,, \qquad T_n(t) \not\equiv 0\,, \tag{1.5}$$

where at least two among the c_ν do not vanish. Then

$$Z_c(T_n) \leqq v_c(c_\nu)\,. \tag{1.6}$$

Proof. By omitting vanishing terms in (1.5) we may assume that $c_\nu \neq 0$ for all ν. Moreover, a change of variable by $t = t' - \pi$ will evidently not alter the left hand side of (1.6). This implies that in our statement (1.6) we may replace $T_n(t)$ by the polynomial $T(t)$ defined by (1.2). By a second appropriate transformation $t = t' + c$ we may replace the conditions (1.4) by the more restrictive inequalities (1.1), at the same time making sure that $T(\pi) \neq 0$. But then

$$Z_c(T_n) = Z_c(T) = Z(T;\ -\pi < t < \pi) \leqq v(c_1, c_2, \cdots, c_m.\ c_1) = v_c(c_\nu)$$

and Lemma 3 is established.

1.2. On the number of variations of a function. The reader may interpret the term "integrable" either according to the definition of Riemann or to that of Lebesgue, or to any other definition that involves the familiar standard properties of the integral. We emphasize the following property: If $f(t)$ and $g(t)$ are integrable and $f(t) \geqq 0$ in the interval I, then

$$\int_I f dt = 0$$

implies

$$\int_I fg dt = 0 .$$

We consider now a real-valued periodic function $f(t)$ with the period 2π, we assume that it is integrable in the interval $(0,2\pi)$ and that $v_c(f)$, as defined in the Introduction, is finite. We consider t (mod 2π), that is, we consider t as attached to a point on the periphery of the unit circle. If $v_c(f) = 2k$, we can, as easily seen, divide the circumference of the unit circle into $2k$ consecutive arcs

$$I_1, I_2, \cdots, I_{2k} \tag{1.7}$$

such that

$$(-1)^{\nu-1} f(t) \geqq 0 \text{ in } I_\nu \tag{1.8}$$

for $\nu = 1, 2, \cdots, 2k$; the arcs (1.7) may be open, or closed, or open from one side and closed from the other, some of them may even reduce to a single point. Now, we *normalize* $f(t)$, that is, we change $f(t)$ (if necessary) as follows: we set $f(t) = 0$ in all points of any interval (1.7) on which $\int f dt$ vanishes; especially, if an interval listed under (1.7) consists of just one point, we set $f(t) = 0$ in that point. This normalization cannot increase (but may decrease) $v_c(f)$ and leaves unchanged the V-means of f (cf. the initial remark of this section). Therefore, it will be sufficient to prove Theorem 1 for normalized functions. If, however, $v_c(f) = 2k$ for a normalized $f(t)$, the intervals (1.7), constructed as above, have the property

$$(-1)^{\nu-1} \int_{I_\nu} f(t) dt > 0 \text{ for } \nu = 1, 2, \cdots, 2k . \tag{1.8'}$$

The foregoing remarks will be useful in the following proof of Theorem 1. Yet we do not need them in establishing the weaker inequality

$$v_c(V_n) \leqq v_c(f) \tag{1.9}$$

for a Riemann-integrable function f.
Indeed, let us consider the integral

$$V_n(t)=\frac{1}{2\pi}\int_0^{2\pi}\omega_n(t-\tau)f(\tau)d\tau$$

and its approximating sums

$$V_{n,m}(t)=\frac{1}{m}\sum_{\nu=0}^{m-1}\omega_n\left(t-\frac{2\pi\nu}{m}\right)f\left(\frac{2\pi\nu}{m}\right).$$

Lemma 3 and definition (2) imply

$$v_c(V_{n,m})\leqq v_c\left(f\left(\frac{2\pi\nu}{m}\right)\right)\leqq v_c(f)$$

or

$$v_c(V_{n,m})\leqq v_c(f).$$

Since $V_{n,m}(t)\to V_n(t)$ for all t, as $m\to\infty$, the last inequality evidently implies (1.9). An "approximation argument" extending (1.9) to a more comprehensive class of functions is easy, but hardly deserves to be presented here.

1.3. A first proof of Theorem 1. The first inequality (13) is immediate and so the essential assertion of Theorem 1 consists in the inequality

$$Z_c(V_n)\leqq v_c(f). \tag{1.10}$$

If $v_c(f)\geqq 2n$ there is nothing to prove; also if $v_c(f)=0$ for then $V_n(t)$ clearly can not vanish. Let us assume, then, that $f(t)$ is "normalized" according to §1.2, and that $0<v_c(f)=2k<2n$, and let us divide the unit circumference into the $2k$ consecutive arcs (1.7) which satisfy the conditions (1.8) and (1.8′). We may then write the Fourier coefficients of $f(t)$ in the form

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(t)dt=\sum_{s=1}^{k}\frac{1}{\pi}\int_{I_{2s-1}}|f(t)|dt-\sum_{s=1}^{k}\frac{1}{\pi}\int_{I_{2s}}|f(t)|dt$$

$$\begin{aligned}a_\nu&=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos\nu t dt=\sum_1^k\frac{1}{\pi}\int_{I_{2s-1}}|f(t)|\cos\nu t dt\\&\quad-\sum_1^k\frac{1}{\pi}\int_{I_{2s}}|f(t)|\cos\nu t dt,\end{aligned} \tag{1.11}$$

$$b_\nu=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin\nu t dt=\sum_1^k\frac{1}{\pi}\int_{I_{2s-1}}|f(t)|\sin\nu t dt$$

$$-\sum_1^k \frac{1}{\pi}\int_{I_{2s}} |f(t)| \sin\nu t dt .$$

$$\nu=1, \cdots, n;$$

Consider in the $2n$-dimensional space E_{2n} the closed curve Γ defined in parametric form by $\cos\nu t$, $\sin\nu t$ ($\nu=1, \cdots, n; 0\leqq t\leqq 2\pi$). To the division (1.7) of the circumference into the arcs I_μ, corresponds a division of Γ into arcs

(1.12) $$\Gamma_1, \Gamma_2, \cdots, \Gamma_{2k} ,$$

where we think of the arc Γ_μ as carrying the *positive* mass

(1.13) $$\frac{1}{\pi}\int_{I_\mu} |f(t)| dt .$$

This mass has a centroid the coordinates of which, multiplied by (1.13), are

(1.14) $$\frac{1}{\pi}\int_{I_\mu} |f(t)|\cos \nu t dt, \frac{1}{\pi}\int_{I_\mu} |f(t) \sin \nu t dt \qquad (\nu=1, \cdots, n) .$$

By a well known theorem of Carathéodory the mass (1.13) of Γ_μ may be concentrated in a finite number of points along Γ_μ so as to produce the same centroid (1.14). This we do for each of the arcs (1.12). Arranging all these points in *cyclic order* along Γ we obtain points $\tau_1, \tau_2, \cdots, \tau_m$ and corresponding coefficients $c_1, c_2, \cdots, c_m$ where $(-1)^{\mu-1}c_j>0$ when τ_j belongs to I_μ. In view of the relations (1.11) we obtain

(1.15) $$a_0=\sum_{j=1}^m c_j,\ a_\nu=\sum_{j=1}^m c_j \cos\nu\tau_j,\ b_\nu=\sum_{j=1}^m c_j \sin \nu\tau_j ,$$

(1.16) $$v_c(c_j)=2k=v_c(f) .$$

We consider now the trigonometric polynomial

(1.17) $$F(t)=\frac{1}{2}\sum_{j=1}^m c_j\omega_n(t-\tau_j)$$

and claim that

(1.18) $$F(t)=V_n(t) .$$

Indeed, by (6′)

$$F(t)=\frac{1}{2}\sum_{j=1}^m c_j\omega_n(t-\tau_j)$$

$$=\sum_j c_j\left\{\frac{1}{2}+\sum_{\nu=1}^n \frac{n!}{(n-\nu)!}\frac{n!}{(n+\nu)!}\cos\nu(t-\tau_j)\right\}$$

$$=\sum_j c_j\left\{\frac{1}{2}+\sum_{\nu=1}\frac{n!}{(n-\nu)!}\,\frac{n!}{(n+\nu)!}\,(\cos\nu t\cos\nu\tau_j+\sin\nu t\sin\nu\tau_j)\right\},$$

and interchanging the order of summations, we obtain by (1.15)

$$F(t)=\frac{1}{2}a_0+\sum_{\nu=1}^{n}\frac{n!}{(n-\nu)!}\,\frac{n!}{(n+\nu)!}\,(a_\nu\cos\nu t+b_\nu\sin\nu t)$$

which is identical with $V_n(t)$ by (12). Finally, by (1.17), (1.18), (1.16) and Lemma 3

$$Z_c(V_n)=Z_c(F)\leqq v_c(c_j)=v_c(f),$$

which proves the inequality (1.10).

2. A second proof of Theorem 1. The foregoing proof is based on Sylvester's result which we stated as Lemma 1. We shall now prove Theorem 1 without assuming the knowledge of this result.

We transform (7) by changing the variables. Setting

$$x=\tan\frac{t}{2},\quad \xi=-\cot\frac{\tau}{2},$$

we obtain from (7) (by steps similar to those exhibited following (1.3)) that

$$(2.1)\quad (1+x^2)^n V_n(2\operatorname{arc}\tan x)=\int_{-\infty}^{\infty}(x-\xi)^{2n}\frac{(n!)^2 2^{2n}}{(2n)!\,\pi}\,\frac{f(-2\operatorname{arc}\cot\xi)}{(1+\xi^2)^{n+1}}d\xi .$$

This relation is contained in the more general

$$(2.2)\quad P(x)=\int_{-\infty}^{\infty}(x-\xi)^m A(\xi)d\xi$$

where m is a positive integer and the integral $\int_{-\infty}^{\infty}\xi^m A(\xi)d\xi$ is absolutely convergent; $P(x)$ is by the structure of the formula (2.2) a polynomial of degree not higher than m.

We consider the following quantities connected with (2.2):

N the number of real zeros of $P(x)$, counted with multiplicity;

v the number of variations of sign of $A(\xi)$ in the open interval $-\infty<\xi<\infty$;

sgn$A(\infty)$ is the constant sign, different from 0, that $A(\xi)$ possesses whenever it is different from 0 in a suitably chosen interval $\omega<\xi<\infty$; waessume here that $A(\xi)$ is normalized in the sense of §1.2;

$\operatorname{sgn} A(-\infty)$ is similarly defined;

$\eta=\frac{1}{2}|\operatorname{sgn} A(\infty)-\operatorname{sgn}(-1)^m A(-\infty)|$

so that η is either 0 or 1;

$V=v+\eta$.

In fitting (2.1) into the more general pattern (2.2), we can assume without loss of generality (by rotating the circle through an appropriate angle) that $V_n(\pi)\neq 0$, that $f(t)$ is normalized in the sense of §1.2, and that 0 is an interior point of one of the intervals of constant sign considered there, so located that, for some positive ε, $f(t)$ takes some nonvanishing values in both intervals $-\varepsilon<t<0$ and $0<t<\varepsilon$. Under these circumstances, in the particular case (2.1),

$$\begin{gathered} m=2n\ , \\ A(-\infty)=A(\infty)\ , \\ \eta=0\ , \\ V=v=v_c(f)\ , \\ N=Z_c(V_n)\ , \end{gathered}$$

and so Theorem I is an immediate consequence of the following.

LEMMA 4. $N\leqq V$.

We need several steps to prove Lemma 4.

(a) There are some particular cases in which Lemma 4 is obvious.

If $P(x)$ vanishes identically there is nothing to prove since in this case, by definition, $N=0$.

If $V\geqq m$ there is nothing to prove since, of course, $N\leqq m$.

If $v=0$ and m is even (so that $V=\eta=0$) then $P(x)$ will have for all real x the constant sign of $A(\xi)$ and so $N=0$ as it should be according to Lemma 4.

If $v=0$ and m is odd (so that $V=\eta=1$) then $m-1$ is even and so

$$P'(x)=\int_{-\infty}^{\infty}(x-\xi)^{m-1}mA(\xi)d\xi$$

has a constant sign for all x, by what we have just said. Therefore, $P(x)$ is monotone and $N=1$ which agrees with Lemma 4.

And so we may and shall assume in the sequel that

$$1\leqq v\leqq V\leqq m-1\ . \tag{2.3}$$

(b) Let c be a point of change of sign for $A(\xi)$; that is, c is the common endpoint of two contiguous intervals in each of which $A(\xi)$ keeps a constant sign, yet the two signs (cf. §1.2) considered are opposite. The number of such points is v and we have assumed (2.3).

We assert that at least one of the $m-1$ quantities $P'(c), P''(c), \cdots, P^{(m-1)}(c)$ is different from 0. If this assertion were wrong, the integral

$$\int_{-\infty}^{\infty} (\xi-c)^{\mu} A(\xi) d\xi$$

would vanish for $\mu=m-1, \cdots, 2,1$ and, as a linear combination of these integrals,

$$\int_{-\infty}^{\infty} (\xi-c) Q(\xi) A(\xi) d\xi \tag{2.4}$$

would vanish for any polynomial $Q(\xi)$ of degree not exceeding $m-2$. Yet this is certainly false if

$$Q(\xi)=(x-c_1)(x-c_2) \cdots (x-c_{v-1}) \tag{2.5}$$

where $c, c_1, c_2, \cdots, c_{v-1}$ are all the points of change of sign of $A(\xi)$; observe (2.3) in computing the degree of $Q(\xi)$. In fact, with (2.5) the integrand in (2.4) has a constant sign and so the integral (2.4) cannot vanish.

We have seen by the way, that under the condition (2.3) $P(x)$ cannot vanish identically.

(c) Set

$$G(x)=P(x)(x-c)^{-m} \tag{2.6}$$

$$\begin{aligned} P^*(x) &= (x-c)^{m+1} G'(x) \\ &= (x-c)P'(x) - mP(x) \\ &= \int_{-\infty}^{\infty} (x-\xi)^{m-1} A^*(\xi) d\xi \end{aligned} \tag{2.7}$$

where

$$A^*(\xi)=m(\xi-c)A(\xi) \tag{2.8}$$

and let $N^*, m^*, v^*, \eta^*, V^*$ be just so connected with $P^*(x)$ and $A^*(\xi)$ as N, m, v, η and V are with $P(x)$ and $A(\xi)$. Obviously

$$\begin{aligned} m^* &= m-1 \\ v^* &= v-1 \\ \operatorname{sgn} A^*(\infty) &= \operatorname{sgn} A(\infty) \\ \operatorname{sgn} A^*(-\infty) &= -\operatorname{sgn} A(-\infty) \end{aligned} \tag{2.9}$$

and so $\eta^*=\eta$. Combining this with (2.9), we obtain

$$V^* = V - 1 . \tag{2.10}$$

We intend to prove Lemma 4 by mathematical induction with respect to V. In fact, we have already proved Lemma 4 in the particular case $V=0$ under (a). We therefore assume $V \geqq 1$, cf. (2.3), and that Lemma 4 has been proved for the preceding value (2.10), and so we take for granted that

$$N^* \leqq V^* . \tag{2.11}$$

(d) Let k denote the number of those zeros of $P(x)$ that coincide with the point c; obviously $k \geqq 0$, and, by (b),

$$k \leqq m-1 . \tag{2.12}$$

Let k^* denote the number of those zeros of $P^*(x)$ that coincide with c. We set

$$N = k + l , \quad N^* = k^* + l^* . \tag{2.13}$$

The quantities l and l^*, defined by (2.13), enumerate those zeros of $P(x)$ and $P^*(x)$, respectively, that fall into one or the other of the two open intervals $-\infty < x < c$ and $c < x < \infty$.

We note the critical term of the expansion of $P(x)$ around the point $x=c$,

$$P(x) = \frac{P^{(k)}(c)(x-c)^k}{k!} + \cdots, \ P^{(k)}(c) \neq 0 .$$

By (2.6) and (2.12), $G(x)$ has a pole at the point c and (2.7) yields

$$P^*(x) = \frac{(k-m)P^{(k)}(c)(x-c)^k}{k!} + \cdots .$$

We infer that $P^*(x)$ has just as many zeros at the point c as $P(x)$:

$$k^* = k . \tag{2.14}$$

By the way, we have seen that $P^*(x)$ does not vanish identically.

(e) It remains to consider the real zeros different from c; $P(x)$ or, which is the same, $G(x)$ has l such zeros, and $P^*(x)$ or, which is the same, $G'(x)$ has l^* such zeros. These zeros are distributed somehow in the two open intervals, $-\infty < x < c$ and $c < x < \infty$.

By the theorem of Rolle, in each of these intervals at most one zero can be lost in the passage from $G(x)$ to $G'(x)$, so that

(2.15) $$l^* \geqq l-2\,;$$

this information is correct, but insufficient for our purpose. We shall obtain, however, additional information by using the following remark (cf. [9, vol. 2, p. 39, problem 14]).

No zero can be lost in the passage from $G(x)$ *to* $G'(x)$ *in the interval* $(-\infty, c)$ *if*

(2.16) $$\operatorname{sgn} G(-\infty) = \operatorname{sgn} G'(-\infty)$$

and no zero can be lost in this passage in the interval (c, ∞) *if*

(2.17) $$\operatorname{sgn} G(\infty) = -\operatorname{sgn} G'(\infty)\,.$$

The signs mentioned in (2.16) and (2.17) refer to a certain neighborhood of $-\infty$ or ∞ and, as $G(x)$ has only a finite number of zeros, they are certainly different from 0.

(f) We know, cf. (b), that the polynomial $P(x)$ does not vanish identically. We set

(2.18) $$P(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$$

and distinguish two cases.

Case I. If $b_0 = 0$, there is an s such that $b_0 = b_1 = \cdots = b_{s-1} = 0$, $b_s \neq 0$ and so we easily find the initial terms in the expansions around ∞:

$$G(x) = \frac{b_s}{x^s} + \cdots\,, \qquad \frac{G'(x)}{G(x)} = -\frac{s}{x} + \cdots\,.$$

In this case, both conditions (2.16) and (2.17) are satisfied, and, by the final remark under (e), we can improve (2.15) to

(2.19) $$l^* \geqq l\,.$$

Case II. Now

(2.20) $$b_0 = \int_{-\infty}^{\infty} A(\xi)\,d\xi \neq 0\,,$$

and the expansions around ∞ begin

(2.21) $$G(x) = b_0 + \frac{mcb_0 + b_1}{x} + \cdots$$

(2.22) $$G'(x) = -\frac{mcb_0 + b_1}{x^2} + \cdots$$

where

$$mcb_0+b_1=m\int_{-\infty}^{\infty}A(\xi)(c-\xi)d\xi\ . \tag{2.23}$$

We again distinguish two cases.

Subcase II, 1. If $v=1$, c is the only point of change of sign of $A(\xi)$, the integrand in (2.23) is of constant sign, and so the integral is different from 0.

Subcase II, 2. If $v\geqq 2$, the integral (2.23) could vanish. Yet in this case $A(\xi)$ has at least another point of change of sign, c_1, and we say that (2.23) and

$$m\int_{-\infty}^{\infty}A(\xi)(c_1-\xi)d\xi$$

cannot vanish simultaneously: in fact, their difference is

$$m(c_1-c)\int_{-\infty}^{\infty}A(\xi)d\xi=m(c_1-c)b_0\neq 0$$

by our present assumption (2.20). Therefore, assuming that the point of change was properly selected from the start (which boils down to a proper choice of notation) we may assume that (2.23) is different from 0, also in the present subcase.

Finally, in both subcases, we conclude from (2.21) and (2.22)

$$\lim_{x\to\pm\infty}\frac{x^2G'(x)}{G(x)}=-\frac{mcb_0+b_1}{b_0}\neq 0$$

and we see that just one of the two conditions (2.16) and (2.17) is fulfilled. Therefore, by the final remark under (e), we can improve (2.15) to

$$l^*\geqq l-1\ . \tag{2.24}$$

Thus, even in the less favorable of the two cases I and II, we have (2.24). Combining this with (2.13) and (2.14), we obtain

$$N^*\geqq N-1$$

and hence and from (2.10) and (2.11) we obtain

$$V-1=V^*\geqq N^*\geqq N-1$$

or $V\geqq N$, which is the desired conclusion of Lemma 4.

The foregoing somewhat involved proof becomes more understandable if it is compared with the proof for Lemma 1 given in [7] or in [9, vol. 2, p. 50, problem 79].

PART II. SOME APPLICATIONS OF THE VARIATION DIMINISHING PROPERTY OF V-MEANS

3. A theorem of Ch. Sturm and A. Hurwitz. Let $f(t)$ be a real-valued, integrable, periodic function of period 2π. Let

$$f(t)=\frac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu t+b_\nu\sin\nu t) \tag{3.1}$$

be its Fourier expansion. Suppose that the partial sum

$$S_n(t)=\frac{1}{2}a_0+\sum_{\nu=1}^{n}(a_\nu\cos\nu t+b_\nu\sin\nu t) \tag{3.2}$$

is known. What can we say about the number $v_c(f)$ of changes of sign of $f(t)$ in a period? An answer is immediate: Knowing (3.2), we can compute (11), the nth V-mean of $f(t)$, and we must have

$$v_c(f)\geqq Z_c(V_n) \tag{3.3}$$

by Theorem 1.

The information provided by this inequality is strongest when the right hand side attains its largest value $2n$. There is a simple sufficient condition for this eventuality which we record as follows.

COROLLARY 1. *If*

$$\begin{aligned}(a_n^2+b_n^2)^{1/2}>\binom{2n}{1}(a_{n-1}^2+b_{n-1}^2)^{1/2}+\binom{2n}{2}(a_{n-2}^2+b_{n-2}^2)^{1/2}+\cdots\\+\binom{2n}{n-1}(a_1^2+b_1^2)^{1/2}+\frac{1}{2}\binom{2n}{n}|a_0|\ ,\end{aligned} \tag{3.4}$$

then every function $f(t)$ having (3.2) *as the* nth *partial sum of its Fourier series, must change sign within a period at least $2n$ times.*

Indeed, it is clear by (3.4) that the last term of the expression (11) for $V_n(t)$ so predominates that $V_n(t)$ has $2n$ simple zeros, hence $Z_c(V_n)=2n$. The statement now follows from (3.3).

We obtain a classical result [2, pp. 572-574] as a very special case.

COROLLARY 2. *If* $a_0=a_1=b_1=\cdots=a_{n-1}=b_{n-1}=0,\ a_n^2+b_n^2>0,$ *then* $v_c(f)\geqq 2n$.

The following is an equivalent formulation. *If* $a_n^2+b_n^2>0$ *then*

$$v_c(f(t)-S_{n-1}(t))\geqq 2n\ . \tag{3.5}$$

This second formulation is especially interesting and intuitive because it shows that the graph of the partial sum $S_{n-1}(t)$ must cross the graph of $f(t)$ at least $2n$ times. Hurwitz's proof of Corollary 2 is direct and elementary. However, his classical argument is no longer available to establish other special cases such as the following.

If

$$f(t)=\frac{1}{2}+\sum_{\nu=1}^{n}\cos\nu t+\sum_{\nu=n+1}^{\infty}(a_\nu\cos\nu t+b_\nu\sin\nu t)$$

then

$$v_c(f)\geqq 2n\ .$$

For in this case $V_n(t)=\frac{1}{2}\omega_n(t)$, hence $Z_c(V_n)=2n$ so that (3.3) implies the result. Such particular examples are easily constructed and we see no other way of proving them except by the fundamental inequality (3.3).

4. The simplest Pólya-Wiener result concerning high order derivatives of periodic functions. Let $f(t)$ be a real function of period 2π which is infinitely often differentiable. Let us consider its zeros and also the zeros of its successive derivatives. Counting multiplicities as usual we set

$$N^{(k)}=Z_c(f^{(k)})\ ,\qquad k=0,\ 1,\ \cdots,$$

and assume all these numbers to be finite. A familiar application of Rolle's theorem shows that

$$N^{(0)}\leqq N^{(1)}\leqq\cdots\leqq N^{(k)}\leqq N^{(k+1)}\leqq\cdots\ . \tag{4.1}$$

Can this sequence remain bounded? This is surely the case if $f(t)$ is a trigonometric polynomial. The truth of the converse is stated by the following proposition due to Pólya and Wiener [8.]

COROLLARY 3. *If the sequence* (4.1) *is bounded and*

$$\lim N^{(k)}=2m\ , \tag{4.2}$$

then $f(t)$ is a trigonometric polynomial of exact order m.

Indeed, let (3.1) be the Fourier series of $f(t)$. It is known to converge under our assumptions and the expansion of $f^{(k)}(t)$ is obtained by formal differentiations of the expansion of $f(t)$. Let us assume that for a certain n

$$a_n^2+b_n^2>0\ . \tag{4.3}$$

It is clear then from the form of the Fourier series for $f^{(k)}(t)$ that this series will satisfy the inequality (3.4) of Corollary 1, provided only that

k is sufficiently large, $k>K$ say. But then by Corollary 1

$$N^{(k)} \geqq v_c(f^{(k)}) \geqq 2n, \text{ (if } k>K) .$$

Thus (4.2) and (4.3) imply that $n \leqq m$, and $f(t)$ must reduce to a trigonometric polynomial of order $q \leqq m$. On the other hand, if $f(t)$ is such a polynomial, $N^{(k)} \leqq 2q$ which implies $2m \leqq 2q$ or $m \leqq q$, hence $q=m$ and the theorem is established.

5. The graphic behavior of V-means. We now wish to discuss the shape-preserving properties of the V-means which are implicitly contained in the fundamental inequality

$$Z_c(V_n) \leqq v_c(f) . \tag{5.1}$$

It shows that $V_n(t)$ can't oscillate about zero more frequently than $f(t)$ does. But there is nothing peculiar about the level zero. Indeed, if γ is any real, then $f(t)=\gamma$ implies $V_n(t)=\gamma$. Thus we may replace in (5.1) f and V_n by $f-\gamma$ and $V_n-\gamma$, respectively, obtaining the inequality

$$Z_c(V_n-\gamma) \leqq v_c(f-\gamma) . \tag{5.2}$$

A second remark is based on the obvious known fact (see [5, p. 191]) that if $f(t)$ is absolutely continuous then $V_n'(t)$ is the V-mean of $f'(t)$. But then (5.1) immediately gives

$$Z_c(V_n') \leqq v_c(f') . \tag{5.3}$$

This operation may naturally be repeated giving

$$Z_c(V_n^{(k)}) \leqq v_c(f^{(k)}) , \tag{5.4}$$

which is valid depending on how many derivatives $f(t)$ possesses. For instance, if

$$f(t) \in C''$$

then the graph of $V_n(t)$ can't have more maxima, minima or points of inflexion than the corresponding numbers for the graph of $f(t)$.

It is desirable, however, to discuss this phenomenon for functions of a lower degree of smoothness and the following developments aim to do that. We consider the class D_0 of real periodic functions $f(t)$, of bounded variation, normalized by $2f(t)=f(t+0)+f(t-0)$. A subclass of D_0 is the class D_1 of functions satisfying the classical *Dirichlet conditions*. By $f(t) \in D_1$ we mean that the circle can be dissected into a finite *number of consecutive open arcs in each of which* $f(t)$ *is monotone in the wide sense.*

With each $f(t) \in D_1$ we associate an even non-negative integer $S(f)$, called the number of *sense-reversals* of $f(t)$ and defined as follows. Consider, for a given natural number k, the periodic sequence of ordinates

$$f_\nu=f\left(\frac{2\pi\nu}{k}\right)=f(\nu h)\ , \qquad h=2\pi/k\ , \tag{5.5}$$

of period k, and the likewise periodic sequence of differences

$$\Delta f_\nu=f\left(\frac{2\pi(\nu+1)}{k}\right)-f\left(\frac{2\pi\nu}{k}\right)\ ,$$

$\nu=0,\ 1,\ 2,\ \cdots,\ n-1$. We now define $S(f)$ by

$$S(f)=\lim_{k\to\infty} v_c(\Delta f_\nu)=\max_k v_c(\Delta f_\nu)\ . \tag{5.6}$$

The reader is urged to supply proof for the statements implied in this definition; it depends on an analysis of the finitely many points which have no neighborhood in which $f(t)$ is monotone. If in addition to $f(t)\in D_1$ we assume that $f(t)\in C'$ then evidently $S(f)=v_c(f')$.

Our substitute for (5.3) for the class D_1 is given by the following.

THEOREM 3. *If* $f(t)\in D_1$ *then*

$$v_c(V_n')=S(V_n)\leqq S(f)\ . \tag{5.7}$$

The proof is very simple. Besides the V-mean

$$V_n(t)=\frac{1}{2\pi}\int_0^{2\pi}\omega_n(t-\tau)f(\tau)d\tau$$

we consider the approximating sums

$$V_{n,k}(t)=\frac{1}{k}\sum_\nu \omega_n(t-\nu h)f_\nu\ , \qquad h=2\pi/k.$$

Replacing t by $t+h$ we obtain

$$V_{n,k}(t+h)=\frac{1}{k}\sum_\nu \omega_n(t-\nu h)f_{\nu+1}$$

and therefore

$$\Delta V_{n,k}/h=(V_{n,k}(t+h)-V_{n,k}(t))/h=\frac{1}{2\pi}\sum_\nu \omega_n(t-\nu h)\Delta f_\nu\ . \tag{5.8}$$

By Lemma 3, in view of (5.6), we obtain $v_c(\Delta V_{n,k}/h)\leqq v_c(\Delta f)\leqq S(f)$ or

$$v_c(\Delta V_{n,k}/h)\leqq S(f)\ .$$

Because the difference quotient (5.8) converges to $V_n'(t)$, as $k\to\infty$, for all t, the last inequality implies (5.7).

There is a similar significant substitute for (5.4) if $k=2$. In order to formulate it we define a class of functions $f(t)$ which we denote by D_2: By $f(t)\in D_2$ we mean that the circle can be dissected into a finite number of consecutive open arcs in each of which $f(t)$ is continuous and convex, or concave, or linear. It is clear that $D_2\subset D_1$.

With each $f(t)\in D_2$ we associate an even non-negative integer $T(f)$, called the number of *turn-reversals* of $f(t)$ and defined as follows: Besides the $\Delta f\nu$ we consider the periodic sequence of second differences

$$\delta^2 f_\nu = f_{\nu+1} - 2f_\nu + f_{\nu-1}$$

and define $T(f)$ by

$$T(f) = \lim_{k\to\infty} v_c(\delta^2 f_\nu) = \sup_k v_c(\delta^2 f_\nu)\ . \tag{5.9}$$

Again a proof of the equality of the last two expressions requires the consideration of the points (finite in number) which have no neighborhood in which $f(t)$ is convex, or concave. If in addition to $f(t)\in D_2$ we assume that $f(t)\in C''$ then evidently

$$T(f) = v_c(f'')\ .$$

A substitute of (5.4) for $k=2$ is given by

THEOREM 4. *If $f(t)\in D_2$ then*

$$v_c(V_n'') = T(V_n) \leqq T(f)\ . \tag{5.10}$$

The proof is so very similar to the proof of Theorem 3 that it suffices to indicate the main points. In place of (5.8) we now start from the second order difference quotient

$$\begin{aligned}\delta^2 V_{n,k}/h^2 &= (V_{n,k}(t+h) - 2V_{n,k}(t) + V_{n,k}(t-h))/h^2 \\ &= \frac{1}{2\pi h}\sum_\nu \omega_n(t-\nu h)\delta^2 f_\nu\ ,\end{aligned}$$

and observe that on the one hand it converges to $V_n''(t)$, on the other hand by Lemma 3 and (5.9)

$$v_c(\delta^2 V_{n,k}/h^2) \leqq T(f)\ .$$

This last inequality implies (5.10) on letting $k\to\infty$.

The following remarks concerning the simplest elements of D_1 and D_2 are called for: 1. If $f(t)=$const. then clearly $S(f)=0$ and $T(f)=0$. Conversely, either of these relations is easily seen to imply that $f(t)=$ const. 2. The first non-trivial case is

$$S(f)=2 . \tag{5.11}$$

Functions $f(t)$ satisfying (5.11) are in a way the simplest non-constant periodic functions and may aptly be called *periodically monotone*. Likewise functions with

$$T(f)=2 \tag{5.12}$$

may be called *periodically convex*.

It is easily shown that (5.12) implies (5.11). That these new terms are appropriate is also shown by the following two statements.

1. If the periodic function $f(t)$ is monotone (non-constant) in $-\pi<t<\pi$ then $S(f)=2$, that is, $f(t)$ is *periodically monotone*.

2. If the periodic function $f(t)$ is convex or concave (non-constant) in $-\pi<t<\pi$ then $T(f)=2$, that is $f(t)$ is *periodically convex*.

Observe that the distinction between "increasing" and "decreasing" as well as between "convex" and "concave", drops out for periodic functions.

We conclude our short excursus into "descriptive function theory" with a few examples:

$$S(\sin t)=T(\sin t)=2 .$$

$$S(|\sin t|)=T(|\sin t|)=4 .$$

If $f(t)=\sin t+1$ in $(-\pi, 0)$ and $f(t)=\sin t$ in $(0, \pi)$ then

$$S(f)=T(f)=6 .$$

If $f(t)=\sin t+t$ in $0<t<2\pi$, then

$$S(f)=2, \ T(f)=4 .$$

From these examples we see that

$$S(f)\leqq T(f) \tag{5.13}$$

and this inequality is generally true. We see this if we observe that for a periodic sequence (5.5) we always have

$$v_c(f_\nu)\leqq v_c(\Delta f_\nu)\leqq v_c(\delta^2 f_\nu) .$$

In view of (5.6), (5.9) and the corresponding relations

$$v_c(f)=\lim_{k\to\infty} v_c(f_\nu)=\max_k v_c(f_\nu) ,$$

we conclude that

$$v_c(f)\leqq S(f)\leqq T(f) . \tag{5.14}$$

It is of some interest to show that the remarkable properties of the third Cesàro means established by L. Fejér in his Theorems 1, 2 and 3 [1, p. 82 and p. 86] are also enjoyed by the de la Vallée Poussin means $V_n(t)$. Thus Fejér's work suggests the following

THEOREM 5. *If $f(t)$ is an odd periodic function which is positive and concave in the range $0<t<\pi$, then*

$$0< V_n(t)\leqq f(t) \text{ if } 0<t<\pi \quad (n\geqq 1) . \tag{5.15}$$

Moreover, the function $V_n(t)$ is also concave in $0<t<\pi$.

The last statement and the first inequality (5.15) are easily proved. Indeed, it is clear that

$$v_c(f)=S(f)=T(f)=2 . \tag{5.16}$$

Observe also that $V_n(t)\not\equiv 0$ if $n\geqq 1$, for $V_n(t)\equiv 0$ would imply $S_n(t)\equiv 0$, hence also $v_c(f)\geqq 2n+2\geqq 4$ (by Corollary 2) which contradicts (5.16). By Theorem 1 and (5.16) surely

$$Z_c(V_n)=2 . \tag{5.17}$$

Since $V_n(t)$ is a sine polynomial it vanishes at 0 and π. By (5.17) these zeros are simple and the only zeros of $V_n(t)$. Also by (5.16) and Theorems 3 and 4 we conclude that

$$v_c(V'_n)=v_c(V''_n)=2 .$$

These remarks show that $V_n(t)$ or perhaps $-V_n(t)$ enjoy the properties to be established. That $V_n(t)$, rather than $-V_n(t)$, has these properties is shown by observing that

$$V'_n(0)=c\int_{-\pi}^{\pi}\left(\cos\frac{\tau}{2}\right)^{2n-1}\sin\frac{\tau}{2}f(\tau)d\tau , \qquad c>0,$$

(obtained from (7) by differentiation) has a positive integrand and is therefore positive.

To establish the second inequality (5.15) or

$$V_n(t)\leqq f(t) \qquad 0<t<\pi, \tag{5.18}$$

is a little more troublesome and we resort to Fejér's own method. We consider the "roof-function"

$$\hat{f}(t)=\begin{cases} \dfrac{b}{a}t & \text{if } 0\leqq t\leqq a \\ b\dfrac{\pi-t}{\pi-a} & \text{if } a\leqq t\leqq \pi \end{cases} \qquad 0<a<\pi,\ b>0, \tag{5.19}$$

and denote again by $\hat{f}(t)$ its odd periodic extension. We now observe that indeed

$$V_n(t) < \hat{f}(t)\,, \qquad 0<t<\pi, \tag{5.20}$$

for these special functions. Since we already know from our previous discussion that $\hat{V}_n(t)$ is positive and concave in $(0, \pi)$, the inequality (5.20) is perfectly clear as soon as we can prove that

$$\hat{V}_n'(0) < \hat{f}'(0)\,, \quad \hat{V}'(\pi) > \hat{f}'(\pi)\,, \tag{5.21}$$

These inequalities, however, follow immediately from previous remarks. Since $\hat{f}(t)$ is continuous, $\hat{V}_n'(t)$ is the V-mean of $\hat{f}'(t)$. Since $\hat{f}'(0)=\sup \hat{f}'(t)$, $\hat{f}'(\pi)=\inf \hat{f}'(t)$, we conclude, for instance from (5.2), that

$$\hat{f}'(\pi) < \hat{V}_n'(t) < \hat{f}'(0) \text{ for all } t\,.$$

The proof of the general inequality (5.18) now follows from the observation that the function $f(t)$ of Theorem 5 may be approximated by appropriate linear combinations of roof-functions with *positive* coefficients.

6. Convex, and star-shaped, conformal maps of the circle. The following introductory remark (previously made by one of us; see [10, pp. 226–227]) applies to any variation diminishing kernel $\Omega(t)$ as defined by the relations (3), (4) and (5) of our Introduction.

Let

$$f(t) = f_1(t) + i f_2(t) \qquad (f_1,\ f_2 \text{ real-valued}) \tag{6.1}$$

be a complex-valued continuous function of period 2π and let

$$g(t) = \frac{1}{2\pi}\int_0^{2\pi} \Omega(t-\tau) f(\tau)\, d\tau \tag{6.2}$$

be its transform; $g(t)$ is evidently also complex-valued periodic and we may write

$$g(t) = g_1(t) + i g_2(t)\,, \qquad (g_1,\ g_2 \text{ real-valued}). \tag{6.3}$$

Since Ω is real and (3) holds it follows that the transforms of $f_1(t)$, $f_2(t)$ and 1 are $g_1(t)$, $g_2(t)$ and 1, respectively. If A, B, C are arbitrary real constants it follows that $Ag_1(t)+Bg_2(t)+C$ is the transform of $Af_1(t)+Bf_2(t)+C$. Since $\Omega(t)$ is assumed to be a variation diminishing kernel, we conclude by (5) that the inequality

$$v_c(Ag_1(t)+Bg_2(t)+C) \leqq v_c(Af_1(t)+Bf_2(t)+C) \tag{6.4}$$

always holds.

The inequality (6.4) admits a remarkable geometric interpretation. Indeed, let us denote by $\{f\}$ the closed curve traced out by $f(t)$ in the complex plane of the variable $z=x+iy$ as t varies in the range $[0, 2\pi]$, and let $\{g\}$ be the corresponding curve described by $g(t)$. Let the following statement, too simple to be called a theorem, be referred to as a

PRINCIPLE. *The curve $\{g\}$ never crosses a straight line more often than the curve $\{f\}$ does.*

For if $Ax+By+C=0$ is the equation of a line L then the two members of the inequality (6.4) are identical with the total numbers of crossings of L by $\{g\}$ and $\{f\}$, respectively. In particular we have the

COROLLARY 4. *If the curve $\{f\}$ is convex then $\{g\}$ is interior to $\{f\}$ and $\{g\}$ is also convex.*

Indeed, $\{f\}$ being convex, it crosses any L at most twice, hence also $\{g\}$ crosses any L at most twice and is therefore convex. That $\{g\}$ has no points outside of $\{f\}$ follows already from the properties

$$\Omega(t)\geqq 0, \frac{1}{2\pi}\int_0^{2\pi}\Omega(t)dt=1 , \tag{6.5}$$

and in no way requires the sophisticated condition that $\Omega(t)$ be variation diminishing. On the other hand the conditions (6.5) are by themselves insufficient to enforce the convexity of $\{g\}$. It is also true, however, that the variation diminishing property of $\Omega(t)$ is sufficient but far from necessary for $\{g\}$ to be convex. As an example we mention the periodic kernel

$$\Omega(t)=\begin{cases}\pi/h & \text{if } -h\leqq t\leqq h\\ 0 & \text{if } -\pi\leqq t\leqq h \text{ or } h\leqq t\leqq\pi\end{cases} \quad (0<h<\pi) , \tag{6.6}$$

which is readily shown to have the "convexity preserving" property of Corollary 4. However, (6.6) is not variation diminishing because it is not periodic totally positive (see [4]).

We now turn to an application of these remarks to conformal maps of the circle, in particular to a proof of Theorem 2 of the Introduction.

Let

$$F(z)=z+c_2z^2+c_3z^3+\cdots \tag{6.7}$$

be regular in the unit circle. For a fixed value of r we consider the complex-valued periodic function

$$f(t\,;\,r)=F(re^{it})=re^{it}+c_2r^2e^{2it}+\cdots , \qquad 0\leqq r\leqq 1. \tag{6.8}$$

By (6.8) and (9) its V-means are

$$V_n(t;r)=\frac{1}{2\pi}\int_0^{2\pi}\omega_n(t-\tau)F(re^{i\tau})d\tau$$

$$=\frac{1}{\binom{2n}{n}}\left\{\binom{2n}{n+1}re^{it}+\binom{2n}{n+2}c_2r^2e^{2it}+\cdots+c_nr^ne^{nit}\right\}$$

or

$$V_n(t;\ r)=\boldsymbol{V}_n(re^{it})\ , \tag{6.9}$$

where $\boldsymbol{V}_n(z)$ are the de la Vallée Poussin means of the power series as defined by (15), with $c_1=1$. We also record the more explicit expression

$$\boldsymbol{V}_n(z)=\frac{n}{n+1}z+\frac{n(n-1)}{(n+1)(n+2)}c_2z^2+\cdots+\frac{n(n-1)\cdots 1}{(n+1)(n+2)\cdots(2n)}c_nz^n\ . \tag{6.10}$$

Our Theorem 2 seems now almost self-evident. Indeed, if $F(z)\in K$ then the curve $\{\boldsymbol{V}_n(re^{it})\}$ is convex by (6.8), (6.9) and Corollary 4. This being true for every $r<1$, we conclude that $\boldsymbol{V}_n(z)\in K$. Conversely, if $\boldsymbol{V}_n(z)\in K$ for every n, then $\{\boldsymbol{V}_n(re^{it})\}$ is a convex curve for all n and all $r<1$. From the relation

$$\lim_{n\to\infty}\boldsymbol{V}_n(re^{it})=F(re^{it})$$

we conclude that also $\{F(re^{it})\}$ is convex. Hence $F(z)\in K$.

REMARK 1. In order to conclude from (17) that $F(z)\in K$ it is not necessary to assume that the power series (6.7) converges in the unit circle or that it converges at all. Rather the converse part of Theorem 2 holds for a *formal* power series (6.7). For it is known (see e.g. [9, vol. II, p. 29]) that the assumptions (17) imply that all coefficients of the polynomial (6.10) are bounded in absolute value by $n/(n+1)$. Letting $n\to\infty$ we obtain $|c_\nu|\leqq 1$ $(\nu=1, 2, \cdots)$ which clearly imply the convergence of (6.7) within the unit circle.

REMARK 2. Let $F(z)\in K$ and hence $\boldsymbol{V}_n(z)\in K$. Let D and D_n denote the convex domains into which the unit circle is mapped by $F(z)$ and $\boldsymbol{V}_n(z)$, respectively. We know by Corollary 4 that

$$D_n\subset \mathrm{D}\ . \tag{6.11}$$

At this point it is natural to suspect that more is true, namely that all the inclusions

$$D_1 \subset D_2 \subset \cdots \subset D_n \subset D_{n+1} \subset \cdots \tag{6.12}$$

are valid, but we are unable to prove or disprove this.

REMARK 3. Since numerous elements of the class K are explicitly known, Theorem 2 is a ready source of polynomials belonging to K. Thus

$$F(z) = \frac{z}{1-z} = z + z^2 + \cdots \tag{6.13}$$

is in K because it maps the unit circle onto the half-plane $\Re z > -\frac{1}{2}$. The corresponding V-means

$$V_n(z) = \frac{1}{\binom{2n}{n}} \left\{ \binom{2n}{n+1} z + \binom{2n}{n+2} z^2 + \cdots + z^n \right\} \tag{6.14}$$

are a remarkable sequence of polynomials some extremal properties of which might be discussed on another occasion. Of course (6.11) holds. Here the convex boundary of D_n touches the line $\Re z = -\frac{1}{2}$ to an order of contact which increases with n. Also the inclusions (6.12) can be verified in this special case.

REMARK 4. Observe that the image D_1 of the unit circle by $V_1(z) = \frac{1}{2} z$ is the circle

$$D_1 : \ |z| < \frac{1}{2} \, . \tag{6.15}$$

By (6.11) we have $D_1 \subset D$ for every $F(z) \in K$. This proves the following proposition: *The circle* (6.15) *is covered by every convex map* D *and* (6.15) *is the largest circle with this property.* That D_1 is the largest circle is shown by the special function (6.13). This theorem is due to Study, [**11**, p. 116], and our proof is really identical with Nehari's proof in [**6**, pp. 223–224].

REMARK 5. A comparison of Theorem 2 with Fejér's Theorem IV [**1**, p. 87] again shows the extent to which the de la Vallée Poussin means of a power series are superior to its third Cesàro means as far as shape-preserving properties are concerned.

REMARK 6. In § 3 we have seen that from a knowledge of the section (3.2) of the Fourier series (3.1) of $f(t)$ we can infer the information (3.3) concerning the zeros of $f(t)$. Is there a similar result for power series? Specifically, let

$$F(z)=\sum_0^\infty c_\nu z^\nu \quad (c_0=1)$$

converge for $|z|<1$ and let, for a certain value of n,

$$V_n(z)=\frac{1}{\binom{2n}{n}}\sum_{\nu=0}^{n}\binom{2n}{n+\nu}c_\nu z$$

be given and known to have a certain number of zeros within the unit circle. Can we then draw any positive conclusion concerning the existence of zeros of $f(z)$ in the unit circle?

That the answer is negative is very simply shown as follows. With the given $c_0=1$, $c_1, \cdots, c_n$ derive the expansion

$$\log(1+c_1z+\cdots+c_nz^n)=b_1z+\cdots+b_nz^n+\cdots.$$

But then

$$F(z)=e^{b_1z+\cdots+b_nz^n}=1+c_1z+\cdots+c_nz^n+\cdots$$

is a *zero-free* entire function whose nth V-mean is precisely the given $V_n(z)$.

In concluding this section we wish to point out similar applications concerning the class Σ of power series $\sum_1^\infty b_\nu z^\nu$ which map the unit circle onto a univalent domain which is *star-shaped* with respect to the origin. It is well known that the two classes K and Σ are related as follows:

LEMMA 5. $\sum_1^\infty a_\nu z^\nu \in \Sigma$ *if and only if*

$$\sum_1^\infty \frac{a_\nu}{\nu}z^\nu \in K.$$

But then Theorem 2 easily implies the following.

COROLLARY 5. *For* $F(z)\in\Sigma$ *it is necessary and sufficient that* $V_n(z)\in\Sigma$ *for* $n=1, 2, \cdots$.

APPENDIX I. THE BERNSTEIN POLYNOMIALS

7. The Bernstein construction is variation diminishing. The purpose of the present appendix is to furnish for functions $f(x)$ defined in a finite interval a theory analogous to that given in Parts I and II for periodic functions. It is remarkable that such a theory is provided by the classical Bernstein polynomials. Indeed, let $f(x)$ be defined in $[0, 1]$ and let

$$B_n(x)=\sum_0^n f\left(\frac{\nu}{n}\right)\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\,, \qquad n\geqq 1\,, \tag{7.1}$$

be the corresponding Bernstein polynomial (see [5]). Let $Z(B_n)$ denote the number of zeros of $B_n(x)$ in the open range $(0, 1)$. We now state the following

THEOREM 6. *Denoting by $v(f)$ the number of changes of sign of $f(x)$ in $[0, 1]$ we have the inequalities*

$$v(B_n)\leqq Z(B_n)\leqq v(f)\,. \tag{7.2}$$

This result, an analogue of Theorem 1, can be derived as a special case from a general theorem of S. Karlin [3]. It admits, however, a very simple direct proof. Indeed, with $z=x/(1-x)$ for $0<x<1$, we have

$$\frac{B_n(x)}{(1-x)^n}=\sum_{\nu=0}^n f\left(\frac{\nu}{n}\right)\binom{n}{\nu}z^\nu\,,$$

hence by Descartes' rule of signs

$$Z(B_n)=Z\left(\frac{B_n(x)}{(1-x)^n}\right)=\underset{(0,\infty)}{Z}\left(\sum_0^n f\left(\frac{\nu}{n}\right)\binom{n}{\nu}z^\nu\right)\leqq v\left(f\left(\frac{\nu}{n}\right)\right)\leqq v(f)\,.$$

8. The graphic behavior of the Bernstein polynomials. If we write $B_n(x)=B_n(x\,;\,f)$ to indicate the dependence on $f(x)$, it is known that

$$B_n(x\,;\,Ax+B)=Ax+B\,. \tag{8.1}$$

But then (7.1) implies that $B_n(x)-Ax-B$ is the Bernstein polynomial of $f(x)-Ax-B$. Now (7.2) implies the

COROLLARY 6. *If $Ax+B$ is an arbitrary linear function then*

$$Z(B_n(x)-Ax-B)\leqq v(f(x)-Ax-B)\,. \tag{8.2}$$

Intersecting the graphs of $f(x)$ and $B_n(x)$ by appropriate straight lines $y=Ax+B$, the inequality (8.2) furnishes a good deal of information concerning the shape of the graph of $B_n(x)$. Notice in particular the following

COROLLARY 7. *If $f(x)$ is convex in* $[0, 1]$, *possibly discontinuous at the endpoints, but not linear in* $[0, 1]$, *then*

1. $B_n(x)$ *is convex,*
2. $B_n(x)>f(x)$ if $0<x<1$,
3. $B_n(0)=f(0)$, $B_n(1)=f(1)$.

We may omit the simple proof based an Corollary 6.

Observe that the relation $B'_n(x\,;f)=B_n(x\,;\,f')$ is *not* valid. However a simple calculation shows that (7.1) implies

$$B'_n(x)=n\sum_0^{n-1}\binom{n-1}{\nu}\Delta f_\nu\, x^\nu(1-x)^{n-1-\nu}$$

and

$$B''_n(x)=n(n-1)\sum_0^{n-2}\binom{n-2}{\nu}\Delta^2 f_\nu\, x^\nu(1-x)^{n-2-\nu}$$

where we have set

$$f_\nu=f\left(\frac{\nu}{n}\right), \qquad (\nu=0, \cdots, n),$$

$$\Delta f_\nu=f_{\nu+1}-f_\nu,\quad \Delta^2 f_\nu=f_{\nu+2}-2f_{\nu+1}+f_\nu\,.$$

(See Natanson [5], p. 179, fifth line from the bottom). The Theorems 3, 4 and 5 have precise analogues as will now be shown with a minimum of details. The function classes D_1 and D_2 have analogues in the present situation and the numbers of sense-reversals and turn-reversals may again be defined by the relations

$$S(f)=\lim_{n\to\infty} v(\Delta f_\nu)=\sup_n v(\Delta f_\nu) \qquad f\in D_1,$$

$$T(f)=\lim_{n\to\infty} v(\Delta^2 f_\nu)=\sup_n v(\Delta^2 f_\nu) \qquad f\in D_2,$$

respectively.

As in the periodic case we obtain the following.

THEOREM 7. *If* $f(x)\in D_1$ *then*

$$S(B_n)=v(B'_n)\leqq S(f)\,.$$

If $f(x)\in D_2$ *then*

$$T(B_n)=v(B''_n)\leqq T(f)\,.$$

If $f(x)$ is odd about the point $x=\frac{1}{2}$, then $B_n(x)$ is found to share this property. As an analogue of Theorem 5 we have the following

THEOREM 8. *If $f(x)$ is odd about $x=\frac{1}{2}$, concave and non-negative in $\frac{1}{2}\leqq x\leqq 1$, positive in $\frac{1}{2}<x<1$, then also $B_n(x)$ is concave in $\left[\frac{1}{2}, 1\right]$ and*

$$0<B_n(x)<f(x) \text{ if } \frac{1}{2}<x<1 . \tag{8.3}$$

Indeed, let us first observe the following. Because of the invariance of linear functions expressed by (8.1), we may subtract from $f(x)$ the linear function whose graph is the chord joining the extreme points $(0, f(0))$ and $(1, f(1))$, without altering the assumptions on $f(x)$. Thus without loss of generality we may assume that $f(0)=f(1)=0$. From this point the proof is entirely similar to the proof of Theorem 5 in all details, including the use of the roof-functions. Finally notice that the equality is excluded in the second inequality (8.3). This is so because of the inequality (8.2) of Corollary 6; in the periodic case we only had the weaker analogue (5.2).

APPENDIX II. A CONJECTURE ON POWER SERIES MAPPING A CIRCLE ONTO A CONVEX DOMAIN

9. Sources and forms of the conjecture. As stated in the Introduction, a power series

$$a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n+\cdots=f(z) \tag{9.1}$$

is said to belong to the class K, if it converges in the circle $|z|<1$ and maps this circle onto a convex domain. We say that the infinite sequence of complex numbers $\lambda_1, \lambda_2, \lambda_3, \cdots \lambda_n, \cdots$ is a *convexity-preserving factor sequence* if the series $\lambda_1a_1z+\lambda_2a_2z^2+\lambda_3a_3z^3+\cdots$ necessarily belongs to K whenever (9.1) belongs to K. Let us apply such a factor sequence to the simplest power series belonging to K, to the geometric series

$$z+z^2+z^3+\cdots=\frac{z}{1-z} . \tag{9.2}$$

We obtain

$$\lambda_1z+\lambda_2z^2+\lambda_3z^3+\cdots+\lambda_nz^n+\cdots ; \tag{9.3}$$

if $\lambda_1, \lambda_2, \lambda_3, \cdots$ is a convexity-preserving factor sequence, the power series (9.3) must necessarily belong to K. We state the conjecture that this obvious necessary condition is also *sufficient*; that is, we formulate

CONJECTURE I. *If both power series*

$$a_1z+a_2z^2+a_3z^3+\cdots$$

$$b_1z+b_2z^2+b_3z^3+\cdots$$

belong to K, also

$$a_1b_1z+a_2b_2z^2+a_3b_3z^3+\cdots$$

belongs to K.[2]

In view of Lemma 5, the conjecture can be restated in other forms, equivalent to the first.

CONJECTURE II. *If the power series*

$$a_1z+a_2z^2+a_3z^3+\cdots$$

belongs to K and

$$b_1z+b_2z^2+b_3z^3+\cdots$$

belongs to Σ, then

$$a_1b_1z+a_2b_2z^2+a_3b_3z^3+\cdots$$

belongs to Σ.

CONJECTURE III. *If both power series*

$$a_1z+a_2z^2+a_3z^3+\cdots$$

$$b_1z+b_2z^2+b_3z^3+\cdots$$

belong to Σ, also

$$\frac{a_1b_1}{1}z+\frac{a_2b_2}{2}z^2+\frac{a_3b_3}{3}z^3+\cdots$$

belongs to Σ.

These three Conjectures I, II and III are completely equivalent, they stand and fall together. The third form brings out most clearly the relation to a conjecture that has been found, years ago and independently of each other, by two of our friends, Professor S. Mandelbrojt and Professor M. Schiffer, and which is published here with their permission :

[2] One of the "intuitive sources" of the conjecture is the feeling that (9.2) plays a "leading role" in K, that it "sets the fashion." Which one of the two authors of this paper is the author of the conjecture will be disclosed if and when the conjecture is proved.

CONJECTURE M. S. *If both power series*

$$a_1z+a_2z^2+a_3z^3+\cdots$$

$$b_1z+b_2z^2+b_3z^3+\cdots$$

are "schlicht" in the unit circle, also

$$\frac{a_1b_1}{1}z+\frac{a_2b_2}{2}z^2+\frac{a_3b_3}{3}z^3+\cdots$$

is "schlicht" in the unit circle.

Whereas III is equivalent to I or II, it appears logically independent of MS. As far as obvious conclusions from the statements go, III could be true but MS false, or MS true yet III false, or both could be true or both false. Still, the conjectures are obviously related and their joint consideration may lead to various suggestions.

The Conjectures I, II and III are more "elementary" than MS and they are certainly more accessible; we succeeded in treating several of their particular cases and consequences.

10. Verification of the conjecture in some particular cases. We shall exhibit several particular series $\sum b_nz^n$ belonging to K which, convoluted with an arbitrary series (9.1) belonging to K, generate a series $\sum a_nb_nz^n$ belonging to K.

(a) The polynomial (6.14) belongs to K. That its convolution with an arbitrary series belonging to K necessarily belongs to K is precisely what Theorem 2 asserts.

(b) If the series (9.1) belongs to K, it belongs, a fortiori, to Σ. Therefore, by Lemma 5, the series

$$\frac{a_1z}{1}+\frac{a_2z^2}{2}+\frac{a_3z^3}{3}+\cdots$$

belongs to K. This is another special case of Conjecture I; that the series

$$\frac{z}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\cdots=\log\frac{1}{1-z}$$

maps the unit circle onto a convex domain follows from its relation to (9.2) and from Lemma 5 but this fact can also be established directly (see [9, vol. 1, p. 106, problem 114]).

(c) The result mentioned under (a) (Theorem 2) is due to the fact that the V-means are variation diminishing; cf. §6. Any variation diminishing transformation on the circle leads to an analogous result, and so we obtain especially the following (cf. [4]). *Let* $g(z)$ *be the product of* $e^{-\gamma z^2}$, *where* $\gamma \geqq 0$, *with an entire function of genus* 1, *all coeffiicients and all zeros of which are reat ; then*

$$\sum_1^\infty \frac{z^n}{g(in)}$$

belongs to K, and, provided that (9.1) *belongs to K, also*

$$\sum_1^\infty \frac{a_n z^n}{g(in)}$$

belongs to K. The term "entire function of genus 1" is used here in the comprehensive sense, that is, it is supposed to include also entire functions of genus 0 and polynomials (but, obviously, not the identically vanishing polynomial); the case in which $g(z)$ reduces to z was mentioned under (b).

(d) Let p and q denote two different given points on the unit circle ($|p|=|q|=1$, $p \neq q$). Assume that (9.1) belongs to K and let z describe a circle concentric with, and interior to, the unit circle. Then $f(z)$ describes a convex curve of which $f(pz)-f(qz)$ represents a moving chord; as it is easy to see geometrically this chord turns all the time in the same sense. The argument of the complex number $f(pz)-f(qz)$ increases steadily. That is, the power series

$$\frac{f(pz)-f(qz)}{p-q}=\sum_1^\infty a_n \frac{p^n-q^n}{p-q} z^n$$

belongs to Σ (maps the unit circle onto a star-shaped domain) and so, by Lemma 5, the power series

$$\sum_{n=1}^\infty \frac{a_n}{n} \frac{p^n-q^n}{p-q} z^n \tag{10.1}$$

belongs to K (cf. [6]). The series (10.1) is the convolution of (9.1) and of that particular case of (10.1) in which $a_n=1$; this particular series maps $|z|<1$ onto an infinite strip bounded by two parallels.

11. Verification of some consequences. In the foregoing, we have dealt mainly with form I of the conjecture, but now we shall consider its form III. We assume, therefore, that the function (9.1) belongs to the class Σ, that is, it maps the circle $|z|<1$ onto a star-shaped domain. We shall say that (9.1) is normalized if

$$a_1=1\ . \tag{11.1}$$

(a) We are given an integer n, $n \geqq 2$. Let us consider the normalized functions of the class Σ and let us seek one for which $|a_n|$ is a maximum. We leave aside the (easy) discussion of the existence and assume that (9.1) is such a function with maximum $|a_n|$. Now we apply Conjecture III with $b_m = a_m$ for $m=1, 2, 3, \cdots$; the resulting series is again normalized and so its nth coefficient cannot have an absolute value exceeding the maximum; that is, $\frac{|a_n|^2}{n} \leqq |a_n|$, from which it follows that

$$|a_n| \leqq n .$$

For series of the class Σ this inequality is well known and easily established independently of the Conjecture III. And so our previous reasoning served only to enhance somewhat the plausibility of Conjecture III. Yet the same reasoning is also applicable to the Conjecture *MS* and reveals one of the essential sources of this Conjecture.

(b) The function $f(z)$ belongs to the class Σ if, and only if,

$$\frac{zf'(z)}{f(z)} = 1 + 2\alpha_1 z + 2\alpha_2 z^2 + 2\alpha_3 z^3 + \cdots \tag{11.2}$$

is regular in the circle $|z|<1$ and has there a positive real part. This will be the case if, and only if, the Hermitian form of the variables $z_0, z_1, \cdots, z_n$

$$\sum_{k=0}^{n} \sum_{l=0}^{n} \alpha_{k-l} z_k \bar{z}_l \tag{11.3}$$

($\alpha_{-\nu} = \bar{\alpha}_\nu$, by definition) is positive (definite or semidefinite) for $n=1, 2, 3, \cdots$. This well known important necessary and sufficient condition is due to Carathéodory and Toeplitz. It can also be expressed in terms of the determinants

$$A_n = \begin{vmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_{-1} & 1 & \alpha_1 & \cdots & \alpha_{n-1} \\ \alpha_{-2} & \alpha_{-1} & 1 & \cdots & \alpha_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{-n} & \alpha_{-n+1} & \alpha_{-n+2} & \cdots & 1 \end{vmatrix} . \tag{11.4}$$

Now (see (9.1)) the relation (11.2) can be written in the form

$$a_1 z + a_2 z^2 + a_3 z^3 + \cdots = a_1 z \exp\left(\frac{2\alpha_1 z}{1} + \frac{2\alpha_2 z^2}{2} + \frac{2\alpha_3 z^3}{3} + \cdots \right) \tag{11.5}$$

or in the form

$$(11.6)\qquad \frac{2\alpha_1 z}{1}+\frac{2\alpha_2 z^2}{2}+\frac{2\alpha_3 z^3}{3}+\cdots=\log\left(1+\frac{a_2 z}{a_1}+\frac{a_3 z^2}{a_1}+\cdots\right)$$

and so we can express both a_n/a_1 as a polynomial in the α and α_n as a polynomial in the a/a_1:

$$(11.7)\qquad \begin{aligned} \frac{a_2}{a_1}&=2\alpha_1\\ \frac{a_3}{a_1}&=\frac{(2\alpha_1)^2+2\alpha_2}{2\,!}\\ \frac{a_4}{a_1}&=\frac{(2\alpha_1)^3+2(2\alpha_3)+3(2\alpha_1)(2\alpha_2)}{3\,!}\\ &\cdots \end{aligned}$$

$$(11.8)\qquad \begin{aligned} 2\alpha_1&=\frac{a_2}{a_1}\\ 2\alpha_2&=-\frac{a_2^2-2a_1a_3}{a_1^2}\\ 2\alpha_3&=\frac{a_2^3-3a_1a_2a_3+3a_1^2a_4}{a_1^3}\\ &\cdots \end{aligned}$$

It would be easy to write down (11.7) or (11.8) for general n, but we shall not enter into details. Using (11.8) we could express the Hermitian form (11.3) and the determinant (11.4) in terms of the coefficients of the series (9.1) and doing so we wonld render more explicit the necessary and sufficient condition for the class Σ. Yet we postpone this consideration.

(c) Now consider, besides (9.1), two other power series with coefficients b_n and c_n respectively, and let β_n and $\boldsymbol{B}_n$ be so linked to the b, and γ_n and $\boldsymbol{C}_n$ so linked to the c, as α_n and $\boldsymbol{A}_n$ are to the a. Thus we have besides (11.5) (in all summations $n=1, 2, 3, \cdots$)

$$(11.9)\qquad \sum b_n z^n=b_1 z\exp\left(2\sum\frac{\beta_n z^n}{n}\right),\quad \sum c_n z^n=c_1 z\exp\left(2\sum\frac{\gamma_n z^n}{n}\right),$$

Set

$$(11.10)\qquad \frac{a_n b_n}{n}=c_n\,.$$

Now express a_n/a_1 in terms of the α from (11.7), and express analogously b_n/b_1 in terms of the β, then c_n/c_1 in terms of the α and β from (11.10) and finally from relations analogous to (11.8), express γ_n in terms of the c/c_1 and so in terms of the α and β. This leads to

$$\begin{aligned}
\gamma_1 &= \alpha_1\beta_1 \\
3\gamma_2 &= \alpha_2\beta_2 + 2\alpha_2\beta_1^2 \\
&\quad + 2\alpha_1^2\beta_2 - 2\alpha_1^2\beta_1^2 \\
6\gamma_3 &= \alpha_3\beta_3 + 3\alpha_3\beta_1\beta_2 + 2\alpha_3\beta_1^3 \\
&\quad + 3\alpha_1\alpha_2\beta_3 + 3\alpha_1\alpha_2\beta_1\beta_2 - 6\alpha_1\alpha_2\beta_1^3 \\
&\quad + 2\alpha_1^3\beta_3 - 6\alpha_1^3\beta_1\beta_2 + 4\alpha_1^3\beta_1^3 \\
&\cdots
\end{aligned} \tag{11.11}$$

Not all details of the general formula for γ_n are obvious; a few features will be discussed under (e). The determinant $\boldsymbol{C}_n$ (expressed in terms of the γ as $\boldsymbol{A}_n$ is in terms of the α, cf. (11.4)) becomes by virtue of (11.11) a polynomial in the $\alpha, \overline{\alpha}, \beta$ and $\overline{\beta}$. By the theory of Carathéodory and Toeplitz, Conjecture III is equivalent to the following.

CONJECTURE IV. *The* $2n$ *inequalities*

$$\boldsymbol{A}_1 > 0,\ \boldsymbol{A}_2 > 0,\ \cdots,\ \boldsymbol{A}_n > 0\ ,$$

$$\boldsymbol{B}_1 > 0,\ \boldsymbol{B}_2 > 0,\ \cdots,\ \boldsymbol{B}_n > 0\ ,$$

imply the n *inequalities*

$$\boldsymbol{C}_1 > 0,\ \boldsymbol{C}_2 > 0,\ \cdots,\ \boldsymbol{C}_n > 0$$

and this holds for $n = 1, 2, 3, \cdots$.

This formulation excludes the case of equality in all the $3n$ inequalities considered. This is due to the fact that, without loss of generality, we may suppose $\Sigma a_n z^n$ and $\Sigma b_n z^n$ regular in $|z| \leqq 1$.

(d) The case $n=1$ of Conjecture IV is trivial. In fact, if we assume that the series are normalized, see (11.1), and introduce the coefficients of the mapping functions, see (11.8), the statement that we have to prove reduces to this:

The inequalities $\qquad |a_2| < 2,\ |b_2| < 2$

imply $\qquad \dfrac{|a_2 b_2|}{2} < 2$

which is obvious.

(e) The case $n=2$ of Conjecture IV was first established by Dr. G. A. Hummel and can be proved as follows.

We take the series as normalized, see (11.1), and set

$$a_2 = a,\ a_3 = A,\ b_2 = b,\ b_3 = B\ ;$$

we suppose, without loss of generality, that $a\geqq 0, b\geqq 0$. We have to show:

The two inequalities

$$\left|A-\frac{3a^2}{4}\right|<1-\frac{a^2}{4}, \quad \left|B-\frac{3b^2}{4}\right|<1-\frac{b^2}{4} \tag{11.12}$$

imply

$$\left|\frac{AB}{3}-\frac{3a^2b^2}{16}\right|<1-\frac{a^2b^2}{16}. \tag{11.13}$$

(The first inequality (11.12) results from the condition $A_2>0$, see (11.4), by virtue of (11.8); it *implies* $a<2$, and so the condition $A_1>0$.)

Let

$$A=\frac{3a^2}{4}+u, \quad B=\frac{3b^2}{4}+v. \tag{11.14}$$

By the hypothesis (11.12) of the theorem that we are about to prove

$$|u|<1-\frac{a^2}{4}, \quad |v|<1-\frac{b^2}{4}. \tag{11.15}$$

We derive from (11.14) and (11.15)

$$AB=\frac{9a^2b^2}{16}+\frac{3a^2}{4}v+\frac{3b^2}{4}u+uv$$

$$\left|AB-\frac{9a^2b^2}{16}\right|<\frac{3a^2}{4}\left(1-\frac{b^2}{4}\right)+\frac{3b^2}{4}\left(1-\frac{a^2}{4}\right)+\left(1-\frac{a^2}{4}\right)\left(1-\frac{b^2}{4}\right). \tag{11.16}$$

We assert that

$$1+\frac{a^2}{2}+\frac{b^2}{2}-\frac{5a^2b^2}{16}<3-\frac{3a^2b^2}{16}; \tag{11.17}$$

in fact, this follows from $a<2, b<2$, since it is equivalent to

$$\left(1-\frac{a^2}{4}\right)\left(1-\frac{b^2}{4}\right)>0.$$

The right hand side of (11.16) is equal to the left hand side of (11.17), and so the combination of these two inequalities immediately yields the desired conclusion (11.13).

(e) We consider now the expression of γ_n in terms of the α and

β for general n; for the cases $n=1, 2, 3$, see (11.11). The procedure that led us to (11.11) shows that γ_n is a polynomial in $\alpha_1, \alpha_2, \cdots, \alpha_n$, $\beta_1, \beta_2, \cdots, \beta_n$ with rational coefficients. Obviously, by virtue of (11.10), γ_n is symmetric in the α and β. If we substitute ρz for z in (11.5) or, which is the same, we change α_n into $\rho^n\alpha_n$ and a_n into $\rho^n a_n$, there results a change, see again (11.10), of c_n into $\rho^n c_n$ and of γ_n into $\rho^n\gamma_n$; therefore, γ_n must be an isobaric polynomial in the α of weight n. Finally, γ_n must be of the form

(11.18) $$\gamma_n=\sum_{k=1}^{p}\sum_{l=1}^{p} j_{kl}^{(n)} A_k B_l ,$$

where

$p=p(n)$ is the number of partitions of the integer n,

$A_1, A_2, \cdots, A_p$ are the products of powers of weight n of $\alpha_1, \alpha_2, \cdots, \alpha_n$, ordered lexicographically so that

(11.19) $$A_1=\alpha_n,\ A_2=\alpha_{n-1}\alpha_1,\ \cdots\ A_p=\alpha_1^n .$$

Generally A_k is of the form

(11.20) $$A_k=\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n} ;$$

its weight $1k_1+2k_2+3k_3+\cdots+nk_n=n$.

$B_1, B_2, \cdots, B_p$ are analogously expressed in terms of $\beta_1, \beta_2, \cdots, \beta_n$, and $j_{ik}^{(n)}$ are rational numbers, $j_{ik}^{(n)}=j_{ki}^{(n)}$.

For example $p(4)=5$ and, for $n=4$

$$A_1=\alpha_4,\ A_2=\alpha_3\alpha_1,\ A_3=\alpha_2^2,\ A_4=\alpha_2\alpha_1^2,\ A_5=\alpha_1^4 ,$$

the B are analogously defined and the matrix of the $j_{ik}^{(4)}$ results from

9	24	9	36	12
24	24	24	−24	−48
9	24	−1	−4	−28
36	−24	−4	−136	128
12	−48	−28	128	−64

if each of the 25 numbers displayed is divided by 90.

We cannot exhibit the law of the dependence of $j_{ik}^{(n)}$ on n in some obviously useful manner, but we note here one property. If $\beta_n=1$ it is easily seen from (11.9) that $b_n/b_1=n$ and, therefore, by (11.10) $c_n/c_1=a_n/a_1$ and so finally

$$\gamma_n=\alpha_n$$

for any choice of the α_n; this must be compatible with (11.18) and so, since $B_1=B_2=\cdots=B_p=1$, by our choice of the β,

$$(11.21)\qquad \sum_{l=1}^{p} j_{kl}^{(n)} = \begin{cases} 1 & \text{for } k=1 \\ 0 & \text{for } k=2, 3, \cdots, p\,. \end{cases}$$

(f) The system of n complex numbers $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, for which we shall also use the more concise notation (α), determines a point in $2n$-dimensional Euclidean space. A point (α) belongs to the *coefficient-domain* if, and only if, it corresponds by virtue of (11.5) to the initial terms of a power series of the class Σ. The most remarkable boundary point of the coefficient domain is the "Koebe-point" which corresponds to the function

$$z+2z^2+3z^3+\cdots=z(1-z)^{-2}\,.$$

Our aim is to show that, for any given n, Conjecture IV is true for two interior points of the coefficient domain which are sufficiently close to the Koebe-point.

Let us choose two arbitrary points (u) and (v) in the interior or the coefficient domain. That is, (cf. under (b)) both Hermitian forms

$$(11.22)\qquad \sum\sum u_{k-l} z_k \bar{z}_l\,,\quad \sum\sum v_{k-l} z_k \bar{z}_l$$

are positive definite. Let α, β and ε denote positive numbers; α and β are arbitrary and ε so small that $\alpha\varepsilon<1$, $\beta\varepsilon<1$. The coefficient domain is convex. Therefore, if we set

$$(11.23)\qquad \alpha_\nu=(1-\varepsilon\alpha)+\varepsilon\alpha u_\nu\,,\quad \beta_\nu=(1-\varepsilon\beta)+\varepsilon\beta v_\nu$$

for $\nu=0, \pm1, \pm2, \cdots, \pm n$, the points (α) and (β) are in the interior of the coefficient domain. If A_k is given by (11.20)

$$A_k=1+\varepsilon\alpha\tilde{u}_k+O(\varepsilon^2)$$

where

$$\tilde{u}_k=k_1(u_1-1)+k_2(u_2-1)+\cdots+k_n(u_n-1)$$

and $O(\varepsilon^2)$ denotes a quantity of order not exceeding ε^2 when ε tends to 0. There is a similar expression for B_l and finally, by (11.21),

$$\begin{aligned}(11.24)\qquad \gamma_n&=\sum_{k=1}^{p}\sum_{l=1}^{p} j_{kl}^{(n)} A_k B_l\\ &=\sum_{k=1}^{p}\sum_{l=1}^{p} j_{kl}^{(n)}[1+\varepsilon\alpha\tilde{u}_k+\varepsilon\beta\tilde{v}_l]+O(\varepsilon^2)\\ &=1+\varepsilon\alpha\tilde{u}_1+\varepsilon\beta\tilde{v}_1+O(\varepsilon^2)\\ &=1+\varepsilon\alpha(u_n-1)+\varepsilon\beta(v_n-1)+O(\varepsilon^2)\,.\end{aligned}$$

By virtue of (11.24)

$$\begin{aligned}\sum\sum \gamma_{k-l} z_k \bar{z}_l &= (1-\varepsilon\alpha-\varepsilon\beta)|z_0+z_1+\cdots+z_n|^2 \\ &+\varepsilon\alpha\sum\sum u_{k-l} z_k\bar{z}_l \\ &+\varepsilon\beta\sum\sum v_{k-l} z_k \bar{z}_l + O(\varepsilon^2)\end{aligned}$$

and this Hermitian form is definite positive for sufficiently small ε, since the forms (11.22) are definite positive. With this, we have proved another infinitesimal part of Conjecture IV.

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ASYMMETRY OF A PLANE CONVEX SET WITH RESPECT TO ITS CENTROID

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A. S. Besicovitch [1] proved that every bounded plane convex set K has a central subset of area at least $2m(K)/3$ where $m(K)$ denotes the area of K. His method is to construct a semi-regular hexagon of center N whose vertices belong to the boundary of K.

Ellen F. Buck and R. C. Buck [2] showed that for every K there exists at least one point X, called a six-partite point, such that there are three straight lines through X dividing K into six subsets each of area $m(K)/6$. H. G. Eggleston [3] showed that any six-partite point of K is the center of a semi-regular hexagon of area $2m(K)/3$ contained in K.

I. Fáry and L. Rédei [4] and S. Stein [5] defined for each point P the subset $S(P)$ of K determined by the intersection of K with its radial reflection in P and considered the function $f(P)=m(S(P))/m(K)$. By use of the Brunn-Minkowski theorem these authors showed that if a is a real number, then the set of points at which $f(P)\geqq a$ is convex; and the maximum f^* of $f(P)$ is attained at a single point. (Moreover, these results apply to an n-dimensional bounded convex set in n-dimensional Euclidean space.) Note that these conclusions may be false if the set K is not convex: for example, consider an L-shaped region formed by deleting one quarter of a square.

The results of Besicovitch and Eggleston imply $f(N)\geqq 2/3$ and $f(X)\geqq 2/3$, hence $f^*\geqq 2/3$.

We obtain the following theorem.

THEOREM. *If G is the centroid of K, then $f(G)\geqq 2/3$.*

To see that this result is not included in the theorems previously mentioned, consider the isosceles trapezoid with vertices $(-4, 0)$, $(4, 0)$, $(2, 2)$, $(-2, 2)$. For this example there is only one point N: $(0, 1)$ and only one point X: $(0, 4-4\sqrt{.6})$ and the closure of these points does not include G: $(0, 8/9)$.

Proof of the theorem. If K has central symmetry, then $f(G)=1$. In any case $S(G)$ has central symmetry about G; hence if K does not have central symmetry, the part M of K outside $S(G)$ has G at its centroid. Then as in Figure 1 let T be any maximal connected subset of M with

Received October 10, 1957.

A and B as terminal points of the boundary curve common to K and T. Let P' denote the reflection of a point P in G. Note that the congruent triangles AGB and $A'GB'$ are contained in $S(G)$.

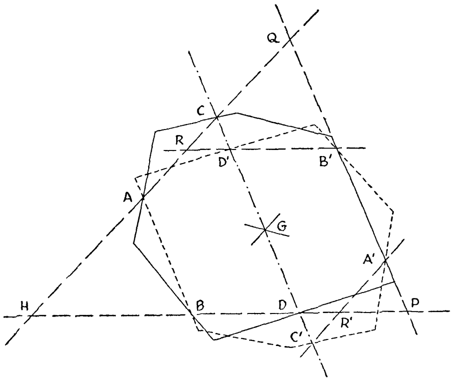

Fig. 1.

If for *every* T the area $m(T)$ is less than or equal to the area Δ of the corresponding triangle AGB, then $m(S(G)) \geqq 2m(M)$. Since

$$m(K) = m(M) + m(S(G)) \leqq 3m(S(G))/2,$$

it follows that $f(G) \geqq 2/3$.

In the contrary case, if we assume for *any* T that $m(T) > \Delta$, we can arrive at a contradiction of the fact that G is the centroid of M.

Let line L through G parallel to AB cut the boundary of K in points C and D. To fix ideas suppose in length $CG > GD$. Let lines BD and AC meet at H and intersect line $A'B'$ in P and Q, respectively. Let AC and $B'D'$ meet at R; then BD and $A'C'$ meet at R'; and R is on the side of L toward T.

Considerations of convexity imply that on the side of L away from T the *maximum* possible moment of M with respect to L is $u + w_2$ where u is the moment of triangle $R'A'P$ and w_2 is the moment of trapezoid $CQB'D'$. On the other side the *minimum* possible moment of M with respect to L is $w_1 + v$ where w_1 is the moment of triangle RCD' and v is the moment of a trapezoid of area $m(T)$ inscribed in triangle ABH and having AB as one base.

We will show that if $m(T) > \Delta$, then $w_1 + v > u + w_2$, in contradiction to G being the centroid of M. It will suffice to show $v > u + w$ where $w = w_2 - w_1$ is the moment of triangle RQB'.

Let $a = AB$, let d be the distance from G to AB and let h be the distance from H to AB. Let $a_1 = A'P$ and $a_2 = QB'$. From similar triangles $(a_1 + a_2 + a)/a = (2d + h)/h$, so that $a_1 + a_2 = 2ad/h$. The combined moments of triangles $R'A'P$ and RQB' are equivalent to those of a single triangle of base $a_1 + a_2$ and altitude d with centroid at a distance $2d/3$ from L, hence $u + w = 2ad^3/3h$.

Let c be the altitude of a trapezoid Z of area Δ inscribed in triangle ABH and having AB as one base. A direct computation shows the moment v' of Z with respect to L to be

$$v' = \frac{ad}{2}\left(d + \frac{c(3h - 2c)}{3(2h - c)}\right).$$

Since $m(T) > \Delta$ implies $v > v'$ the inequality $v > u + w$ will hold if

$v'>u+w$. Since $m(T)>\Delta$ also implies $h>d>c$, the inequality $v'>u+w$ reduces to

$$(6hd-3cd+3ch-2c^2)h>4d^2(2h-c) .$$

Comparison of the areas of Z and triangle ABG shows $c^2=2ch-hd$. Then the previous inequality may be rearranged and factored to obtain the equivalent inequality

$$8hd(h-d)>c(h+4d)(h-d)$$

whose truth follows readily from $h>d>c$.

The case that length $CG=GD$ may be treated in the same manner (even if BD and AC are parallel). This completes the proof of the theorem.

We do not see how to extend the theorem about $f(G)$ to higher dimensions. Possibly the lower limit for $f(G)$ for the general bounded convex set is the same as $f(G)$ for a simplex of corresponding dimension. The value of the latter is given in [4] (but incorrectly given in Theorem 6 of [5], an error for which Professor Stein wishes this note to serve in lieu of a formal corrigendum).

Note that for as simple an example as a trapezoid $f^*>f(G)$. Some necessary conditions for determining P such that $f(P)=f^*$ have been given in [6].

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LOWER BOUNDS FOR HIGHER EIGENVALUES BY FINITE DIFFERENCE METHODS

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1. Introduction. This paper gives lower bounds for all the eigenvalues of an arbitrary second order self-adjoint elliptic differential operator on a bounded domain R with zero boundary conditions in terms of the eigenvalues of an associated finite difference problem. When R is sufficiently smooth, the lower bounds converge to the eigenvalues themselves as the mesh size approaches zero. A certain class of self-abjoint systems of elliptic differential equations containing no mixed derivatives is also treated.

Upper bounds for the eigenvalues of a differential operator can always be found by the Rayleigh-Ritz method. That is, one puts piecewise differentiable functions vanishing on the boundary into the Poincaré inequality [14]. It was pointed out by Courant [2] that in the case of second order operators one can reduce the problem of upper bounds to a finite difference eigenvalue problem by using piecewise linear functions (see § 6).

Lower bounds are more difficult to find. The only known method giving arbitrarily close lower bounds for the eigenvalues is that of A. Weinstein [20], which is usually quite difficult to apply. It was shown by G. E. Forsythe [5, 6, 7] that if the eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \cdots$ of the two-dimensional problem

$$\Delta u + \lambda u = 0 \quad \text{in } R \tag{1.1}$$

with $u=0$ on the boundary are approximated by the eigenvalues $\lambda_1^{(h)} \leqq \lambda_2^{(h)} \leqq \cdots$ of a certain finite difference problem on a mesh of size h, then there exist constants $\gamma^{(1)}_{,}\gamma^{(2)}_{,}\cdots$ such that

$$\lambda_k^{(h)} \leqq \lambda_k - \gamma^{(k)} h^2 + o(h^2) \, . \tag{1.2}$$

The $\gamma^{(k)}$ cannot be computed, but are positive for convex R. However, the $o(h^2)$ term is completely unknown, so that this asymptotic formula cannot be used to bound λ_k below.

It was shown independently by J. Hersch [8] and the author [18, 19] that if λ_1 is the lowest eigenvalue of (1.1) and if $\lambda_1^{(h)}$ is the lowest eigenvalue of a finite difference problem on a mesh that is slightly arger than R, then $\lambda_1^{(h)}$ and, in fact, a quantity slightly larger than $\lambda_1^{(h)}$ are lower bounds for λ_1.

Received September 17, 1957. Sponsored by the United States Army under Contract No. DA-11-022-ORD-2059.

This result is here extended to higher eigenvalues, higher dimensions, and variable coefficients by a modification of the method previously used by the author. The basic idea is to define a mesh function by an average over mesh squares of a linear combination of the first k eigenfunctions of (1.1). One then defines the finite difference eigenvalue problem in such a way that its Rayleigh quotient evaluated for this mesh function can be estimated in terms of the unknown eigenvalue λ_k. By the Poincaré inequality this leads to an upper bound for the eigenvalue $\lambda_k^{(h)}$ in terms of λ_k, which serves as a lower bound for λ_k in terms of $\lambda_k^{(h)}$.[1]

For the sake of clarity, the method is first presented for the problem (1.1) in § 2. It must be noted that while the lower bound (2.25) holds for all λ_k, it is not as good for λ_1 as the bound previously given either by Hersch [8] or the author [19]. It is smaller, rather than larger, than $\lambda_1^{(h)}$ by a term of order h^2.

The method extends easily to an equation in N dimensions with variable coefficients when the operator contains no mixed derivatives. This extension is made in § 3. Again the lower bound is smaller than $\lambda_k^{(h)}$ by a term of order h^2.

In § 4 the general second order self-adjoint operator is considered. The presence of mixed derivatives introduces complications. The lower bound becomes $\lambda_k^{(h)}$ reduced by a term of order $h^{1/2}$. Furthermore, it becomes necssary to assume that R has no re-entrant cusps, corners, or edges, and that it does not have infinite oscillations.

Section 5 presents an extension of the lower bound to a self-adjoint system of second order equations with no mixed derivatives. The extension to a system with mixed derivatives appears to be very difficult, and is not done.

In § 6 the difference between upper and lower bound is discussed. It is estimated explicitly for convex R. At the same time this discussion serves to show when the lower bounds converge to the eigenvalues.

In § 7 we take account of the fact that the solution vanishing on the boundary of a non-homogeneous differential equation can be characterized by a minimum principle (Dirichlet's principle). Using the methods developed for eigenvalues, we give a method for finding a lower bound for this minimum. It is, of course, true that in this case one can get a get a lower bound by Thomson's principle. However, this principle involves solutions of the differential equation which may be difficult to find as well as difficult to compute with. Finite difference methods are more amenable to high speed computation. The upper and lower bounds so obtained, together with the function that gives the upper bound, can be used to find upper and lower bounds for the solution at an

[1] A similar idea was used by L. Collatz [1] to establish the order of magnitude of $|\lambda_k^{(h)}-\lambda_k|$.

interior point by the method of Diaz and Greenberg [3, 4].

Section 8 indicates the extension of our method to an important class of higher order operators. This extension is applied to the problem of the vibrating clamped plate.

2. The basic bound.

Let the eigenvalues of

$$\Delta u + \lambda u = 0 \quad \text{in } R, \qquad u = 0 \quad \text{on the boundary } \dot{R} \tag{2.1}$$

be denoted by

$$\lambda_1 \leqq \lambda_2 \leqq \cdots . \tag{2.2}$$

Let the corresponding eigenfunctions, normalized so that

$$\int_R u^2\, dx\, dy = 1 \tag{2.3}$$

be denoted by $u_1, u_2, \cdots$.

Consider the x-y plane divided into squares by lines $x=mh$, $y=nh$, $m, n=0, \pm 1, \pm 2 \cdots$. Let R_h be a region consisting of a union of entire squares of this grid and having the property of containing not only R, but also all its left and downward translates of distances up to h:[2]

$$R_h \supset \{(x, y) | (x+\alpha, y+\beta) \in R \text{ for some } 0 \leqq \alpha \leqq h,\ 0 \leqq \beta \leqq h\} . \tag{2.4}$$

We consider the class M_h of functions $v(mh, nh)$ defined at mesh points (mh, nh) in R_h and vanishing at boundary points of R_h. The eigenvalues (2.2) are to be approximated by the eigenvalues

$$\lambda_1^{(h)} \leqq \lambda_2^{(h)} \leqq \cdots \tag{2.5}$$

of the finite difference problem

$$\Delta_h v + \lambda^{(h)} v = 0 \tag{2.6}$$

where v is a mesh function of the class M_h, and

$$\Delta_h v = h^{-2}[v(mh+h, nh) + v(mh-h, nh) + v(mh, nh+h) + v(mh, nh-h) - 4v(mh, nh)] . \tag{2.7}$$

The eigenvalues (2.5) are bounded above by the Poincaré (Rayleigh-Ritz) inequality [14], which states that for $v_1, v_2, \cdots, v_k$ of class M_h and linearly independent

[2] Equivalently, if the intersection of R and the square $mh<x<(m+1)h$, $nh<y<(n+1)h$ is non-empty, then (mh, nh) is an interior point of R_h.

$$\lambda_k^{(h)} \leqq \max_{\xi_1,\cdots,\xi_k} \frac{D_h(\xi_1 v_1+\cdots+\xi_k v_k)}{h^2 \sum_{(mh,nh)\in R_h} (\xi_1 v_1+\cdots+\xi_k v_k)^2}\,, \tag{2.8}$$

where

$$D_h(v) \equiv \sum_{(mh,nh)\in R_h} \{[v(mh+h,nh)-v(mh,nh)]^2+[v(mh,nh+h)-v(mh,nh)]^2\}\,. \tag{2.9}$$

Let $u(x, y)$ be a continuous piecewise continuously differentiable function in the whole x-y plane which vanishes outside R. We define the mesh function

$$v(mh,\ nh) = h^{-2}\int_0^h\int_0^h u(mh+\alpha,\ nh+\beta)\,d\alpha\,d\beta\,. \tag{2.10}$$

Because of (2.4) this function belongs to M_h. We note that

$$\begin{aligned}&\int\int_R u^2 dx\,dy - h^2 \sum_{(mh,nh)\in R_h} v(mh,nh)^2 \\ &= \sum_{(mh,nh)\in R_h} \int_0^h\int_0^h [u(mh+\alpha,\ nh+\beta) - v(mh,\ nh)]^2\,d\alpha\,d\beta\,.\end{aligned} \tag{2.11}$$

By definition (2.10)

$$\int_0^h\int_0^h [u(mh+\alpha,\ nh+\beta) - v(mh,\ nh)]d\alpha\,d\beta = 0\,. \tag{2.12}$$

Consequently, each integral on the right of (2.11) is bounded by the integral of the gradient of u times the reciprocal of the second free membrane eigenvalue for the square of side h:

$$\begin{aligned}&\int_0^h\int_0^h [u(mh+\alpha,\ nh+\beta) - v(mh,\ nh)]^2\,d\alpha\,d\beta \\ &\leqq \frac{h^2}{\pi^2}\int_0^h\int_0^h |\text{grad}\, u(mh+\alpha,\ nh+\beta)|^2\,d\alpha\,d\beta\,.\end{aligned} \tag{2.13}$$

Replacing this in (2.11) and summing over all the squares, we have

$$\int\int_R u^2\,dx\,dy - h^2\sum_{R_h} v^2 \leqq \frac{h^2}{\pi^2}\int\int_R |\text{grad}\, u|^2\,dx\,dy\,. \tag{2.14}$$

Now let

$$u = \xi_1 u_1 + \cdots + \xi_k u_k\,, \tag{2.15}$$

where the u_i are the normalized eigenfunctions of (2.1), and the ξ_i are any real numbers. Then we have

$$v = \xi_1 v_1 + \cdots + \xi_k v_k\,, \tag{2.16}$$

where the v_i are defined in terms of the u_i just as v is defined in terms of u by (2.10). Inequality (2.14) can be written in the form

$$h^2 \sum_{R_h} (\xi_1 v_1 + \cdots + \xi_k v_k)^2 \geqq \sum_{i=1}^{k} \xi_i^2 - \frac{h^2}{\Pi^2} \sum_{i=1}^{k} \lambda_i \xi_i^2 . \tag{2.17}$$

This gives a lower bound for the denominator of the ratio in (2.8). In order to be certain that the mesh functions v_i are linearly independent, we assume that h is chosen so small that this lower bound is always positive. That is, we take

$$h^2 < \pi^2/\lambda_k . \tag{2.18}$$

We now turn to the numerator in (2.8). We note that if u and v are again related by (2.10), we have

$$\begin{aligned} &v(mh+h,\ nh) - v(mh,\ nh) \\ &= h^{-2} \int_0^{2h} d\alpha \int_0^h d\beta\, \psi(\alpha) \frac{\partial u}{\partial x}(mh+\alpha,\ nh+\beta)\,, \end{aligned} \tag{2.19}$$

with a similar formula for $v(mh,\ nh+h) - v(mh,\ nh)$. Here we have put

$$\psi(\alpha) = \begin{cases} \alpha & 0 \leqq \alpha \leqq h \\ 2h-\alpha & h \leqq \alpha \leqq 2h\ . \\ 0 & \text{elsewhere} \end{cases} \tag{2.20}$$

so that

$$\psi(\alpha) + \psi(\alpha+h) + \psi(\alpha-h) = h, \quad \int_0^{2h} \psi(\alpha)\, d\alpha = h^2 . \tag{2.21}$$

Consequently, we can write

$$\begin{aligned} &\iint_R |\operatorname{grad}\ u|^2\, dx\, dy - D_h(v) \\ &= h^{-1} \sum_{R_h} \int_0^{2h} d\alpha \int_0^h d\beta\, \psi(\alpha) \Big[\Big\{ \frac{\partial u}{\partial x}(mh+\alpha,\ nh+\beta) \\ &\quad - h^{-1} \langle v(mh+h,\ nh) - v(mh,\ nh) \rangle \Big\}^2 \\ &\quad + \Big\{ \frac{\partial u}{\partial y}(mh+\beta,\ nh+\alpha) - h^{-1} \langle v(mh,\ nh+h) - v(mh,\ nh) \rangle \Big\}^2 \Big] \geqq 0\,. \end{aligned} \tag{2.22}$$

Again making the substitutions (2.15) and (2.16), we have

$$D_h(\xi_1 v_1 + \cdots + \xi_k v_k) \leqq \sum_{i=1}^{k} \lambda_i \xi_i^2 . \tag{2.23}$$

Inserting (2.17) and (2.23) in the bound (2.8) yields

$$(2.24)\qquad \lambda_k^{(h)} \leq \max_{\xi_1,\cdots,\xi_k} \frac{\sum_{i=1}^{k} \lambda_i \xi_i^2}{\sum_{i=1}^{k}\left(1-\frac{h^2\lambda_i}{\pi^2}\right)\xi_i^2} = \frac{\lambda_k}{1-\frac{h^2\lambda_k}{\pi^2}} .$$

Solving for λ_k we find the lower bound

$$(2.25)\qquad \lambda_k \geqq \frac{\lambda_k^{(h)}}{1+\frac{h^2\lambda_k^{(h)}}{\pi^2}} .$$

This bound was derived under the assumption that (2.18) holds. However, if (2.18) is violated, (2.25) is trivially true. Thus, the lower bound (2.25) holds for all k such that $\lambda_k^{(h)}$ is defined (k at most equal to the number of interior mesh points of R_h). The same type of consideration will apply in all the derivations to follow. That is, one derives the lower bound by assuming an inequality like (2.18) to hold, and then finds that the lower bound also holds when the inequality is violated. We shall suppress this argument in what follows.

3. **Variable coefficients, no mixed derivatives.** We now extend the results of the preceding section to an eigenvalue problem in N dimensions. We consider the problem

$$(3.1)\qquad -\sum_{i=1}^{N} \frac{\partial}{\partial x^i}\left(p^i \frac{\partial u}{\partial x^i}\right) + qu = \lambda r u \quad \text{in } R,$$

$$u=0 \qquad \text{on the boundary } \dot{R} .$$

Here R is a bounded N-dimensional domain. The functions p^i, q, and r are assumed to be piecewise continuously differentiable. We assume p^i and r to be positive and q non-negative in the closure of R. The eigenvalues are arranged in increasing order

$$(3.2)\qquad \lambda_1 \leqq \lambda_2 \leqq \cdots$$

and the corresponding eigenfunctions, normalized by

$$(3.3)\qquad \int_R r u^2\, dv = 1 ,$$

are called u_1, u_2, $\cdots$.

The space is divided into N-cubes by the planes $x^i = m^i h$, $m^i = 0$, ± 1, ± 2, $\cdots$.

We again denote by R_h a region consisting of the union of mesh cubes, and containing not only R but all its translates in negative x-directions of distances up to h. We denote by M_h the class of functions

$v(m^1h, \cdots, m^Nh)$ defined at mesh points and vanishing at all such points on the bounday of or exterior to R_h.

Let $u(x^1, \cdots, x^N)$ be a continuous piecewise differentiable function vanishing outside R. Then by definition of R_h the mesh function

$$v(m^1h, \cdots, m^Nh) = h^{-N}\int_{0\leqq\alpha^i\leqq h} u(m^ih+\alpha^i)d\alpha^1 \cdots d\alpha^N \tag{3.4}$$

is in M_h. We define the mesh function[3]

$$\overline{r}(m^ih) = \left[h^{-N}\int_{0\leqq\alpha^i\leqq h} \frac{d\alpha^1 \cdots d\alpha^N}{r(m^ih+\alpha^i)}\right]^{-1} . \tag{3.5}$$

Analogous to (2.11) we have the identity

$$\begin{aligned}&\int_R ru^2dV - h^N\sum \overline{r}(m^ih)v(m^ih)^2 \\ &= \sum_{R_h}\int_{0\in\alpha^i\leqq h}[r(m^ih+\alpha^i)u(m^ih+\alpha^i) - \overline{r}(m^ih)v(m^ih)]^2 \frac{d\alpha^1 \cdots d\alpha^N}{r(m^ih+\alpha^i)} .\end{aligned} \tag{3.6}$$

Also, by (3.4) and (3.5)

$$\int_{0\leqq\alpha^i\leqq h}[r(m^ih+\alpha^i)u(m^ih+\alpha^i) - \overline{r}(m^ih)v(m^ih)] \frac{d\alpha^1 \cdots d\alpha^N}{r(m^ih+\alpha^i)} = 0 . \tag{3.7}$$

Thus, we are again led to a free membrane problem, and we find

$$\int_{0\leqq\alpha^i\leqq h}[ru-\overline{r}v]^2 \frac{d\alpha^1 \cdots d\alpha^N}{r} \leqq \frac{h^2}{\pi^2 r_m}\int_{0\leqq\alpha^i\leqq h}|\text{grad } ru(m^ih+\alpha^i)|^2 d\alpha^1 \cdots d\alpha^N, \tag{3.8}$$

where we have put

$$r_m = \min_{x\in R} r(x^1, \cdots, x^N) . \tag{3.9}$$

By the triangle inequality

$$\left\{\int_R|\text{grad } ru|^2 dV\right\}^{1/2} \leqq \left\{\int_R r^2|\text{grad } u|^2dV\right\}^{1/2} + \left\{\int_R u^2|\text{grad } r|^2dV\right\}^{1/2} . \tag{3.10}$$

Hence we have

[3] The definition of $r(x)$ outside R is rather arbitrary. We choose it in such a way that the term in the bracket is the mean value of r over the intersection of the domain of integration with R. Since $\lambda_k^{(h)}$ decreases with increasing R_h, we can assume without loss of generality that R_h is minimal with respect to the analogue of (2.4), so that for squares corresponding to interior points of R_h this intersection is not empty. Similar considerations will apply to the mesh functions formed from the other coefficients.

$$h^N \sum_{R_h} \overline{r} v^2 \geqq \int_R ru^2 dV - \frac{h^2}{\pi^2 r_m}\left[K\int_R\left[p^1\left(K\frac{\partial u}{\partial x^1}\right)^2 + \cdots + p^N\left(\frac{\partial u}{\partial x^N}\right)^2 + qu^2\right]dV\right\}^{1/2} + \left\{L\int_R ru^2\, dV\right\}^{1/2}\Bigg]^2 \tag{3.11}$$

where

$$K = \max_{\substack{x\in R\\ i=1,\cdots,N}}\left(\frac{r^2}{p^i}\right), \tag{3.12}$$

$$L = \max\left(\frac{|\text{grad } r|^2}{r}\right).$$

We also find

$$\int_R qu^2 dV - h^N \sum_{R_h} \overline{q} v^2 = \sum_{R_h}\int_{0\leqq\alpha^i\leqq h}[qu-\overline{q}v]^2\,\frac{d\alpha^1\cdots d\alpha^N}{q} \geqq 0, \tag{3.13}$$

where we have put[4]

$$\overline{q}(m^i h) = \left[h^{-N}\int_{0\leqq\alpha^i\leqq h}\frac{d\alpha^1\cdots d\alpha^N}{q(m^i h+\alpha^i)}\right]^{-1}. \tag{3.14}$$

Using the function $\psi(\alpha)$ defined by (2.20), we find that

$$\int_R p^1\left(\frac{\partial u}{\partial x^1}\right)^2 dV - h^{N-2}\sum_{R_h}\overline{p}^1(m^i h)\,[v(m^1h+h,\ m^2h,\cdots,\ m^N h) - v(m^1h,\cdots,m^N h)]^2 = h^{-1}\sum_{R_h}\int_{\substack{0\leq\alpha_1\leq 2h\\ 0\leqq\alpha_j\leqq h,\ j>1}}\Big\{p^1(m^i h+\alpha^i)\,\frac{\partial u}{\partial x^1}\,(m^i h+\alpha^i) - h^{-1}\overline{p}^1[v(m^1h+h) - v(m^1h)]\Big\}^2\psi(\alpha^1)\,\frac{d\alpha^1\cdots d\alpha^N}{p^1(m^i h+\alpha^i)} \geqq 0, \tag{3.15}$$

where we have put[3]

$$\overline{p}^1(m^i h) = \left[h^{-N-1}\int_{\substack{0\leqq\alpha_1\leqq 2h\\ 0\leqq\alpha_j\leqq h, j>1}}\frac{\psi(\alpha^1)d\alpha^1\cdots d\alpha^N}{p^1(m^i h+\alpha^i)}\right]^{-1}. \tag{3.16}$$

In this way we find that if we define the quadratic form

$$Q(w) = h^N\sum_{R_h}\left\{h^{-2}\sum_{j=1}^N \overline{p}^j[w(m^i h+\delta_{ij}h) - w(m^i h)]^2 + \overline{q}w^2\right\} \tag{3.17}$$

for mesh functions w in M_h, where[3]

$$\overline{p}^j = \left[h^{-N-1}\int_{\substack{0\leq\alpha^j\leq 2h\\ 0\leqq\alpha^l\leqq h,\ l\neq j}}\frac{\psi(\alpha^j)d\alpha^1\cdots d\alpha^N}{p^j(m^i h+\alpha^i)}\right]^{-1} \tag{3.18}$$

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[4] See footnote 3. We make the convention that $\overline{q}=0$ if the integral diverges or if $q=0$ in an open set.

and $\bar{q}$ is defined by (3.14), then

(3.19) $$Q(v) \leqq \int_R \left[p^1\left(\frac{\partial u}{\partial x^1}\right)^2 + \cdots + p^N\left(\frac{\partial u}{\partial x^N}\right)^2 + qu^2 \right] dV .$$

We now define the numbers $\lambda_1^{(h)} \leqq \lambda_2^{(h)} \leqq \cdots$ as the successive minima of a ratio of quadratic forms:

(3.20) $$\lambda^{(h)} = \min_{w \in R_h} \frac{Q(w)}{h^N \sum_{R_h} \bar{r} w^2} .$$

The $\lambda_i^{(h)}$ are eigenvalues of the finite difference problem

(3.21) $$L^{(h)} w + \bar{q} w = \lambda^{(h)} \bar{r} w , \quad w \in M_h ,$$

where

(3.22) $$L^{(h)} w(m^i h) = - h^{-2} \sum_{j=1}^{N} \{ \bar{p}^j(m^i h) [w(m^i h + \delta_{ij} h) - w(m^i h)] - \bar{p}^j(m^i h - \delta_{ij} h) [w(m^i h) - w(m^i h - \delta_{ij} h)] \} .$$

The equation (3.21) is clearly a finite difference analogue of (3.1).

We now proceed exactly as in § 2 to let

(3.23) $$u = \xi_1 u_1 + \cdots + \xi_k u_k$$

where the u_i are the normalized eigenfunctions of (3.1). Then

(3.24) $$v = \xi_1 v_1 + \cdots + \xi_k v_k$$

where the v_i are related to the u_i by (3.4). We apply the Poincaré inequality

(3.25) $$\lambda_k^{(h)} \leqq \max_{\xi_1, \cdots, \xi_k} \frac{Q(\xi_1 v_1 + \cdots + \xi_k v_k)}{h^{-N} \sum_{R_h} \bar{r}(\xi_1 v_1 + \cdots + \xi_k v_k)^2}$$

together with the inequalities (3.19) and (3.11) to find

(3.26) $$\lambda_k^{(h)} \leqq \frac{\lambda_k}{1 - \dfrac{h^2}{\pi^2 r_m} \left[\sqrt{K \lambda_k} + \sqrt{L} \right]^2} .$$

Solving for λ_k we obtain the lower bound

$$\lambda_k \geqq \lambda_k^{(h)} \left[\frac{\left\{1+\frac{h^2}{\pi^2 r_m}(K\lambda_k^{(h)}-L)\right\}^{1/2} - \frac{h^2}{\pi^2 r_m}\{KL\lambda_k^{(h)}\}^{1/2}}{1+\frac{h^2}{\pi^2 r_m}K\lambda_k^{(h)}} \right]^2 . \tag{3.27}$$

Clearly this lower bound differs from $\lambda_k^{(h)}$ only by a term of order h^2. It should be noted that it is independent of N and, except for $\lambda_k^{(h)}$ itself, of k. For the case of the Laplace operator treated in section 2, $K=r_m=1$ and $L=0$. Then (3.27) reduces to (2.25).

We note that (3.27) simplifies considerably when the function r is constant so that $L=0$.

4. The general self-adjoint case. In the preceding section we restricted ourselves to the differential equation (3.1), where no mixed derivatives occur. In this section we shall treat the general case

$$\begin{aligned} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x^i}\left(a^{ij}\frac{\partial u}{\partial x^j}\right) + qu &= \lambda r u \quad \text{in } R\,, \\ u &= 0 \quad \text{on } \dot{R}\,. \end{aligned} \tag{4.1}$$

Here a^{ij} is assumed to be a uniformly positive definite symmetric matrix in R, r is assumed positive, and q non-negative. All coefficients are taken as piecewise differentiable.

We keep the notation of § 3. In particular, we consider the continuous function u vanishing outside R, and the mesh function v in M_h defined by (3.4).

The inequalities (3.11) and (3.13) can be used almost without change. The problem is to find a quadratic form in v which can be bounded from above in terms of the quadratic form

$$\int_R \left[\sum_{i,j=1}^{N} a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} + qu^2\right] dV\,, \tag{4.2}$$

and which approximates this form for small h.

We begin with the identity

$$\begin{aligned} &\int_R \sum_{i,j=1}^{N} a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}dV - h^N \sum_{R_h}\sum_{i,j=1}^{N} \overline{a}^{ij}w_i w_j \\ &= \sum_{R_h}\int_{0\leqq\alpha^p\leqq h}\sum_{i,j,k,l=1}^{N} a_{ij}\left[a^{ik}\frac{\partial u}{\partial x^k}(m^p h+\alpha^p) - \overline{a}^{ik}w_k\right]\left[a^{jl}\frac{\partial u}{\partial x^l}(m^p h+\alpha^p) \right. \\ &\quad \left. - \overline{a}^{jl}w_l\right] d\alpha^1\cdots d\alpha^N \geqq 0\,. \end{aligned} \tag{4.3}$$

Here a_{ij} is the inverse matrix of a^{ij}, and we have defined the mesh matrix[3]

$$(4.4)\qquad \overline{a}^{ij}(m^l h) \equiv \left[h^{-N}\int_{0\leq\alpha^l\leq h} a_{ij}(m^l h+\alpha^l)d\alpha^1\cdots d\alpha^N\right]^{-1},$$

i.e.,

$$(4.5)\qquad \sum_{p=1}^{N}\left[h^{-N}\int_{0\leq\alpha^l\leq h} a_{ip}(m^l h+\alpha^l)d\alpha^1\cdots d\alpha^N\right]\overline{a}^{pj}(m^l h) = \delta_{ij},$$

and the mesh vector

$$(4.6)\qquad w_k(m^l h) \equiv \left[h^{-N}\int_{0\leq\alpha^l\leq h} \frac{\partial u}{\partial x^k}(m^l h+\alpha^l)d\alpha^1\cdots d\alpha^N\right].$$

While w_k is clearly an approximation to $\partial u/\partial x^k$, it cannot be obtained from v or any other mesh function. Therefore, (4.3) does not give a quadratic form in v. However, since the finite difference

$$(4.7)\qquad d_k[v](m^i h) \equiv h^{-1}[v(m^i h+\delta_{ik}h) - v(m^i h)]$$

also approximates $\partial u/\partial x^k$, it must approximate w_k. We estimate the error introduced by using $d_k[v]$ instead of w_k. It follows from the triangle inequality that

$$(4.8)\qquad \left\{h^N\sum_{R_h}\left[\sum_{i,j=1}^{N}\overline{a}^{ij}d_i[v]d_j[v] + \overline{q}v^2\right]\right\}^{1/2} \leq \left\{h^N\sum_{R_h}\left[\sum_{i,j}\overline{a}^{ij}w_i w_j + \overline{q}v^2\right]\right\}^{1/2}$$
$$+\left\{h^N\sum_{R_h}\sum_{i,j}\overline{a}^{ij}(w_i-d_i[v])(w_j-d_j[v])\right\}^{1/2}.$$

It can be seen from the definition (4.4) that largest and smallest eigenvalues of $\overline{a}^{ij}$ lie between the maximum of the largest eigenvalue and minimum of the smallest eigenvalue of a^{ij} in the cube of definition. Hence, $\overline{a}^{ij}$ is still positive definite so that the triangle inequality applies. The first term on the right of (4.8) is bounded by means of (4.3). The second term is the error due to replacing w_k by $d_k[v]$. We shall bound it.

Let the constant a be a uniform upper bound for the eigenvalues of a^{ij}; that is,

$$(4.9)\qquad a \equiv \max_{\substack{\xi_1,\cdots,\xi_N\\ x\in R}} \frac{\sum_{i,j=1}^{N} a^{ij}\xi_i\xi_j}{\xi_1^2+\cdots+\xi_N^2}.$$

Then the same bound holds if a^{ij} is replaced by $\overline{a}^{ij}$. Hence,

$$(4.10)\qquad h^N\sum_{R_h}\sum_{i,j=1}^{N}\overline{a}^{ij}(w_i-d_i[v])(w_j-d_j[v]) \leq ah^N\sum_{R_h}\sum_{i=1}^{N}(w_i-d_i[v])^2.$$

We use the identity

$$w_i(m^l h) - d_i[v](m^l h) = \int_{\substack{0 \leq \alpha^i \leq 2h \\ 0 \leq \alpha^l \leq h, l \neq i}} \varphi(\alpha^i) \frac{\partial^2 u}{\partial x^{i2}} (m^l h + \alpha^l) d\alpha^1 \cdots d\alpha^N - \oint \varphi(\alpha^i) \frac{\partial u}{\partial x^i} \frac{\partial \alpha^i}{\partial n} dS_\alpha , \tag{4.11}$$

where

$$\varphi(\alpha) = \begin{cases} \frac{1}{2} h^{-N-1} \alpha(2h-\alpha) & 0 \leqq \alpha \leqq h , \\ \frac{1}{2} h^{-N-1}(2h-\alpha)^2 & h \leqq \alpha \leqq 2h . \end{cases} \tag{4.12}$$

The volume integral is actually over the intersection of the rectangular parallelepiped with R. On the boundaries of the parallelepiped the integrand of the surface integral vanishes by the construction of φ. Thus, the last integal is only over the part of $\dot{R}$ cut by the parallelepiped.

We apply Schwarz's inequality and the triangle inequality to (4.11), and note that R is covered twice by each set of parallelepipeds. Using the fact that $u=0$ on $\dot{R}$ we have

$$\left\{h^N \sum_{R_h} \sum_{i=1}^N (w_i - d_i[v])^2\right\}^{1/2} \leqq \left\{\frac{11}{30} h^2 \sum_{i=1}^N \int_R \left(\frac{\partial^2 u}{\partial x^{i2}}\right)^2 dV\right\}^{1/2} + \left\{h^N \sum_{R_h} \sum_{i=1}^N \left[\oint F\left(\frac{\partial u}{\partial n}\right)^2 dS_\alpha \oint \frac{\varphi^2}{F}\left(\frac{\partial \alpha^i}{\partial n}\right)^4 dS_\alpha\right\}^{1/2} . \tag{4.13}$$

Here F is an arbitrary positive function defined on $\dot{R}$. To estimate the last term on the right, we note that $\oint |\partial \alpha^i / \partial n| dS_\alpha$ represents the projection perpendicular to the x^i-axis of the total surface. We call ν_h the maximum number of intersections of $\dot{R}$ with any line segment of length $2h$ parallel to one of the coordinate axes. Clearly, ν_h is a monotone increasing function of h. If $\dot{R}$ is at all regular, ν_h is bounded, and equals 2 for sufficiently small h. Noting that $\varphi^2 \leqq \frac{1}{4} h^{2-2}{}_N$, $|\partial \alpha^i / \partial n| \leqq 1$, and that the projection of any one layer of area within the parallelepiped in the x^i-direction is at most h^{N-1} we have

$$\oint \frac{\varphi^2}{F}\left(\frac{\partial \alpha^i}{\partial n}\right)^4 dS_\alpha \leqq \frac{\nu_h h^{1-N}}{4F_m} , \tag{4.14}$$

where

$$F_m = \min_{x \in \dot{R}} F , \tag{4.15}$$

Again taking account of the fact that R is covered twice by each set of parallelepipeds, we have

$$\left\{h^N \sum_{R_h} \sum_{i=1}^{N} (w_i - d_i[v]^2)\right\}^{1/2} \leqq \left\{\frac{11}{30} h^2 \sum_{i=1}^{N} \int_R \left(\frac{\partial^2 u}{\partial x^{i2}}\right)^2 dV\right\}^{1/2} + \left\{\frac{N\nu_h h}{2F_m} \oint_{\dot{R}} F\left(\frac{\partial u}{\partial n}\right)^2 dS\right\}^{1/2}. \tag{4.16}$$

It thus becomes necessary to bound the integral of the sum of squares of the second derivatives of u, and a boundary integral of the square of the normal derivative of u. We begin with the latter.

We utilize an identity which was found for the Laplace operator by F. Rellich [**16**], for hyperbolic operators by L. Hörmander [**9**], and which was extensively used for purposes similar to the present one by L. E. Payne and the author [**11, 12, 13**]. Let $f^1(x), \cdots, f_N(x)$ be an arbitrary piecewise differentiable vector field in R. The identity is

$$\begin{aligned} &\oint \left(\sum_{k=1}^{N} f^k n_k\right)\left(\sum_{i,j=1}^{N} a^{ij} n_i n_j\right)\left(\frac{\partial u}{\partial n}\right)^2 dS \\ &= \int_R \sum_{i,j,k=1}^{N} \left(\frac{\partial f^k}{\partial x^k} a^{ij} - 2\frac{\partial f^i}{\partial x^k} a^{jk} + f^k \frac{\partial a^{ij}}{\partial x^k}\right) \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} dV \\ &\quad - \int_R 2\mathscr{A}(u) \sum_{i=1}^{N} f^i \frac{\partial u}{\partial x^i} dV, \end{aligned} \tag{4.17}$$

where we have written

$$\mathscr{A}(u) \equiv \sum_{i,j=1}^{N} \frac{\partial}{\partial x^i}\left(a^{ij}\frac{\partial u}{\partial x^j}\right), \tag{4.18}$$

and n_k is the outward unit normal on $\dot{R}$.

We now assume that the vector field f^k has the property that its outward normal component on $\dot{R}$ is positive:

$$\sum_{k=1}^{N} f^k n_k > 0$$

Then we can put

$$F \equiv \sum_{k=1}^{N} f^k n_k \sum_{i,j=1}^{N} a^{ij} n_i n_j \tag{4.19}$$

in (4.16). For example, if R is star-shaped with respect to the origin, we may take $f^k = x^k$. More generally, if $\dot{R}$ is represented by an equation $R(x)=0$ where $R(x)$ is a twice differentiable function in R whose outward normal derivative on $\dot{R}$ is positive, we may take $f^k = \partial R/\partial x^k$. It still remains to bound the right hand side of (4.17). For this purpose, we restrict ourselves to the function

$$u = \xi_1 u_1 + \cdots + \xi_k u_k \tag{4.20}$$

where $u_1, \cdots, u_k$ are the first k eigenfunctions of (4.1) normalized by (3.3). Then

$$\mathscr{A}(u) = \sum_{n=1}^{k} \xi_n (q - \lambda_n r) u \,. \tag{4.21}$$

The integrand of the first integral on the right of (4.17) is a quadratic form in the gradient of u. Since the lowest eigenvalue of a^{ij} is assumed to be positive and bounded away from zero, there exists a constant c defined by

$$c \equiv \max_{\eta_1, \cdots, \eta_N} \frac{\sum_{i,j,k=1}^{N} \left(\frac{\partial f^k}{\partial x^k} r^{ij} - 2 \frac{\partial f^i}{\partial x^k} a^{jk} + f^k \frac{\partial a^{ij}}{\partial x^k} \right) \eta_i \eta_j}{\sum_{i,j=1}^{N} a^{ij} \eta_i \eta_j} \,. \tag{4.22}$$

Thus, the first integral on the right of (4.17) is bounded by

$$c \int_R \sum_{i,j=1}^{N} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} dV \leqq c \int_R \left[\sum_{i,j=1}^{N} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} + q u^2 \right] dV \tag{4.23}$$
$$= c(\lambda_1 \xi_1^2 + \cdots + \lambda_k \xi_k^2) \,.$$

Substituting (4.21) in the second integral and using Schwarz's inequality, we find the bound

$$M_1 \{\lambda_1 \xi_1^2 + \cdots + \lambda_k \xi_k^2\}^{1/2} \{\lambda_1^2 \xi_1^2 + \cdots + \lambda_k^2 \xi_k^2\}^{1/2} \tag{4.24}$$
$$+ M_2 \{\lambda_1 \xi_1^2 + \cdots + \lambda_k \xi_k^2\}^{1/2} \{\xi_1^2 + \cdots + \xi_k^2\}^{1/2} \,,$$

where

$$M_1 = 2 \max \left\{ r \sum_{i,j=1}^{N} a_{ij} f^i f^j \right\}^{1/2} , \qquad M_2 = 2 \max \left\{ \frac{q^2}{r} \sum_{i,j=1}^{N} a_{ij} f^i f^j \right\}^{1/2} . \tag{4.25}$$

Thus we find

$$\oint_{\dot{R}} F \left(\frac{\partial u}{\partial n} \right)^2 dS \leqq c(\lambda_1 \xi_1^2 + \cdots + \lambda_k \xi_k^2) \tag{4.26}$$
$$+ M_1 \{\lambda_1 \xi_1^2 + \cdots + \lambda_k \xi_k^2\}^{1/2} \{\lambda_1^2 \xi_1^2 + \cdots + \lambda_k^2 \xi_k^2\}^{1/2}$$
$$+ M_2 \{\lambda_1 \xi_1^2 + \cdots + \lambda_k \xi_k^2\}^{1/2} \{\xi_1^2 + \cdots + \xi_k^2\}^{1/2} \,.$$

We now estimate the first integral on the right of (4.16). For this purpose we extend an argument used in the case of the two-dimensional Laplace operator by L. E. Payne [10].

We let

$$a^{ij} = \sqrt{g}\, g^{ij} \tag{4.27}$$

where

$$\begin{aligned} g &= \det [g_{ij}] \,, \\ g_{ij} &= [g^{ij}]^{-1} \,. \end{aligned} \tag{4.28}$$

In three or more dimensions one can solve (4.27):

$$\begin{aligned} g &= \{\det [a^{ij}]\}^{-(2/N-2)} \,, \\ g^{kl} &= \{\det [a^{ij}]\}^{1/N-2} a^{kl} \,. \end{aligned} \tag{4.29}$$

In two dimensions (4.27) implies $\det [a^{ij}]=1$. If this is satisfied, one takes $g^{ij}=a^{ij}$. If not, one must make a change of dependent and independent variables to arrive at $\det [a^{ij}]=1$. We assume this to have been done.

We consider g_{ij} as the metric tensor of a Riemannian space. We derive the tensor identity (using summation convention)

$$\begin{aligned} \sqrt{g}\, g^{kl}(g^{ij}u_{|i}u_{|j})_{|kl} &= 2\sqrt{g}\; g^{kl}g^{ij}[u_{|ik}u_{|jl}+u_{|i}u_{|jkl}] \\ &= 2\sqrt{g}\; g^{kl}g^{ij}[u_{|ik}u_{|jl}+u_{|i}u_{|klj}+u_{|i}R^{p}_{kjl}u_{|p}] \\ &= 2\sqrt{g}\; g^{kl}g^{ij}u_{|ik}u_{|jl}+2\sqrt{g}\, g^{ij}u_{|i}(g^{kl}u_{|kl})_{|j} \\ &\quad - 2\sqrt{g}\; R^{ij}u_{|i}u_{|j} \,. \end{aligned} \tag{4.30}$$

Here we have used symbol $_{|i}$ for covariant differentiation. R^{p}_{kjl} is the Riemann curvature tensor, and R^{ij} is the contravariant Ricci tensor (see, for example, [17]):

$$\begin{aligned} R^{ij} \equiv g^{il}g^{jm}\Big[&\frac{1}{2}\frac{\partial^2}{\partial x^l \partial x^m}\ln g \\ &- \frac{1}{2}\left\{ {p \atop l\, m} \right\}\frac{\partial}{\partial x^p}\ln g - \frac{\partial}{\partial x^p}\left\{ {p \atop l\, m} \right\} + \{{}_l p_q\}\left\{ {q \atop n\, p} \right\}\Big] \,, \end{aligned} \tag{4.31}$$

where

$$\left\{ {p \atop l\, m} \right\} \equiv \frac{1}{2} g^{pq}\left[\frac{\partial g^{ql}}{\partial x^m} + \frac{\partial g^{qm}}{\partial x^l} - \frac{\partial g^{lm}}{\partial x^q}\right] \tag{4.32}$$

is the Christoffel symbol of the second kind. We have

$$\begin{aligned} u_{|i} &\equiv \frac{\partial u}{\partial x^i} \,, \\ u_{|ij} = u_{|ji} &\equiv \frac{\partial^2 u}{\partial x^i \partial x^j} - \left\{ {p \atop i\;\, j} \right\}\frac{\partial u}{\partial x^p} \,, \end{aligned} \tag{4.33}$$

and consequently

(4.34) $$g^{kl}u_{|kl}=\mathscr{A}(u)/\sqrt{g}$$

where $\mathscr{A}(u)$ is the operator (4.18). The left-hand side of (4.30) is a perfect divergence. Integrating (4.30) over R, applying the divergence theorem, and transposing terms gives, after use of (4.34),

(4.35) $$\int_R \sqrt{g}\, g^{kl}g^{ij}u_{|ik}u_{|jl}\,dV = \frac{1}{2}\oint_{\dot{R}} a^{kl}\frac{\partial}{\partial x^k}\left(g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\right)n_l\,dS$$
$$-\int_R a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\mathscr{A}(u)}{\sqrt{g}}\right)dV + \int_R \sqrt{g}\,R^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\,dV\,.$$

Here we have used Euclidean elements of volume and area.

We now restrict ourselves to functions u of the form (4.20), so that $\mathscr{A}(u)=0$ on the boundary. Then by the divergence theorem and (4.27)

(4.36) $$-\int_R a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\mathscr{A}(u)}{\sqrt{g}}\right)dV = \int_R \frac{1}{\sqrt{g}}\mathscr{A}(u)^2\,dV\,.$$

But when u is given by (4.20), $\mathscr{A}(u)$ is given by (4.21). By the triangle inequality we find the bound

(4.37) $$\left\{\int_R \frac{1}{\sqrt{g}}\mathscr{A}(u)^2\,dV\right\}^{1/2} \leqq \{l_1(\xi_1^2+\cdots+\xi_k^2)\}^{1/2} + \{l_2(\lambda_1^2\xi_1^2+\cdots+\lambda_k^2\xi_k^2)\}^{1/2}$$

with

(4.38) $$l_1 = \max\left(\frac{q^2}{r\sqrt{g}}\right)$$

and

(4.39) $$l_2 = \max\left(\frac{r}{\sqrt{g}}\right).$$

For the last term on the right of (4.35) we put

(4.40) $$d \equiv \max_{\substack{\eta_1,\cdots,\eta_N\\ x\in R}}\left(\frac{\sqrt{g}\,R^{ij}\eta_i\eta_j}{a^{ij}\eta_i\eta_j}\right).$$

Then

(4.41) $$\int_R \sqrt{g}\,R^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\,dV \leqq d\int_R a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\,dV \leqq d(\lambda_1\xi_1^2+\cdots+\lambda_k\xi_k^2).$$

We come now to the surface integral in (4.35). We suppose that in some neighborhood of the surface $\dot{R}$ there is defined a differentiable function $R(x)$ vanishing on $\dot{R}$ and such that the outward normal derivative is positive. Since $R(x)$ vanishes on $\dot{R}$ we may put

$$u(x) = R(x)\varphi(x) \tag{4.42}$$

in the neighborhood of $\dot{R}$. Then we see that on $\dot{R}$

$$\begin{aligned} &a^{kl}\frac{\partial}{\partial x^k}\left(g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\right)n_l \\ &=\left(\frac{\partial R}{\partial n}\right)^{-1}\left[4\varphi g^{kl}\frac{\partial R}{\partial x^k}\frac{\partial R}{\partial x^l}\,a^{ij}\frac{\partial R}{\partial x^i}\frac{\partial \varphi}{\partial x^j}+\varphi^2 a^{ij}\frac{\partial R}{\partial x^i}\frac{\partial}{\partial x^j}\left(g^{kl}\frac{\partial R}{\partial x^k}\frac{\partial R}{\partial x^l}\right)\right]. \end{aligned} \tag{4.43}$$

Also on $\dot{R}$

$$\mathscr{A}(u) = \varphi\mathscr{A}(R) + 2a^{ij}\,\frac{\partial R}{\partial x^i}\frac{\partial \varphi}{\partial x^j}\,. \tag{4.44}$$

Since u is taken of the form (4.20) and the u_i satisfy (4.1), $\mathscr{A}(u)$ vanishes on $\dot{R}$. Hence, we may eliminate the derivaties of φ occurring in (4.43) by setting (4.44) equal to zero. Finally, to identify φ in terms of u we take the normal derivative of (4.42) to find

$$\frac{\partial u}{\partial n} = \varphi\frac{\partial R}{\partial n}\,. \tag{4.45}$$

Thus, we arrive at

$$\begin{aligned} &a^{kl}\frac{\partial}{\partial x^k}\left(g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\right)n_l \\ &=-\,2[g^{pq}n_p n_q]^{3/2}\,\frac{\partial}{\partial x^i}\left(\left\{g^{kl}\frac{\partial R}{\partial x^k}\frac{\partial R}{\partial x^l}\right\}^{-1/2}\sqrt{g}\,g^{ij}\frac{\partial R}{\partial x^j}\right)\left(\frac{\partial u}{\partial n}\right)^2. \end{aligned} \tag{4.46}$$

The coefficient of $(\partial u/\partial n)^2$ is clearly independent of the particular function $R(x)$ used to represent $\dot{R}$. It is a local geometric property of $\dot{R}$. In fact, if g^{ij} is the unit matrix, the coefficient is just $-2(N-1)$ times the mean curvature of $\dot{R}$, as can be seen by taking for $R(x)$ the distance from $\dot{R}$. If g_{ij} is the metric of a flat space, the divergence term is still proportional to the mean curvature of $\dot{R}$ in this space. The first part of the coefficient arises from the fact that we are mixing a Euclidean and a non-Euclidean metric.

Setting

$$e \equiv \max\left\{-F^{-1}\,[g^{pq}n_p n_q]^{3/2}\,\frac{\partial}{\partial x^i}\left(\left\{g^{kl}\frac{\partial R}{\partial x^k}\frac{\partial R}{\partial x^l}\right\}^{-1/2}\sqrt{g}\;g^{ij}\frac{\partial R}{\partial x^j}\right)\right\} \tag{4.47}$$

where F is defined by (4.19) in terms of the arbitrary vector field pointing outward on $\dot{R}$, we have by (4.26)

$$(4.48)\qquad \frac{1}{2}\oint_{\dot R} a^{kl}\frac{\partial}{\partial x^k}\left(g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\right)n_l\,dS \leqq e\oint F\left(\frac{\partial u}{\partial n}\right)^2 dS$$
$$\leqq ce(\lambda_1\xi_1^2+\cdots+\lambda_k\xi_k^2)+M_1e\{\lambda_1\xi_1^2+\cdots+\lambda_k\xi_k^2\}^{1/2}\{\lambda_1^2\xi_1^2+\cdots+\lambda_k^2\xi_k^2\}^{1/2}$$
$$+M_2e\{\lambda_1\xi_1^2+\cdots+\lambda_k\xi_k^2\}^{1/2}\{\xi_1^2+\cdots+\xi_k^2\}^{1/2}.$$

We note that in order to have a finite e it is necessary to assume that the coefficient of $(\partial u/\partial n)^2$ in (4.46) is bounded above. Since this coefficient, at least in a flat space, is proportional to the negative of the mean curvature, one sees that this implies that $\dot R$ has no re-entrant corners, edges, or cusps. On the other hand, non-re-entrant corners, edges, and cusps cause no difficulty. It is easily ascertained from the asymptotic form of a solution of (4.1) that the integrals of the squares of the second derivatives, which we are seeking to bound, actually diverge at re-entrant corners, edges, and cusps.

Having bounded the right-hand side of (4.35), we turn to the left-hand side. The positive definite symmetric matrix g^{ij} may be expanded in terms of its eigenvalues $0<\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_N$ and orthonormal eigenvectors in the form

$$(4.49)\qquad g^{ij}=\sum_{p=1}^{N}\mu_p c_p^i c_p^j\,.$$

Then

$$(4.50)\qquad g^{kl}g^{ij}u_{|ik}u_{|jl}=\sum_{p,q,i,k,j,l=1}^{N}\mu_p\mu_q(c_p^ic_q^ku_{|ik})(c_p^jc_q^lu_{|jl})$$
$$\geqq\mu_1^2\sum_{p,q=1}^{N}\left(\sum_{i,k=1}^{N}c_p^ic_q^ku_{|ik}\right)^2$$
$$=\mu_1^2\sum_{i,k=1}^{N}u_{|ik}^2\,,$$

the last equality being due to the orthonormality of the eigenvectors. Now by virtue of (4.33) and the triangle inequality

$$(4.51)\qquad \left\{\int_R\sum_{i=1}^{N}\left(\frac{\partial^2u}{\partial x^{i2}}\right)^2dV\right\}^{1/2}\leqq\left\{\int_R\sum_{i,k=1}^{N}u_{|ik}^2\,dV\right\}^{1/2}$$
$$+\left\{\int_R\sum_{i=1}^{N}\left[\sum_{p=1}^{N}\left\{{p\atop i\;i}\right\}\frac{\partial u}{\partial x^p}\right]^2dV\right\}^{1/2}.$$

Thus, letting

$$(4.52)\qquad b\equiv\max\left(\frac{1}{\mu_1^2\sqrt{g}}\right)=\max_{\substack{\eta_1,\cdots,\eta_N\\ x\in R}}\left(\frac{\eta_1^2+\cdots+\eta_N^2}{\sum_{i,j=1}^{N}a^{ij}\eta_i\eta_j}\right)^2\sqrt{g}$$

and

$$m = \max \sum_{i,p,q=1}^{N} a_{pq} \left\{ {p \atop i\, i} \right\} \left\{ {q \atop i\, i} \right\} \tag{4.53}$$

and applying Schwarz's inequality we have

$$\left\{ \int_R \sum_{i=1}^{N} \left(\frac{\partial^2 u}{\partial x^{i2}} \right)^2 dV \right\}^{1/2} \leqq \left\{ b \int_R \sqrt{g}\; g^{kl} g^{ij} u_{|ik} u_{|jl}\, dV \right\}^{1/2} + \left\{ m \int_R a^{pq} \frac{\partial u}{\partial x^p} \frac{\partial u}{\partial x^q}\, dV \right\}^{1/2} . \tag{4.54}$$

We return now to the original problem. We define the quadratic functional $Q(w)$ of mesh function w by

$$Q(w) \equiv h^N \sum_{R_h} \left\{ \sum_{i,j=1}^{N} \bar{a}^{ij} d_i\, [w]\, d_j[w] + g w^2 \right\} \tag{4.55}$$

where d_i is the first difference operator in the x^i direction defined by (4.7), and $\bar{a}^{ij}$ and $\bar{q}$ are the average functions defined by (4.9) and (3.14).

We let $\lambda_1^{(h)} \leqq \lambda_2^{(h)} \leqq \cdots$ be the successive minima of the ratio

$$\frac{Q(w)}{h^N \sum_{R_h} \bar{r} w^2} \tag{4.56}$$

with respect to mesh functions w in M_h. Here $\bar{r}$ is defined by (3.5). The minimizing functions and the minima satisfy the finite difference equation

$$- \sum_{i,j=1}^{N} d_i [\bar{a}^{ij} d_j[w]\,](m^l h - \delta_{il} h) + \bar{q}(m^l h) w(m^l h) = \lambda^{(h)} \bar{r}(m^l h) w(m^l h) . \tag{4.57}$$

This is, of course, a finite difference analogue of (4.1).

The Poincaré inequality (3.25) still holds. Taking for v the mesh function defined by (3.4) and for u the linear combination $\xi_1 u_1 + \cdots + \xi_k u_k$ of the first k eigenfunctions of (4.1), we get v in the form $\xi_1 v_1 + \cdots + \xi_k v_k$.

We now put together the inequalities (4.8), (3.13), (4.3), (4.10), (4.16), (4.54), (4.35), (4.48), (4.37), (4.41), and (4.26) to find

$$\begin{aligned} \{Q(\xi_1 v_1 + \cdots + \xi_k v_k)\}^{1/2} &\leqq \left\{ \sum_{i=1}^{k} \lambda_i \xi_i^2 \right\}^{1/2} \\ &+ h \left\{ \frac{11}{30} ab \right\}^{1/2} \{(ce+d) \textstyle\sum \lambda_i \xi_i^2 + M_1 e [\sum \lambda_i \xi_i^2 \sum \lambda_j^2 \xi_j^2]^{1/2} \\ &+ M_2 e [\textstyle\sum \lambda_i \xi_i^2 \sum \xi_j^2]^{1/2} + [(l_1 \sum \xi_i^2)^{1/2} + (l_2 \sum \lambda_i^2 \xi_i^2)^{1/2}]^2\}^{1/2} \\ &+ h \left\{ \frac{11}{30} am \textstyle\sum \lambda_i \xi_i^2 \right\}^{1/2} + h^{1/2} \left\{ \frac{N \nu_h a}{2 F_m} \right\}^{1/2} \\ &\times \{c \textstyle\sum \lambda_i \xi_i^2 + M_1 [\sum \lambda_i \xi_i^2 \sum \lambda_j^2 \xi_j^2]^{1/2} + M_2 [\sum \lambda_i \xi_i^2 \sum \xi_j^2]^{1/2}\}^{1/2} . \end{aligned} \tag{4.58}$$

The denominator is bounded by the generalization of (3.11), namely

$$h^N \sum_{R_h} \overline{r v^2} \geqq 1 - \frac{h^2}{\pi^2 r_m}\left[\left\{K \sum_{i=1}^{k} \lambda_i \xi_i^2\right\}^{1/2} + \left\{L \sum \xi_i^2\right\}^{1/2}\right]^2 \tag{4.59}$$

where

$$K = \max_{\substack{\eta_1, \cdots, \eta_N \\ x \in R}} \frac{r^2(\eta_1^2 + \cdots + \eta_N^2)}{\sum_{i,j=1}^{N} a^{ij}\eta_i\eta_j}, \tag{4.60}$$

$$L = \max(|\operatorname{grad} r^2/\nu)$$

Inserting these bounds in the Poincaré inequality (3.25) yields

$$\begin{aligned}\lambda_k^{(h)} \leqq \Big[& \lambda_k^{1/2} + h^{1/2}\{M_2\lambda_k^{1/2} + c\lambda_k + M_1\lambda_k{}^{1/2}\}^{1/2}\left\{\frac{N\nu_h a}{2F_m}\right\}^{1/2} + h\{l_1 + M_2 e\lambda_k^{1/2} \\ & + (ce + d + 2\sqrt{l_1 l_2})\lambda_k + M_1\lambda_k^{3/2} + l_2\lambda_k^2\}^{1/2}\left\{\frac{11}{30}ab\right\}^{1/2} \\ & + h\left\{\frac{11}{30}am\lambda_k\right\}^{1/2}\Big]^2\left[1 - \frac{h^2}{\pi^2 r_m}(\sqrt{K\lambda_k} + \sqrt{L})^2\right]^{-1}.\end{aligned} \tag{4.61}$$

This is an implicit lower bound for λ_k. We note that the lower bound differs from $\lambda_k^{(h)}$ by a term of order $h^{1/2}$, rather than h^2 as in the absence of mixed derivatives. The inequality (4.61) does not reduce to (3.27) when a^{ij} is diagonal.

5. Systems with no mixed derivatives. The process used in § 3 is easily extended to a self-adjoint system of elliptic equations. We must only consider the unknown function in (3.1) as a vector and the coefficients as symmetric matrices. Thus we have

$$\sum_{\beta=1}^{n}\left\{-\sum_{i=1}^{N}\frac{\partial}{\partial x^i}\left(p_{\alpha\beta}^{(i)}\frac{\partial u^\beta}{\partial x^i}\right) + q_{\alpha\beta}u^\beta\right\} = \lambda\sum_{\beta=1}^{n} r_{\alpha\beta}u^\beta, \qquad \alpha = 1, \cdots, n \tag{5.1}$$

We assume the matrices $p_{\alpha\beta}^{(1)} \cdots p_{\alpha\beta}^{(N)}$ and $r_{\alpha\beta}$ to be positive definite and $q_{\alpha\beta}$ semi-definite, and all their components piecewise differentiable. We put

$$v^\beta(m^i h) \equiv h^{-N}\int_{0 \leqq \alpha^i \leqq h} u^\beta(m^i h + \alpha^i)d\alpha^1 \cdots d\alpha^N \tag{5.2}$$

and, writing $r^{\alpha\beta}$ for the inverse of $r_{\alpha\beta}$,[3]

$$\overline{r}_{\alpha\beta}(m^i h) \equiv \left[h^{-N}\int_{0 \leqq \alpha^i \leqq h} r^{\alpha\beta}(m^i h + \alpha^i)d\alpha^1 \cdots d\alpha^N\right]^{-1}. \tag{5.3}$$

Then we have, analogous to (3.6)

$$(5.4)\qquad \int_R \sum_{\alpha,\beta=1}^{N} r_{\alpha\beta}u^\alpha u^\beta dV - h^N \sum_{R_h} \sum_{\alpha,\beta=1}^{n} \bar{r}_{\alpha\beta}v^\alpha v^\beta$$
$$= \sum_{R_h} \int_{0\leqq\alpha^i\leqq h} \sum_{\alpha,\beta,\gamma,\delta=1}^{n} r_{\alpha\beta}\left[r_{\alpha\gamma}u^\gamma - \bar{r}_{\alpha\gamma}v^\gamma\right]\left[r_{h\delta}u^\delta - \bar{r}_{\beta\delta}v^\delta\right]d\alpha^1\cdots d\alpha^N .$$

Thus, putting

$$(5.5)\qquad r_m = \min_{\substack{\eta^1,\dots,\eta^n\\ x\in R}}\left[\frac{\sum_{\alpha,\beta=1}^{n} r_{\alpha\beta}\eta^\alpha\eta^\beta}{(\eta^1)^2+\cdots+(\eta^n)^2}\right] = \min_{\substack{\xi^1,\dots,\xi^n\\ x\in R}}\left[\frac{\xi_1^2+\cdots+\xi_n^2}{\sum_{\alpha,\beta=1}^{n} r^{\alpha\beta}\xi_\alpha\xi_\beta}\right],$$
$$K = \max_{\substack{i=1,\cdots N\\ x\in R}}\left[\sum_{\alpha,\beta,\gamma=1}^{n} p^{(i)\alpha\beta}r_{\alpha\gamma}r_{\beta\gamma}\right],$$
$$L = \max_{x\in R}\left[\sum_{\alpha,\beta,\gamma=1}^{n} r^{\alpha\beta}\operatorname{grad} r_{\alpha\gamma}\cdot\operatorname{grad} r_{\beta\gamma}\right],$$

where we have written $p^{(i)\alpha\beta}$ for the inverse matrix of $p^{(i)}_{\alpha\beta}$, we have the analogue of (3.11)

$$(5.6)\qquad h^N \sum_{R_h}\sum_{\alpha,\beta=1}^{n} \bar{r}_{\alpha\beta}v^\alpha v^\beta \geqq \int_R \sum_{\alpha,\beta=1}^{n} r_{\alpha\beta}u^\alpha u^\beta dV$$
$$-\frac{h^2}{\pi^2 r_m}\left[\left\{K\int_R \sum_{\alpha,\beta=1}^{n}\left(\sum_{i=1}^{n} p^{(i)}_{\alpha\beta}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^i} + q_{\alpha\beta}u^\alpha u^\beta\right)dV\right\}^{1/2}\right.$$
$$\left.+\left\{L\int_R \sum_{\alpha,\beta=1}^{n} r_{\alpha\beta}u^\alpha u^\beta dV\right\}^{1/2}\right]^2 .$$

Similarly, defining[3]

$$(5.7)\qquad \bar{q}_{\alpha\beta}(m^i h) \equiv \left[h^{-N}\int_{0\leqq\alpha^i\leqq h} q^{\alpha\beta}(m^i h + \alpha^i)d\alpha^1\cdots d\alpha^N\right]^{-1}$$

where $q^{\alpha\beta}$ is the inverse matrix of $q_{\alpha\beta}$ and[3]

$$(5.8)\qquad \bar{p}^{(i)}_{\alpha\beta}(m^j h) \equiv \left[h^{-N}\int_{0\leqq\alpha^j\leqq h} p^{(i)\alpha\beta}(m^j h + \alpha^j)d\alpha^1\cdots d\alpha^N\right]^{-1},$$

we find the analogues of (3.13) and (3.15). Thus, if we define the finite difference eigenvalues $\lambda_1^{(h)}\leqq\lambda_2^{(h)}\leqq\cdots$ as the successive minima of the ratio

$$(5.9)\qquad \frac{\sum_{\alpha,\beta=1}^{n}\left(\sum_{i=1}^{n}\bar{p}^{(i)}_{\alpha\beta}d_i[w^\alpha]d_i[w^\beta]+\bar{q}_{\alpha\beta}w^\alpha w^\beta\right)}{\sum_{\alpha,\beta=1}^{n}\bar{r}_{\alpha\beta}w^\alpha w^\beta}$$

among sets of mesh functions $(w^1,\cdots,w^n)$ in class M_h, we find again the lower bound (3.27) for the eigenvalues of (5.1) in terms of those of (5.9).

$$\lambda_k \geqq \lambda_k^{(h)} \left[\frac{\left\{1+\frac{h^2}{\pi^2 r_m}(K\lambda_k^{(h)}-L)\right\}^{1/2} - \frac{h^2}{\pi^2 r_m}\{KL\lambda_k^{(h)}\}^{1/2}}{1+\frac{h^2}{\pi^2 r_m}K\lambda_k^{(h)}} \right]^2 . \tag{5.10}$$

The considerations of § 4 do not appear capable of extension to systems of elliptic differential equations containing mixed derivatives.

6. Error estimation. As has already been mentioned in the introduction, it is rather easy to get upper bounds for the eigenvalues λ_k by means of another finite difference problem. Thus in order to determine the error, one must first calculate the eigenvalues of two finite difference problems. If the error turns out to be too large, one must reduce the mesh size and recalculate the eigenvalues. It is a great saving of labor to have an a priori estimate of the error in terms of the mesh size. For then one can pick a mesh size to give at most a given error and do only one eigenvalue computation.

We proceed to estimate the error by considering the scheme for obtaining upper bounds. For the sake of clarity we begin with the two-dimensional Laplace operator case treated in § 2.

Following a method suggested by R. Courant [2] (and already implicitly contained in a paper of L. Collatz [1]), we divide each square of the finite difference mesh into two triangles by means of a diagonal in a fixed direction. Then, given any mesh function v of class M_h, we can associate with it a piecewise differentiable function u by specifying that it coincides with v at the mesh points, and is linear in each triangle. This function vanishes on the boundary of the domain R_h. Furthermore, if $v_1, \cdots, v_k$ are linearly independent mesh functions, the corresponding functions $u_1, \cdots, u_k$ are linearly independent; and to the linear combination $\xi_1 v_1 + \cdots + \xi_k v_k$ corresponds the linear combination $\xi_1 u_1 + \cdots + \xi_k u_k$. Letting $\mu_k(R_h)$ be the kth eigenvalue of the fixed membrane problem

$$\Delta u + \mu u = 0 \qquad \text{in } R_h \tag{6.1}$$

with $u=0$ on the boundary of R_h, we have the Poincaré inequality

$$\mu_k(R_h) \leqq \max_{\xi_1,\cdots,\xi_k} \frac{\int_{R_h} |\operatorname{grad}(\xi_1 u_1 + \cdots + \xi_k u_k)|^2 dxdy}{\int_{R_h} (\xi_1 u_1 + \cdots + \xi_k u_k)^2 dxdy} . \tag{6.2}$$

Since n depends linearly on its mesh values v, both the numerator and denominator in (6.2) are quadratic forms in the mesh function v. They have been explicitly determined by G. Polya [15], who finds that

(6.3) $$\int_{R_h} |\text{grad } u|^2 dxdy = D_h(v)$$

defined by (2.9), while

(6.4) $$\int_{R_h} u^2 dxdy = I(v) \equiv h^2 \sum_{R_h} \Big\{ v(x_m, y_n)^2 - \frac{1}{12}[v(x_m+h, y_n) - v(x_m, y_n)]^2 - \frac{1}{12}[v(x_m, y_n+h) - v(x_m, y_n)] - \frac{1}{12}[v(x_m+h, y_n+h) - v(x_m, y_n)]^2 \Big\} .$$

We now let $\mu_1^{(h)} \leqq \mu_2^{(h)} \leqq \cdots$ be the successive minima of the ratio

(6.5) $$\frac{D_h(v)}{I(v)} .$$

Letting $v_1, \cdots, v_k$ be the first k minimizing functions, we see from (6.2) that

(6.6) $$\mu_k(R_h) \leqq \mu_k^{(h)} .$$

Thus, we have upper bounds for the $\mu_k(R_h)$ in terms of the minimum problem (6.5), which can again be formulated as a finite difference problem. However, noting that

(6.7) $$I(v) \geqq h^2 \sum_{R_h} v(x_m, y_n)^2 - \frac{1}{4} h^2 D_h(v) ,$$

we can bound the $\mu_k^{(h)}$ in terms of the eigenvalues $\lambda_k^{(h)}$ by

(6.8) $$\mu_k^{(h)} \leqq \frac{\lambda_k^{(h)}}{1 - \frac{1}{4} h^2 \lambda_k^{(h)}} ,$$

assuming, of course, that h is so small that $h^2\lambda_k^{(h)} < 4$. Thus, we have the upper bound

(6.9) $$\mu_k(R_h) \leqq \frac{\lambda_k^{(h)}}{1 - \frac{1}{4} h^2 \lambda_k^{(h)}} .$$

This process is easily extended to N dimensions. Here each mesh cube is divided in an arbitrary but fixed manner into simplices with vertices at the corners. Then the values of the mesh function v determine a function u coinciding with v at the mesh points and linear in each simplex. We again find the bound (6.9) with the factor 1/4 replaced by a constant c_N depending on the dimensionality.

In the case of variable coefficients an extra error occurs because the coefficients appearing in the quadratic forms for the upper bound are different averages of the coefficients of the differential problem from those used in finding upper bounds. However, both are averages over cubes of size at most $2h$. Thus the differences will be at most h times a constant depending on the maximum gradients of the coefficients. This constant can be calculated. Thus we find in general

$$\mu_k(R_h) \leqq \lambda_k^{(h)} + hf(h, \lambda_k^{(h)}) \tag{6.10}$$

where $f(h, \lambda_k^{(h)})$ is an explicitly known bounded non-decreasing function of h and $\lambda_k^{(h)}$.

Now since R is contained in R_h, $\mu_k(R_h) < \lambda_k$. However, if R_h is close to R, we expect the $\mu_k(R_h)$ to be close to λ_k. The estimation of this closeness depends on the geometry of R. For example, if R contains a cut, the domains R_h will never have this cut, and so the $\mu_k(R_h)$ will not approach the λ_k. However, if R is so smooth that the boundary of R_h approaches that of R as $h \to 0$, then it is easy to show that $\mu_k(R_h) \to \lambda_k$ and the inequality (6.10) together with the lower bound for λ_k proves that $\lambda_k^{(h)} \to \lambda_k$.

If R is convex and contains a circle of radius $\bar{r}$, then one can see that the image of R under a dilatation of the ratio $(1+3h\bar{r}^{-1})$: 1 about the center of this circle contains a region R_h. The eigenvalues of this image are $(1+3h\bar{r}^{-1})^{-2}\lambda_k$, and they now lie below the $\mu_k(R_h)$. Thus, using (6.10) we have

$$\lambda_k \leqq (1 + 3h\bar{r}^{-1})^2 (\lambda_k^{(h)} + hf(h, \lambda_k^{(h)})) \,. \tag{6.11}$$

In other words, we have an upper bound for λ_k differing from $\lambda_k^{(h)}$ by a term of order h. The difference between this and the lower bound thus approaches zero with h. In order to make this difference explicit, we need only bound $\lambda_k^{(h)}$ in terms of λ_k by the inequality (3.26), (4.61), or (5.10) and use some upper bound for λ_k.

For another error estimate when R is not convex the reader is referred to § 5 of our previous paper [19]. While the argument is given there only for the lowest eigenvalue, it applies equally well to higher eigenvalues.

7. The non-homogeneous problem. We consider the elliptic differential equation

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(a^{ij}\frac{\partial u}{\partial x^j}\right) + qu = G \qquad \text{in } R\,, \tag{7.1}$$

$$u = 0 \qquad \text{on } \dot{R}\,. \tag{7.2}$$

Here the coefficients a^{ij} and q satisfy the hypotheses of § 4, and G is a given continuous function.

By the well-known Dirichlet principle, u minimizes the ratio

$$(7.3)\qquad \frac{\int_R\left(\sum_{i,j=1}^N a^{ij}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}+q\varphi^2\right)dV}{\left(\int_R \varphi G dV\right)^2}$$

among functions φ vanishing on the boundary. Let the value of the minimum be $(1/\lambda)$. It is easily seen from the equation (7.1) that

$$(7.4)\qquad \lambda=\int_R uGdV=\int_R\left(\sum_{i,j=1}^N a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}+qu^2\right)dV\,.$$

An upper bound for $(1/\lambda)$ is easily found from the minimum principle. We proceed to find a lower bound. We define the mesh domain R_h as before, the mesh function v in terms of u by (3.4), the mesh coefficients $\overline{a^{ij}}$ and $\overline{g}$ by (4.4) and (3.14), and the mesh function[3]

$$(7.5)\qquad \overline{G}(m^ih)\equiv h^{-N}\int_{0\leqq\alpha^i\leqq h}G(m^ih+\alpha^i)d\alpha^1\cdots d\alpha^N\,.$$

Then, by Schwarz's inequality, the free membrane problem for the cube, and (7.4)

$$\begin{aligned}
&\left[\int_R uGdV-h^N\sum_{R_h}v\overline{G}\right]^2\\
&=\left[\sum_{R_h}\int_{0\leqq\alpha^i\leqq h}\{u(m^ih+\alpha^i)-v(m^ih)\}G(m^ih+\alpha^i)d\alpha^1\cdots d\alpha^N\right]^2\\
&\leqq\left[\sum_{R_h}\int_{0\leqq\alpha^i\leqq h}\{u(m^ih+\alpha^i)-v(m^ih)\}^2\,d\alpha^1\cdots d\alpha^N\right]\\
&\quad\times\left[\sum_{R_h}\int_{0\leqq\alpha^i\leqq h}G(m^ih+\alpha^i)^2\,d\alpha^1\cdots d\alpha^N\right]\\
&\leqq\frac{h^2}{\pi^2}\int_R|\operatorname{grad}u|^2dV\int_R G^2dV\leqq\frac{h^2\lambda}{\pi^2A}\int_R G^2dV
\end{aligned}$$

where

$$(7.6)\qquad A\equiv\min_{\substack{\eta_1,\dots,\eta_N\\ x\in R}}\frac{\sum_{i,j=1}^N a^{ij}\eta_i\eta_j}{\eta_1^2+\cdots+\eta_N^2}\,.$$

The inequality (7.5) gives the lower bound

$$(7.7)\qquad \left(h^N\sum_{R_h}v\overline{G}\right)^2\geqq\left[\lambda-\frac{h}{\pi}\left(\frac{\lambda}{A}\int G^2dV\right)^{1/2}\right]^2.$$

We derive an upper bound for the form $Q(v)$ defined by (4.55) in the same way as we derived (4.58). We must, however, use the differential equation (7.1) instead of (4.1) and the single function u instead of the linear combination (4.20). Thus, we find that the bound for the first integral on the right of (4.17) is, by inequality (4.23), just $c\lambda$ with c defined by (4.22). However, the bound for the second integral on the right of (4.17) becomes (we again introduce the summation convention)

$$P\lambda + 2\left[\int_R a_{ij} f^i f^j G^2 dV\right]^{1/2} \lambda^{1/2} \tag{7.8}$$

instead of (4.24). Here we have defined

$$P \equiv \max\,[2(q a_{ij} f^i f^j)^{1/2}]\,. \tag{7.9}$$

Thus, (4.26) is replaced by

$$\oint_{\dot R} F\left(\frac{\rho u}{\partial n}\right)^2 dS \leqq (c+P)\lambda + 2\left(\int_R a_{ij} f^i f^j G^2 dV\right)^{1/2} \lambda^{1/2}\,. \tag{7.10}$$

Since $\mathscr{A}(u)$ does not necessarily vanish on the boundary, (4.36) becomes

$$\begin{aligned} -\int_R a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j}\left(\frac{\mathscr{A}(u)}{\sqrt{g}}\right) dV \\ = \int_R \frac{1}{\sqrt{g}} \mathscr{A}(u)^2 dV - \oint_{\dot R} g^{ij} \frac{\partial u}{\partial x^i} n_j \mathscr{A}(u) dS\,. \end{aligned} \tag{7.11}$$

Using the differential equation and the triangle inequality we bound the first term on the right.

$$\left\{\int_R \frac{1}{\sqrt{g}} \mathscr{A}(u)^2 dV\right\}^{1/2} \leqq (l_3\lambda)^{1/2} + \left(\int_R \frac{G^2}{\sqrt{g}} dV\right)^{1/2} \tag{7.12}$$

where

$$l_3 \equiv \max\,(q/\sqrt{g})\,. \tag{7.13}$$

If we eliminate the derivatives of φ between (4.43) and (4.44), without assuming $\mathscr{A}(u)$ to vanish on the boundary, we find

$$\begin{aligned} \frac{1}{2} a^{kl} \frac{\partial}{\partial x^k}\left(a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}\right) n_l \\ = -[g^{pq} n_p n_q]^{3/2} \frac{\partial}{\partial x^i}\left(\left\{g^{kl} \frac{\partial R}{\partial x^k} \frac{\partial R}{\partial x^l}\right\}^{-(1/2)} \sqrt{g} g^{ij} \frac{\partial R}{\partial x^j}\right)\left(\frac{\partial u}{\partial n}\right)^2 \\ + g^{ij} \frac{\partial u}{\partial x^i} n_j \mathscr{A}(u)\,. \end{aligned} \tag{7.14}$$

The integral of the second term just cancels the boundary terms of (7.11) when we substitute in (4.35). The first terms on the right of (7.14) is bounded as before by $e\oint F(\partial u/\partial n)^2 dS$ where e is defined by (4.47).

Inequality (4.41) remains unchanged. Thus we derive in the same way as (4.58) that

$$\begin{aligned}(7.15)\qquad \{Q(v)\}^{1/2} \leqq \lambda^{1/2} + h\left\{\frac{11}{30}ab\right\}^{1/2}\Big\{(ce+Pe+l_3+d)\lambda \\ +2\left[e\left(\int_R a_{ij}f^i f^j G^2 dV\right)^{1/2} + \left(l_3\int_R \frac{G^2}{\sqrt{g}}\,dV\right)^{1/2}\right]\lambda^{1/2} + \int_R \frac{G^2}{\sqrt{g}}\,dV\Big\}^{1/2} \\ +h\left\{\frac{11}{30}\,am\lambda\right\}^{1/2} + h^{1/2}\left\{\frac{N\nu_h a}{2F_m}\right\}^{1/2}\Big\{(c+P)\lambda \\ +2\left(\int_R a_{ij}f^i f^j G^2 dV\right)^{1/2}\lambda^{1/2}\Big\}^{1/2}.\end{aligned}$$

We now define

$$(7.16)\qquad \frac{1}{\lambda^{(h)}} = \min_{w\in M_h} \frac{Q(w)}{\left(h^N \sum_{R_h} w\bar{G}\right)^2}.$$

This quantity may be computed by a finite difference analogue of (7.1).

By the minimum property,

$$(7.17)\qquad \frac{1}{\lambda^{(h)}} \leqq \frac{Q(v)}{\left(h^N \sum_{R_h} v\bar{G}\right)^2}.$$

But the right-hand side is bounded by an explicit function of λ and h of the form $(1/\lambda)+o(h^{1/2})$ by means of (7.7) and (7.15). This gives a lower bound for $1/\lambda$ in terms of $1/\lambda(h)$.

The absence of mixed derivatives results in a great simplification. Inequality (3.19) is valid, and we find

$$(7.18)\qquad \frac{1}{\lambda} \geqq \frac{\lambda^{(h)}}{\left[\lambda^{(h)} + \dfrac{h}{\pi}\left\{\dfrac{\lambda^{(h)}}{A}\displaystyle\int_R G^2 dV\right\}^{1/2}\right]^2}.$$

The upper bound for $1/\lambda$ can again be obtained by means of a finite difference method using piecewise linear functions. Once this piecewise linear function and the error (difference between upper and lower bounds) is known, one can find a pointwise approximation to u at any interior point by the method of Diaz and Greenberg [3, 4].

8. **Higher order operators.** The methods of § 3 are easily extended to the eigenvalue problem

$$(8.1)\qquad Lu = \lambda ru \quad \text{in } R$$

where L is an elliptic operator of order $2m$, and all derivatives of orders up to m-1 of u vanish on $\dot{R}$, provided the numerator of the corresponding Rayleigh quotient is the integral of a linear combination of squares of derivatives of u.

We illustrate the extension by applying it to the problem of the vibrating clamped plate

$$\begin{aligned} \Delta\Delta u &= \lambda u && \text{in } R\,, \\ u &= \partial u/\partial n = 0 && \text{on } \dot{R} \end{aligned} \tag{8.2}$$

in two dimensions. The Rayleigh quotient may be written as

$$\frac{\iint_R \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2\right] dxdy}{\iint_R u^2\, dxdy} \tag{8.3}$$

The domain R_h is defined as before. M_h is the set of mesh functions vanishing everywhere except at the interior mesh points of R_h. The finite difference eigenvalues $\lambda_k^{(h)}$ are defined as the successive minima of the ratio

$$\frac{Q(w)}{h^2 \sum_{R_h} w^2} \tag{8.4}$$

with $w \in M_h$ and

$$\begin{aligned} h^2 Q(w) \equiv \sum_{R_h} \{&[w(mh+h, nh) - 2w(mh, nh) + w(mh-h, nh)]^2 \\ &+ 2[w(mh+h, nh+h) - w(mh+h, nh) \\ &\qquad - w(mh, nh+h) + w(mh, nh)]^2 \\ &+ [w(mh, nh+h) - 2w(mh, nh) + w(mh, nh-h)]^2\}\,. \end{aligned} \tag{8.5}$$

The mesh function v is related by means of (2.10) to the function u having continuous first derivatives and piecewise continuous second derivatives and vanishing outside R.

We now find

$$\begin{aligned} &\iint_R \left(\frac{\partial^2 u}{\partial x^2}\right) dxdy - h^{-2} \sum_{R_n} [v(mh+h, nh) - 2v(mh, nh) + v(mh-h, nh)]^2 \\ &= h^{-2} \sum_{R_h} \int_{-h}^{2h} d\alpha \int_0^h d\beta \tilde{\psi}(\alpha) \left[\frac{\partial^2 u}{\partial x^2}(mh+\alpha, nh+\beta) - h^{-2}\{v(mh+h, nh) \right. \\ &\qquad \left. - 2v(mh, nh) + v(mh-h, nh)\}\right]^2 \geqq 0\,. \end{aligned} \tag{8.6}$$

We have put

$$(8.7)\qquad 2\tilde{\psi}(\alpha) = \begin{cases} (\alpha+h)^2 & -h \leqq \alpha \leqq 0 \\ h^2+2h\alpha-2\alpha^2 & 0 \leqq \alpha \leqq h \\ (2h-\alpha)^2 & h \leqq \alpha \leqq 2h\,. \end{cases}$$

A similar inequality holds for $\partial^2 u/\partial y^2$. For the mixed derivative we have

$$(8.8)\qquad \iint_R \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 dxdy - h^{-2} \sum_{R_h}$$
$$\times [v(mh+h, nh+h) - v(mh+h, nh) - v(mh, nh+h) + v(mh, nh)]^2$$
$$= h^{-2} \sum_{R_h} \int_0^{2h}\int_0^{2h} \psi(\alpha)\psi(\beta)$$
$$\times \left[\frac{\partial^2 u}{\partial x \partial y}(mh+\alpha, nh+\beta) - h^{-2}\{v(mh+h, nh+h) - v(mh+h, nh)\right.$$
$$\left. - v(mh, nh+h) + v(mh, nh)\}\right]^2 d\alpha d\beta$$
$$\geqq 0$$

with $\psi(\alpha)$ defined by (2.20).

Thus $Q(v)$ is bounded by the numerator of (8.3). For the denominator of (8.4) we use the inequality (2.14) together with Green's theorem and Schwarz's inequality to give

$$(8.9)\qquad \iint_R u^2 dxdy - h^2 \sum_{R_h} v^2 \leqq \frac{h^2}{\pi^2}\left|\iint_R u^2 dxdy \iint_R u\Delta\Delta u dxdy\right|^{1/2}.$$

The substitution (2.15) and Poincare's inequality then give

$$(8.10)\qquad \lambda_k^{(h)} \leqq \frac{\lambda_k}{1-(h^2/\pi^2)\lambda_k^{1/2}},$$

which is a lower bound for λ_k.

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LOCALLY COMPACT DIVISION RINGS

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Let K be a division ring with a non-discrete topology T with respect to which both the additive group K^+ and the multiplicative group K^* of K are locally compact topological groups.[1] If m is Haar measure for K^+ and $a \in K$, the function $m'(E) = m(aE)$ is clearly an invariant Borel measure for K^+. Hence there exists a real number $\phi(a)$ such that $m'(E)=\phi(a)m(E)$ for all Borel subsets E of K^+. The real-valued function ϕ on K (which is essentially the Radon-Nikodym derivative of m with respect to left-invariant Haar measure on K^*) evidently has the first two of the following three properties.

(1) $\phi(a)\geqq 0$; $\phi(a)=0$ if and only if $a=0$.

(2) $\phi(ab)=\phi(a)\phi(b)$.

(3) There exists $M>0$ such that $\phi(a)\leqq 1$ implies $\phi(1+a)\leqq M$.

We shall show that ϕ satisfies (3) also, i. e., is a valuation for K, and that the topology T_ϕ for K defined by ϕ coincides with T.[2] The classification of K then follows from known results.

LEMMA 1. *ϕ is continuous.*

Proof. Let ε be a positive number and let E be a compact set of positive measure. By the regularity of Haar measure we may choose an open set U containing E such that $m(U)-m(E)<\varepsilon m(E)$. Choose a neighborhood V of 1 with $V=V^{-1}$ and $V\cdot E\subset U$. Then for x in V, $\phi(x)=m(xE)/m(E)\leqq m(U)/m(E)<1+\varepsilon$; since $x^{-1}\in V$, $\phi(x)=(\phi(x^{-1}))^{-1}>(1+\varepsilon)^{-1}$. Hence $1-\varepsilon<\phi(x)<1+\varepsilon$ and the continuity of ϕ on K^* follows,[2*] Now choose an open set U with $m(U)<\varepsilon m(E)$ and a neighborhood V of 0 with $V\cdot E\subset U$. Then for a in V, $\phi(a)=m(aE)/m(E)\leqq m(U)/m(E)<\varepsilon$ and ϕ is continuous at 0.

LEMMA 2. *$S=\{a\in K:\ \phi(a)\leqq 1\}$ is compact.*

Received February 13, 1958 and in revised form April 1, 1958,

The research in this paper was supported jointly by the Army, Navy and Air Force under contract with the Massachusetts Institute of Technology. The first author is a consultant, Lincoln Laboratory, M. I. T. The second author is a staff member, Lincoln Laboratory, M. I. T.

[1] Continuity of the inverse multiplicative operation need not be assumed; cf. the concluding remark. The continuity of multiplication implies that $a\to -a=(-1)\cdot a$ is continuous.

[2] This idea was suggested by some work of Tate, [12].

[2*] Cf. Halmos [3, §60.6, p. 265].

Proof. Let C be a compact neighborhood of 0 and choose a neighborhood V of 0 such that $V\cdot C \subset C$. Let $a \in V\cap C$ such that $0<\phi(a)<1$. If $a^nS\subset C$ holds for no $n = 1, 2, \cdots$, we select for each n an $s_n \in S$ such that $a^ns_n \notin C$. Since $\phi(a^k)\to 0$ and all the a^k lie in the compact set C, $a^k \to 0$ and hence $a^ks_n \in C$ for sufficiently large k. We may therefore choose $k_n \geqq n$ such that $a^{k_n}s_n \notin C$ but $a^{k_n+1}s_n \in C$. Then the sequence $\{a^{k_n}s_n\}$ of elements of the compact set $a^{-1}C$ has a cluster point c in $a^{-1}C$. Hence $\phi(a^{k_n}s_n)=\phi(a)^{k_n}\phi(s_n)\leqq\phi(a)^{k_n}$ has $\phi(c)$ as a cluster point by the continuity of ϕ; thus $\phi(c)=0$ and $c=0$, which contradicts $a^{k_n}s_n\notin C$. It follows that S is a subset of the compact set $a^{-n}C$ for some n and so, being closed by virtue of the continuity of ϕ, is compact.

COROLLARY. *ϕ is a valuation.*

Proof. $\phi(1+S)$, the continuous image of the compact set $1+S$, is bounded.

LEMMA 3. $T_\phi=T$.

Proof. Let $V \in T-\{\phi\}$, $a \in V$ and $B_n=\{b \in K:\ \phi(b-a)<2^{-n}\}$. Suppose we can choose $b_n \in B_n$ with $b_n \notin V$ for each $n=1, 2, \cdots$. But then the points b_n-a, all of which lie in the compact set S, have a cluster point c in S which must be 0 since $\phi(c)=0$. Hence $b_n\to a$ contrary to our assumption and it follows that $T \subset T_\phi$. Since the opposite inclusion is an immediate consequence of the continuity of ϕ, the proof is complete.

If K is connected[3], it is the real, complex or quaternion field (Pontrjagin [**10**]); in particular, ϕ is archimedean. Conversely, if ϕ is archimedean, the theorem of Ostrowski [8, p. 278] asserts that the center of K is either the real or complex field and so K, not being totally disconnected, is connected.[5]

If K is totally disconnected, ϕ is non-archimedean (and conversely, according to the above) and results due to van Dantzig [2], Hasse [4], Hasse and Schmidt [5], Jacobson and Taussky [6] and Jacobson [7] assert that K is of one of the following three types;[4]

(i) the completion of an algebraic number field at a finite prime,
(ii) the completion of an algebraic function field in one variable

[3] K is either connected or totally disconnected: if the component C of 0 contains $a \notin 0$ then $ba^{-1}C$ is a connected set containing 0 and $b \in K$.

[4] Otobe [**9**] shows that $a\to a^{-1}$ need not be assumed to be continuous; cf. our final remark in this connection.

[5] Alternatively, if K is connected, it is not difficult to show that ϕ is archemedian; then K is a vector space over the reals (Ostrowski) with ϕ as a norm, hence is the real, complex or quaternion field (Arens [**1**] Tornheim [**13**]), proving Pontrjagin's theorem.

over a finite field H,

(iii) a division ring D obtained from a field F of type (ii) by redefining $x.\ a=a^{\sigma}.\ x$, $a \in H$, σ a fixed non-trivial automorphism of H, the elements of D and F being regarded as power series $\sum_{i=n}^{\infty} a_i x^i$ in an indeterminate x over H with coefficients in H.

REMARK. Continuity of $a \to a^{-1}$ need not be assumed, for it appears in the connected case only in the proof that K is not compact in the proof of the Pontrjagin theorem [11, p. 173, Theorem 45.]. If K were compact, $\phi(a)=m(aK)/m(K)\leqq 1$ for all $a \in K$. But, as in the proof of the continuity of ϕ at 0 in Lemma 1, we can find $a \in K$ such that $0<\phi(a)<1$; then $\phi(a^{-1})>1$ and it follows that K is not compact. If K is totally disconnected we have only to apply to T, K^* the following unpublished theorem of A. M. Gleason: Let G be a group with a totally disconneted topology T under which the group operation is continuous from $G\times G$ to G. Then T, G is a topological group.

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HOMOMORPHISMS ON NORMED ALGEBRAS

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1. **Introduction** Let B_1 and B be real normed Q-algebras (not necessarily complete) and T be a homomorphism of B_1 into B. Our main object is to show that, for certain algebras B, T will always be either continuous or closed if the range $T(B_1)$ contains "enough" of B. If B is the algebra of all bounded linear operators on a Banach space $\mathfrak{X}$ and $T(B_1)$ contains all finite-dimensional operators then T is continuous. If B is primitive with minimal one-sided ideals, $T(B_1)$ is dense in B and intersects at least one minimal ideal of B then T is closed. Other examples are given. In these results we can obtain the conclusion for ring homomorphism as well as algebra homomorphism if we assume that $\rho(T(x)) \leqq \rho(x)$, $x \in B_1$, where $\rho(x)$ is the spectral radius of x. Note that this is a necessary condition for real-homogeneity. For the application of these results it is desirable to have examples of algebras which are Q-algebras in all possible normed algebra norms. Examples are given in § 2. For previous work on the continuity of homomorphisms and the homogeneity of isomorphisms on Banach algebras see [8], [9], [11], [12] and [14].

2. **Normed Q-algebras and continuity of homomorphisms.** For the algebraic notions used see [6]. Let B be a normed algebra over the real field (completeness is not assumed). As in [8], [11] a complex number $\lambda \neq 0$ is in the spectrum of $x \in B$ if it is in the usual complex algebra spectrum of $(x, 0)$ in the complexification of B. If B is already a complex algebra then the spectrum of x in this sense is the smallest set in the complex plane symmetric with respect to the real axis which contains the spectrum of x in the complex algebra sense. Let $\rho(x)$ be the *spectral radius* of x, $\rho(x) = \sup |\lambda|$ for λ in the spectrum of x. B is called a *Q-algebra* if the set of quasi-regular elements of B is open. Every regular maximal one-sided or two-sided ideal in a Q-algebra is closed. Hence the radical of a Q-algebra is closed and so also is any primitive ideal. See [10 ; 77].

2.1. LEMMA. *For a normed algebra B the following statements are equivalent.*

(a) *B is a Q-algebra.*

(b) $\rho(x) = \lim \|x^n\|^{1/n}$, $x \in B$.

(c) $\rho(x) \leqq \|x\|$, $x \in B$.

Received January 14, 1958. This research was supported in part by the National Science Foundation, research grant NSF-G 2573.

Suppose (a). Then there exists a number $c>0$ such that x is quasi-regular for all x, $\|x\|<c$. Set $k=[(1+c)^{1/2}-1]^{-1}$. Let $x \in B$ and $\lambda=a+bi$ be any complex number $\neq 0$ where $|\lambda|>k\|x\|$. Then

$$|\lambda|^{-2}\|2ax-x^2\| \leqq |\lambda|^{-2}(2|\lambda|\ \|x\|+\|x\|^2) < 2k^{-1}+k^{-2} < c$$

This shows that $\rho(x) \leqq k\|x\|$. Thus

$$\rho(x) = \rho(x^n)^{1/n} \leqq k^{1/n}\|x^n\|^{1/n}$$

for every positive integer n. Letting $n \to \infty$ we see that $\rho(x) \leqq \lim \|x^n\|^{1/n}$. But $\lim \|x^n\|^{1/n} = \rho(x|B^c)$, the spectral radius of x in the completion B^c of B. Hence $\rho(x) \leqq \rho(x|B^c)$. Since $\rho(x|B^c) \leqq \rho(x)$, (b) follows. Clearly (b) implies (c). Suppose that (a) is false. Then there exists a sequence $\{x_n\}$, $x_n \to 0$ where x_n is not quasi-regular. Then $\rho(x_n) \geqq 1$ for each n and (c) is false.

Let $\mathfrak{X}$ be a Banach space and let $\mathfrak{E}(\mathfrak{X})$ be the Banach algebra of all bounded linear operators on $\mathfrak{X}$ in the uniform topology. Let $\mathfrak{F}(\mathfrak{X})$ be the ideal of all elements of $\mathfrak{E}(\mathfrak{X})$ with finite dimensional range.

2.2. LEMMA. *Let j be an idempotent in a normed algebra B. Then the non-zero spectrum of an element in jBj is the same whether computed in jBj or B.*

This is given in [9; 375] in the complex case. The real case offers no new difficulty.

2.3. THEOREM. *Let U be a ring homomorphism or anti-homomorphism of a normed Q-algebra B_1 into $\mathfrak{E}(\mathfrak{X})$ where $U(B_1) \supset \mathfrak{F}(\mathfrak{X})$ and $\rho[U(V)] \leqq \rho(V)$, $V \in B_1$. Then U is continuous.*

Suppose that U is not continuous. By the additivity of U (see [2; 54]) there exists a sequence $\{T_n\}$ in B_1 such that $\|T_n\|_1 \to 0$ and $\|U(T_n)\| \to \infty$ where $\|T\|_1$ is the norm in B_1 and $\|T\|$ is the usual norm in $\mathfrak{E}(\mathfrak{X})$. Consider any idempotent J of $\mathfrak{E}(\mathfrak{X})$ such that $J\mathfrak{E}(\mathfrak{X})$ is a minimal right ideal of $\mathfrak{E}(\mathfrak{X})$. By the work of Arnold [1] these elements J are the linear operators on $\mathfrak{X}$ of the form $J(x)=x^*(x)y$ where $x^* \in \mathfrak{X}^*$, $y \in \mathfrak{X}$ and $x^*(y)=1$. Let $U(W)=J$ and $U(T_n)=V_n$. Since $\|WT_nW\|_1 \to 0$ we have, by Lemma 2.1, $\rho(WT_nW) \to 0$ and therefore $\rho(JV_nJ) \to 0$. By Lemma 2.2 and the Gelfand-Mazur theorem, $\|JV_nJ\| \to 0$. Note that $JV_nJ(x)=x^*(x)x^*[V_n(y)]y$. Hence $x^*[V_n(y)] \to 0$. Fix $y \neq 0$ in $\mathfrak{X}$. Then $x^*[V_n(y)] \to 0$ for all $x^* \in K = \{x^* \in \mathfrak{X}^* \mid x^*(y) \neq 0\}$. Let $z^* \in \mathfrak{X}^*$, $z^*(y)=0$. Since z^* can be written as the sum of two elements of K, $x^*[V_n(y)] \to 0$ for all $x^* \in \mathfrak{X}^*$. Hence $\sup \|V_n(y)\| < \infty$ for each $y \in \mathfrak{X}$. By the uniform boundedness theorem, $\sup \|V_n\| < \infty$. This is a contradiction.

2.4. THEOREM. *Let T be a ring homomorphism or anti-homomorphism of a normed Q-algebra onto a dense subring of a semi-simple* *finitedimensional normed algebra B where $\rho[T(x)] \leqq \rho(x)$, $x \in B_1$. Then T is continuous.*

By [7 ; 698] B is strongly semi-simple and so, by Theorem proved below, T is real-homogenous and closed. Let $||x||_1$ ($||x||$) denote the norm in $B_1(B)$. Suppose that T is not continuous. Then there exists a sequence $\{x_n\}$ in B_1 such that $||x_n||_1 \to 0$ and $||T(x_n)||=1$, $n=1, 2, \cdots$. There exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $||T(y_n)-w|| \to 0$ for some $w \in B$. Since $||w||=1$ we contradict the fact that T is a closed mapping.

A normed algebra B is called a *permanent Q-algebra* if it is a Q-algebra in all normed algebra norms. We say that the normed algebra B has the *spectral extension property* if the spectral radius of $x \in B$ is the same as the spectral radius of x considered as an element of any Banach algebra B_1 in which B may be algebraically imbedded. Examples of algebras with this property are B^*-algebras [13] and annihilator Banach algebras [3]. To test if a normed algebra B has this property it is sufficient to consider the completions of B in all possible normed algebra norms.

2.5. LEMMA. *A normed algebra B is a permanent Q-algebra if and only if B has the spectral extension property.*

Let B be a permanent Q-algebra, $x \in B$. Then $\lim ||x^n||^{1/n}$ has the same value $\rho(x)$, by Lemma 2.1, for any normed algebra norm for B. Thus B has the spectral extension property. If B has the latter property then for any norm $||x||$, $\rho(x)=\lim ||x^n||^{1/n}$ and B is a permanent Q-algebra by Lemma 2.1.

2.6. THEOREM. *Any two sided ideal I of $\mathfrak{E}(\mathfrak{X})$ where $I \supset \mathfrak{F}(\mathfrak{X})$ and any closed subalgebra B of $\mathfrak{E}(\mathfrak{X})$, $B \supset \mathfrak{F}(\mathfrak{X})$ have the spectral extension property.*

Let R be any such ideal I or closed subalgebra B. Let $||T||_1$ be a normed algebra norm for R and $||T||$ the usual norm. For $T \in R$ let $\rho(T)$ be its spectral radius as an element of R, $\rho_1(T)$ as an element of the completion of R in the norm $||T||_1$ and $\rho_2(T)$ as an element $\mathfrak{E}(\mathfrak{X})$. In the ideal case if $U \in R$ has a quasi-inverse V in $\mathfrak{E}(\mathfrak{X})$ then $V \in R$. In every case $\rho(T)=\rho_2(T)$.

It is enough to show the identity imbedding of R (with norm $||T||_1$) into $\mathfrak{E}(\mathfrak{X})$ (with norm $||T||$) is continuous. For then there exists $c>0$, $||T|| \leqq c||T||_1$, $T \in R$, whence

$$||T^n||^{1/n} \leqq c^{1/n}||T^n||_1^{1/n}$$

for all positive integers n. Consequently $\rho(T) \leqq \rho_1(T)$. Since $\rho_1(T) \leqq \rho(T)$ we would have $\rho(T)=\rho_1(T)$.

Theorem 2.3 cannot be applied since it is not known *a priori* that R is a Q-algebra in the norm $||T||_1$. If, however, the imbedding is discontinuous there exists a sequence $\{T_n\}$ in R such that $||T_n||_1 \to 0$ and $||T_n|| \to \infty$. By the arguments of [1], the minimal ideals of R are the same as the minimal ideals of $\mathfrak{E}(\mathfrak{X})$. For each idempotent generator J of a minimal right ideal of R, JRJ is a normed division algebra and hence has a unique norm topology by the Gelfand-Mazur theorem. Since $||JT_nJ||_1 \to 0$ we have $||JT_nJ|| \to 0$. The remainder of the proof may be handled as in Theorem 2.3.

For a ring B and a subset $A \subset B$ we denote the left (right) annihilator of A by $L(A)$ ($R(A)$). Bonsall and Goldie [4] have considered topological rings called annihilator rings in which for each proper right (left) closed ideal I, $L(I) \neq (0)$ ($R(I) \neq (0)$). We consider the related purely algebraic concept of a *modular annihilator ring* which is defined to be a ring in which $L(M) \neq (0)$ ($R(M) \neq (0)$) for every regular maximal right (left) ideal. From the standpoint of algebra these rings appear to be a natural class containing H^*-algebras, etc. In view of what follows it is natural to ask if the two concepts agree for semi-simple normed Q-algebras or semi-simple Banach algebras. A affirmative answer would settle an unsolved problem in the theory of annihilator algebras.

2.7. LEMMA. *Let B be a semi-simple normed annihilator Q-algebra and I be a closed two-sided ideal in B. Then I is a modular annihilator Q-algebra.*

Thus if we had affirmative answer to the above question, any closed two-sided ideal of a semi-simple annihilator Banach algebra would also be one. The analogous result is known for dual algebras [7; 690].

Let M be a regular maximal right ideal of I. Since I is a Q-algebra (as an ideal in B), M is closed in B. Since $L(I) = R(I)$, ([4; 159]), $L(I + R(I)) = (0)$ so that $I + R(I)$ is dense. The arguments of [7; Theorem 2] show that M is a right ideal in B. We must show $L(M) \cap I \neq (0)$. Suppose the contrary. Then $I\ L(M) = (0)$ and $L(M) \subset R(I) = L(I)$. As $M \subset I$, $L(M) \supset L(I)$. Therefore $L(M) = L(I)$. $R(M)M = (0)$ since it is a nilpotent ideal in B. Thus $R(M) \subset L(M) = R(I)$. Then since $R(M) \supset R(I)$ we see that $R(M) = L(M)$. If $x \in L(M + R(M))$ then $x \in L(M) = R(M)$ and $x \in LR(M)$. Thus $x^2 = 0$ and, by semi-simplicity and the annihilator property, $M + R(M)$ is dense in B. Then $(M + R(M))\ I = (M + L(I))I \subset M$ and $BI \subset M$. Let j be a left identity for I modulo M. Then $jx - x \in M$, $x \in I$ and $jx \in M$, $x \in I$. Hence $I \subset M$ which is a contradiction.

2.8. LEMMA. *In a semi-simple modular annihilator ring, every proper right (left) ideal contains a minimal right (left) ideal. A normed* *modular annihilator algebra B has the spectral extension property.*

Since the first statement is shown by stripping the arguments of Bonsall and Goldie [4] of all topological connotations, a sketch of the argument is sufficient. As in [4, Lemma 2], if j is not right (left) quasi-regular there exists $x\neq 0$ in B where $xj=x(jx=x)$. The arguments of [4, Theorem 1] show that if M is a regular maximal right (left) ideal of B then $L(M)$ $(R(M))$ is a minimal left (right) ideal generated by an idempotent. Also the left (right) annihilator of a minimal right (left) ideal is a regular maximal left (right) ideal. Consider the socle K of B. By the reasoning of [4, Theorem 4], $L(K)=R(K)=(0)$. Let I be a proper right ideal of B. If I contained no minimal right ideals of B then, as in the proof of [4, Lemma 4], $I\subset L(K)$, which is impossible.

Let $x\in B$ and let B' be the completion of B in the normed algebra norm $\|x\|_1$. Consider $\lambda=a+bi\neq 0$ in $sp(x|B)$. Then $u=|\lambda|^{-2}(2ax-x^2)$ has no quasi-inverse in B. As in [3 ; p 159] there exists $y\neq 0$ such that $uy=y$ and u has no quasi-inverse in B'. Then $\rho(x|B')=\rho(x|B)$.

3. **Closure of homomorphisms and anti-homomorphisms.** Throughout this section the following notation is assumed. Let $B_1(B)$ be a real normed algebra with norm $\|x\|_1$ $(\|x\|)$. T is a ring homomorphism or anti-homomorphism of B_1 *onto a dense subset of* B. T is called closed if $\|x_n-x\|_1\to 0$, $\|T(x_n)-y\|\to 0$ imply that $y\in T(B_1)$ and $y=T(x)$. By the *separating set* S of T we mean the set of all $y\in B$ such that there exists a sequence $\{x_n\}$ in B_1 where $\|x_n\|_1\to 0$ and $\|y-T(x_n)\|\to 0$. We assume that $\rho[T(x)]\leqq\rho(x)$, $x\in B_1$. Note that this condition is automatic if T is real-linear.

The next lemma is an adaptation of results of Rickart [11].

3.1. LEMMA. *T is closed and real-homogeneous if and only if $S=(0)$. S is a closed two-sided ideal in B and $T^{-1}(S)$ a closed two-sided ideal in B_1. If B_1 is a normed Q-algebra then every element of S is a topological divisor of zero in B.*

Clearly T is rational-homogeneous. Let $x\in B_1$ and $r_n\to r$ where each r_n is rational and r is real. Then $\|r_nx-rx\|_1\to 0$ and $\|rT(x)-T(rx)-T(r_nx-rx)\|\to 0$. Hence $rT(x)-T(rx)\in S$. The first statement follows by a straightforward argument.

Let $y_n\in S$, $\|w-y_n\|\to 0$. There exists, for each n, an element $z_n\in B_1$ such that $\|y_n-T(z_n)\|<n^{-1}$ and $\|z_n\|_1<n^{-1}$. Then $\|w-T(z_n)\|\to 0$ so that $w\in S$. Hence S is closed in B. Since $x\in S$ and r rational imply $rx\in S$ it follows that S is a real linear manifold. To show that S is an ideal in B it is enough to show that xy and $yx\in S$ for $x\in S$ and $y=T(z)\in T(B_1)$. This, however, is a simple matter. Suppose next that $\|x_n-x\|_1\to 0$ where each $x_n\in T^{-1}(S)$. For each n there exists $y_n\in B_1$ such that $\|T(x_n)-T(y_n)\|<n^{-1}$ and $\|y_n\|_1<n^{-1}$. Then $\|x-(x_n-y_n)\|_1\to 0$ while

$||T(x)-T[x-(x_n-y_n)]||\to 0$ whence $T(x)\in S$. Hence $T^{-1}(S)$ is closed. It is readily seen to be a two-sided ideal in B_1.

Let B^c be the completion of B where we use $||x||$ to denote the norm in B^c and $\rho(x)$ the spectral radius there. To show that $s\in S$ is a topological divisor of zero in B it is sufficient to show that it is one in B^c. Choose a sequence $\{x_n\}$ in B_1 such that $||s-T(x_n)||\to 0$ and $||x_n||_1\to 0$. If B_1 is a normed Q-algebra s is the limit of quasi-regular elements of B^c by Lemma 2.1. Hence so also is λs for any real λ. By the arguments of [11 ; 621] it suffices to rule out the possibility that both B^c has an identity 1 and that s has a two-sided inverse in B^c.

Suppose this is the case. Let S_0 be the separating set for T considered as a mapping of B_1 into B^c. Clearly $S\subset S_0$. Then as S_0 is an ideal in B^c, $S_0=B^c$ and $1\in S_0$. There exists a sequence $\{u_n\}$ in B_1 such that $||1-T(u_n)||\to 0$ and $||u_n||_1\to 0$. Since $1-T(u_n)$ and $T(u_n)$ permute we have by Lemma 2.1,

$$1=\rho(1)\leqq\rho(1-T(u_n))+\rho(T(u_n))\leqq||1-T(u_n)||+\rho(u_n|B_1)\to 0$$

This contradication completes the argument.

If B_1 and B are Banach algebras, by the closed graph theorem [2 ; 41] $S=(0)$ will imply that T is continuous. In every case $S=(0)$ will imply real-homogeneity for T and the closure of $T^{-1}(0)$.

3.2. LEMMA. *Let B_1 be a normed Q-algebra and B be semi-simple with minimal one-sided ideals. Suppose that there exists a minimal one-sided ideal I of B_1 such that $T(B_1)\cap I\neq 0$. Then $S\cap I=(0)$.*

We consider the case where I is a right ideal and T is a homomorphism. The other cases follow by the reasoning employed. Set $I_1=T^{-1}(I)$. I_1 is a right (ring) ideal of B_1. Let $I=jB$, $j^2=j$ and consider $x_0\in I_1$ where $T(x_0)=jv\neq 0$. By the semi-simplicity of B, $jvB\neq(0)$ and, as jB is minimal, $jvB=jB$. Then $jvT(B_1)$ is dense in I. It follows that $T(I_1^2)\neq(0)$ for otherwise $[jvT(B_1)]^2=(0)$ and $I^2=(0)$. Select $x\in I_1$, $T(x)=jw\neq 0$ and $T(x^2)\neq 0$. Let R be the set of elements y in B for which $jy\in T(I_1)$. As observed, jR is dense in jB. Hence jRj is dense in jBj. But jBj is a normed division algebra and therefore, by the Gelfand-Mazur theorem, finite-dimensional in B. Thus $jRj=jBj$. There exists $z\in R$ such that $jzjwj=jwjzj=j$. For some $x_1\in I_1$, $T(x_1)j=jzj$. Then $T(x_1x)=jzjw=T((x_1x)^2)$. Set $jzjw=h$ and $x_1x=u$. Then h is a non-zero idempotent in $I\cap T(B_1)$. Clearly $hB=I$ so that hBh is a division algebra hence isomorphic to the reals, complexes or quaternions.

We show that $h\notin S$. For suppose otherwise. Then there exists a sequence $\{y_n\}$ in B_1 such that $||h-T(y_n)||\to 0$ and $||y_n||_1\to 0$. Thus $||uy_nu||_1\to 0$ and $||h-T(uy_nu)||\to 0$. By Lemma 2.2 and the fact that hBh is the reals, complexes or quaternions, $||hT(y_n)h||\to 0$. This is a contradiction as $h\neq 0$. Now $S\cap I$ is a right ideal of B, $S\cap I\neq I$. Since I is minimal, $S\cap I=(0)$.

3.3. THEOREM. *Let B_1 be a normed Q-algebra and B be primitive with minimal one-sided ideals. If $T(B_1)\cap I\neq(0)$ for a minimal one-sided ideal I of B then T is closed and real-homogeneous.*

Let K be the socle of B. If $S\neq(0)$ then $K\subset S$ by [6; 75]. Then $I\subset S$ which is impossible by Lemma 3.2.

3.4. COROLLARY. *Let B be any subalgebra of $\mathfrak{E}(\mathfrak{X})$ closed in the uniform norm $\|T\|$ where $B\supset\mathfrak{F}(\mathfrak{X})$. Let $\|T\|_1$ be any normed algebra norm for B such that the completion B^c of B in this norm is primitive. Then the two norms are equivalent.*

By Theorem 2.6 and Lemma 2.5, B is a Q-algebra in the norm $\|T\|_1$. By Theorem 2.3, there exists $c>0$ such that $\|T\|\leqq c\|T\|_1$, $T\in B$. Consider the embedding mapping I of B (with norm $\|T\|$) into B^c. B is a primitive algebra with a minimal right ideal JB, $J^2=J$. Then $I(J)I(B)I(J)$ a normed division algebra and, by the Gelfand-Mazur theorm, closed in B^c. Since $I(J)$ is an idempotent, its closure in B^c is $I(J)B^cI(J)$. Therefore $I(J)B^c$ is a minimal right ideal of B^c. From Theorem 3.3, I is closed. The closed graph theorem [2; 41] shows that I is continuous. Hence there exists $c_1>0$ such that $\|T\|_1\leqq c_1\|T\|$, $T\in B$.

3.5. THEOREM. *Let B_1 and B be normed Q-algebras. Then S is contained in the Brown-McCoy radical of B. If B is strongly semi-simple then T is closed and real-homogeneous.*

The Brown-McCoy radical [5] coincides with the intersection of the regular maximal two-sided ideals of B. Let M be such an ideal of B. Since B is a normed Q-algebra, M is closed. Let π be the natural homomorphism of B onto B/M. Since $T(B_1)$ is dense in B, then $\pi\, T(B_1)$ is dense in B/M. Also $\rho[\pi T(x)]\leqq\rho[T(x)]\leqq\rho(x)$, $x\in B_1$. Hence our theory applies to the mapping πT.

Let S_0 be the separating set for πT. Since B/M is simple with an identity, $S_0=(0)$ by Lemma 3.1. Let $y\in S$, $\|x_n\|_1\to 0$, $\|y-T(x_n)\|\to 0$. Then $\|\pi(y)-\pi T(x_n)\|\to 0$ or $\pi(y)\in S_0$. Therefore $S\subset M$. B is called *strongly semi-simple* if its Brown-McCoy radical is (0).

3.6. THEOREM. *Let B_1 and B be semi-simple normed Q-algebras where B_1 has a dense socle K and B has an identity Let T be real-linear. Then $T$is closed.*

Let P be a primitive ideal of B and π be the natural homomorphism of B onto B/P. Since B is a Q-algebra then P is closed, π is continuous and $\pi T(B_1)$ is dense in B/P. Let S_0 be the separating set for πT

as a mapping of B_1 into B/P. We show first that $T(K)\subset P$ is impossible. Suppose $T(K)\subset P$. Since $K\subset(\pi T)^{-1}(S_0)$, by Lemma 3.1 we have $B_1=(\pi T)^{-1}(S_0)$ and $S_0=B/P$. Since B/P has an identity this is contrary to Lemma 3.1. Hence there exists a minimal right ideal jB_1 of B_1, $j^2=j$ such that $T(j)\notin P$. Set $\pi T(j)=u$, $\pi T(B_1)=B_2$. πT is an isomorphism or anti-isomorphism of the division algebra jB_1j onto uB_2u. Hence uB_2u is a normed division algebra and thus, by the Gelfand-Mazur theorem closed in B/P. Since u is an idempotent, $u(B/P)$ is a minimal right ideal of B/P. By Theorem 3.3, πT is closed from which we obtain $S\subset P$. Since B is semi-simple, $S=(0)$.

3.7. THEOREM. *Let B_1 be a normed Q-algebra and B semi-simple where either B is a modular annihilator algebra or has dense socle. If $T(B_1)$ contains the socle of B then T is closed and real-homogeneous.*

By Lemma 3.2, $S\cap I=(0)$ for every minimal one-sided ideal of B. Let I be a minimal right ideal. Then $SI=(0)$. Thus S annihilates the socle. It follows (see the proof of Lemma 2.8) that $S=(0)$ in the first case. In the second case we have $S^2=(0)$ and $S=(0)$ by semi-simplicity.

Consider further a semi-simple normed modular annihilator algebra B. B is a permanent Q-algebra by Lemma 2.5 and 2.8. From Theorem 3.7 we see that any algebraic homomorphism or anti-homomorphism of B onto B is closed no matter which two norms are used for B.

Let B be a real normed algebra. By an *involution* on B we mean a mapping $x\to x^*$ of B onto B which is a real-linear automorphism or anti-automorphism of period two. Let $H(K)$ be the set of self-adjoint (skew) elements of B with respect to the involution $x\to x^*$. B is the direct sum $H\oplus K$ of the linear manifolds H and K.

The mapping $x\to x^*$ of B onto B is subject to the above analysis. Here S is the set of all $x\in B$ for which there exists a sequence $\{x_n\}$ in B with $||x_n||\to 0$ and $||x-x_n^*||\to 0$.

3.8. LEMMA. $S=\overline{H}\cap\overline{K}$. *$S=(0)$ if and only if H and K are closed.*

Let $w\in S$. Then there exist sequences $\{h_n\}$ and $\{k_n\}$ in H and K respectively such that $||w-(h_n-k_n)||\to 0$ and $||h_n+k_n||\to 0$. Therefore $||w-2h_n||\to 0$ and $||w+2k_n||\to 0$ so $w\in\overline{H}\cap\overline{K}$. Conversely suppose that $||z-h_n||\to 0$, $||z-k_n||\to 0$ where each $h_n\in H$, $k_n\in K$. Then $||z-(h_n+k_n)/2||\to 0$ and $||(h_n-k_n)/2||\to 0$ and $z\in S$.

If H and K are closed, clearly $S=(0)$. Suppose $S=(0)$. Let $h_n\to u+v$ where $h_n\in H$, $u\in H$ and $v\in K$. Then $h_n-u\to v$ and $v\in\overline{H}\cap\overline{K}$. Then $v=0$ and H is closed. Similarly K is closed.

Let B be a semi-simple normed annihilator algebra, for example an H^*-algebra. Then it follows from the above that H and K are closed in B for any involution on B and any normed algebra norm on B. For

B^*-algebras we have been able to show only the following weaker result.

3.9. THEOREM. *Let B be a B^*-algebra with $H(K)$ as the set of self-adjoint (skew) elements in the defining involution for B. Then H and K are closed in any normed algebra norm topology for B.*

B has the spectral extension property [13] and is therefore a permanent Q-algebra by Lemma 2.5. The arguments of [14; § 3] can be adapted to show that H and K are closed in any given normed algebra norm $||x||_1$.

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Pacific Journal of Mathematics

Vol. 8, No. 2 April, 1958