Pacific Journal of Mathematics

ON A COMMUTATIVE EXTENSION OF A COMMUTATIVE BANACH ALGEBRA

CIPRIAN FOIAS

Vol. 8, No. 3 May 1958

ON A COMMUTATIVE EXTENSION OF A COMMUTATIVE BANACH ALGEBRA

CIPRIAN FOIAS

Let A be a commutative Banach algebra without identity such that

(1.a) there exists an approximate identity (i.e. there exists a net $\{u_\alpha\} \subset A$, so that $\|u_\alpha\| = 1$ and $u_\alpha x \to x$ for all $x \in A$);

(1.b) if $\hat{A}$ designates Gelfand's representation of A [3], and M the space of regular maximal ideals of A, then the boundary of M with respect to $\hat{A}$, is equal to M[1].

Let $\mathscr{L}(A)$ be the algebra of all bounded linear operators on A; the mapping $x \to T_x$ of A into $\mathscr{L}(A)$, where $T_x y = xy$, $y \in A$, is isomorphic and isometric (by (1.a)) onto a subalgebra $\tilde{A}$ of $\mathscr{L}(A)$,

Let $\mathscr{A}$ be the set of those operators $T \in \mathscr{L}(A)$ which commute with all $T_x \in \tilde{A}$, that is such that

$$T(xy) = (Tx)y = x(Ty), \qquad x, y \in A. \tag{1}$$

LEMMA (i). *For all $T \in \mathscr{A}$, we have $T = \lim T_{Tu_\alpha}$, the limit being considered in the strong operator topology.*

(ii) *$\mathscr{A}$ is the closure of $\tilde{A}$ in the strong operator topology.*

(iii) *$\mathscr{A}$ is the largest commutative subalgebra of $\mathscr{L}(A)$ which contains $\tilde{A}$.*

(iv) *$\tilde{A}$ is an ideal in $\mathscr{A}$.*

Proof. From (1) and (1.a), it follows that

$$T_{Tu_\alpha} y = Tu_\alpha \cdot y = T(u_\alpha y) \to Ty$$

for all $T \in \mathscr{A}$ and $y \in A$, hence (i) is proved. (ii) results from (i). Concerning (iii), it is enough to prove that $\mathscr{A}$ is commutative; or, by (i) and (1)

$$T_1 T_2 x = \lim T_{T_1 u_\alpha} T_2 x = T_2 \lim T_{T_1 u_\alpha} x = T_2 T_1 x, \qquad T_1, T_2 \in \mathscr{A}, \; x \in A.$$

If $T \in \mathscr{A}$ and $x, y \in A$, then

$$TT_x y = T(xy) = (Tx)y = T_{Tx} y ,$$

hence

Received March 21, 1958.

[1] For example this condition is satisfied if $\mathscr{A}$ is regular or selfadjoint, see [3, p. 81].

$$TT_x = T_xT = T_{Tx}, \tag{2}$$

whence (iv) follows.

Now, let $\mathscr{M}$ be the space of the maximal ideals of $\mathscr{A}$. We can pass to the main result of our note.

THEOREM 1. *There is a homeomorphism $m \to \tilde{m}$ of M, on an open subset $\tilde{M}$ of $\mathscr{M}$, such that for all $m \in M$, and $x \in A$,*

$$\hat{T}_x(\tilde{m}) = \hat{x}(m)\,;$$

if $\tilde{m}_0 \notin \tilde{M}$ then $\hat{T}_x(\tilde{m}_0) = 0$.

Proof. Observe that by (1.b) and by a theorem of Neumark [4][2] to every $m \in M$ there corresponds an $\tilde{m} \in \mathscr{M}$ such that $\hat{x}(m) = \hat{T}_x(\tilde{m})$ for all $x \in A$. We shall show that $\tilde{m}$ is uniquely determined. If $\hat{T}_x(\tilde{m}_1) = \hat{x}(m) = \hat{T}_x(\tilde{m}_2)$ for all $x \in A$, then by (2)

$$\begin{aligned}\hat{T}(\tilde{m}_1)\hat{x}(m) &= \hat{T}(\tilde{m}_1)\hat{T}_x(\tilde{m}_1) = \widehat{TT_x}(\tilde{m}_1) = \hat{T}_{Tx}(\tilde{m}_1) = \hat{T}x(m) \\ &= \hat{T}_{Tx}(\tilde{m}_2) = \widehat{TT_x}(\tilde{m}_2) = \hat{T}(\tilde{m}_2)\hat{x}(m)\,,\end{aligned}$$

where $x \in A$ and $T \in \mathscr{A}$ are arbitrary. Choose $x \in \mathscr{A}$ such that $\hat{x}(m) \neq 0$; then $\hat{T}(\tilde{m}_1) = \hat{T}(\tilde{m}_2)$ for all $T \in \mathscr{A}$; hence $\tilde{m}_1 = \tilde{m}_2$.

Let $\hat{T}_x(\tilde{m}_0) \not\equiv 0$; then the homomorphism $x \to \hat{T}_x(\tilde{m}_0)$ has as kernel a regular maximal ideal m_0 of A, and from $\hat{x}(m_0) = \hat{T}_x(\tilde{m}_0)$ it follows that $\tilde{m}_0 \in \tilde{M}$. Thus, if $\tilde{m}_0 \notin \tilde{M}_0$, then necessarily $\hat{T}_x(\tilde{m}_0) \equiv 0$. This result shows also that $\tilde{M}$ is open in $\mathscr{M}$. In fact, if $\tilde{m}_0 \in \tilde{M}$, there exists an $x \in A$ such that $\hat{T}_x(\tilde{m}_0) \neq 0$; but then $\hat{T}_x(\tilde{m}) \neq 0$ in a neighborhood V of $\tilde{m}_0$; hence $V \subset \tilde{M}$.

The mapping $\tilde{m} \to m$ being evidently continuous, it remains to prove the continuity of the direct mapping $m \to \tilde{m}$. It is enough to show that the topology of $\tilde{M} \subset \mathscr{M}$ is the weak topology generated on $\tilde{M}$ by the functions $\hat{T}_x(\tilde{m})$, $x \in A$; this results from Theorem 5 G of [3], because that the functions $\hat{T}_x(\tilde{m})$ are continuous on $\tilde{M}$, vanish at infinity (with respect to $\tilde{M}$), separate the points of $\tilde{M}$ and do not all vanish at any point of $\tilde{M}$. (These facts are direct consequences of the preceding results).

In this manner, M can be considered identical with $\tilde{M}$; in what follows we consider $M \subset \mathscr{M}$ and $\hat{T}_x(m) = \hat{x}(m)$.

From now on, we suppose that A is *semi-simple*. Then we have the following

[2] In fact we use a slight extension of the Theorem 3, p. 195.

COROLLARY. (i) *If* $\hat{T}_1(m) = \hat{T}_2(m)$ *for* $m \in M$ *then* $T_1 = T_2$ (ii) $\mathscr{A}$ *is semi-simple.*

Proof. (ii) results from (i), and (i) results from the relation

$$\widehat{T_1 x}(m) = \widehat{T_1 T_x}(m) = \hat{T}_1(m)\hat{T}_x(m) = \hat{T}_2(m)\hat{T}_x(m) = \widehat{T_2 T_x}(m) = \widehat{Tx}(m) ;$$

A being semi-simple, we conclude that $T_1 x = T_2 x$ for all $x \in A$, that is $T_1 = T_2$.

THEOREM 2. *A function f defined on M is a factor function of* $\hat{A}$ (that is $f\hat{x} = \hat{y} \in \hat{A}$ for all $\hat{x} \in \hat{A}$) *if and only if there is a* $T \in \mathscr{A}$, *such that* $f(m) = \hat{T}(m), m \in M$.

Proof. If $f(m) = \hat{T}(m)$ then by (2)

$$f(m)\hat{x}(m) = \hat{T}(m)\hat{x}(m) = \widehat{TT_x}(m) = \hat{T}_{Tx}(m) = \widehat{Tx}(m) \in \hat{A} .$$

Conversely, if f is a factor function of $\hat{A}$, then the operator T_f defined by $T_f x = y$ where $\hat{y} = f\hat{x}$ is a linear closed operator defined on A, since A is semi-simple. Hence T_f is bounded. But $f\hat{x}\hat{y} = \hat{x}f\hat{y}$, so that $T_f \in \mathscr{A}$. Thus for all $m \in M$ we have

$$\hat{T}_f(m)\hat{x}(m) = \hat{T}_f(m)\hat{T}_x(m) = \widehat{T_f x}(m) = \hat{y}(m) = f(m)\hat{x}(m) ,$$

for arbitrary $x \in A$. It follows that $\hat{T}_f(m) = f(m)$.

To understand the sense of these results, let us consider the case $A = L^1(G)$ where G is a locally compact abelian group which is not discrete. Let $M'(G)$ be the algebra of all bounded complex measures on G. Then, if $T_\mu x = \mu * x, x \in L^1(G)$ then T_μ is a linear bounded operator on A, and the mapping $\mu \to T_\mu$ is isomorphic and isometric on $M^1(G)$ into $\mathscr{A}$ [1]. Observing that $M = \hat{G}$ one may see easily that

$$\hat{T}_\mu(m) = \int_G \overline{(m, s)} d\mu(s) . \tag{3}$$

THEOREM 3. $\mathscr{A}$ *is isomorphic and isometric with* $M'(G)$.

Proof. It remains to show that for every $T \in \mathscr{A}$, there is a $\mu \in M^1(G)$ such that $T = T_\mu$. For the measures $\{\mu_\alpha\}$, where $d\mu_\alpha(s) = Tu_\alpha(\gamma)ds$, we have $\| \mu_\alpha \| \leqq \| T \|$. But the sphere of radius $\| T \|$ of $M'(G)$ (considered as the conjugate space of $K(G)$ or $C(G \cup \{\infty\})$) is weakly compact. Hence there is a $\mu \in M^1(G)$, which is a weak cluster point of $\{\mu_\alpha\}$. Consequently, by Lemma (i),

$$\hat{T}(m) = \lim \hat{T}u_\alpha(m) = \lim \int_G \overline{(m, s)} Tu_\alpha(s) ds = \int_G \overline{(m, s)} d\mu(s) = \hat{T}_\mu(m) .$$

By Corollary (i) we conclude that $T = T_\mu$.

Let us give some known corollaries of these results. From Theorems 1 and 3, we may obtain directly that every maximal ideal of $M^1(G)$ which does not contain $L^1(G)$ corresponds to a character of the group G, a fact established by H. Cartan and R. Godement [1]. In the same manner, Theorems 2,3 and (3) show that every factor function for the Fourier transform is the Fourier transform of a bounded measure (both the definition of a factor function and this result in the special case of the additive group of the real numbers are due to E. Hille [2]; the extension to the general case of a locally compact abelian group was done by R.S. Edwards, Pacific J. Math. 1953 and independently by I. Cuculescu).

REFERENCES

1. J. Dieudonné, *Análize Harmônica*, Rio de Janeiro, 1952.
2. E. Hille, *Functional analysis and semigroups*, New-York, 1948, Theorem 18.2.2, p. 362.
3. L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953.
4. M. A. Neumark, *Normed rings*, Moskwa, 1956 (in Russian).

UNIVERSITY OF BUCHAREST
RUMANIA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics
Vol. 8, No. 3 May, 1958

Michael Israel Aissen, *A set function defined for convex plane domaines* 383
Robert Ellis, *Distal transformation groups* 401
Ciprian Foias, *On a commutative extension of a commutative Banach algebra* 407
Jerry William Gaddum, *Linear inequalities and quadratic forms* 411
Allen A. Goldstein and Elliott Ward Cheney, Jr., *A finite algorithm for the solution of consistent linear equations and inequalities and for the Tchebycheff approximation of inconsistent linear equations* 415
William L. Hart and T. S. Motzkin, *Proof of the fundamental theorem on implicit functions by use of composite gradient corrections* 429
Henry Berge Helson, *Conjugate series and a theorem of Paley* 437
Wu-Chung Hsiang, *Abelian groups characterized by their independent subsets* 447
John W. Lamperti, *On the isometries of certain function-spaces* 459
Karel DeLeeuw and Walter Rudin, *Extreme points and extremum problems in* H_1 467
Eugene Lukacs, *Some extensions of a theorem of Marcinkiewicz* 487
George W. Mackey, *Multiplicity free representations of finite groups* 503
Eben Matlis, *Injective modules over Noetherian rings* 511
John William Neuberger, *Continuous products and nonlinear integral equations* 529
Lawrence Edward Payne and Hans F. Weinberger, *New bounds for solutions of second order elliptic partial differential equations* 551
William T. Reid, *A Prüfer transformation for differential systems* 575
Howard L. Rolf, *The free lattice generated by a set of chains* 585
K. M. Saksena, *Inversion and representation theorems for a generalized Laplace integral* 597
Daniel Shanks, *Two theorems of Gauss* 609
Paul Slepian, *On the Lebesgue area of a doubled map* 613
Otto Szász and Nelson Paul Yeardley, Jr., *The representation of an analytic function by general Laguerre series* 621
Alan C. Woods, *On two-dimensional convex bodies* 635