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**PROOF OF THE FUNDAMENTAL THEOREM ON IMPLICIT
FUNCTIONS BY USE OF COMPOSITE GRADIENT
CORRECTIONS**

WILLIAM L. HART AND T. S. MOTZKIN

PROOF OF THE FUNDAMENTAL THEOREM ON IMPLICIT FUNCTIONS BY USE OF COMPOSITE GRADIENT CORRECTIONS

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1. Introduction. Many methods have been employed for establishing the classical result, Theorem 2.1, concerning the existence of functions $x_i(t)$ satisfying a system

$$(1.1) \quad f_j(x; t) = 0, \quad (j = 1, 2, \dots, n)$$

of n equations in n unknowns $(x_1, \dots, x_n) = x$ with $(t_1, \dots, t_p) = t$, where all variables and functions are real valued, and $f_j(\alpha; \beta) = 0$. The object of this article is to present a new proof of the theorem by a constructive method of successive approximations involving corrections related to the gradients in x -space of the functions $f_j(x; \beta)$.

To establish Theorem 2.1, a sequence $x^{(m)}(t)$ with $x^{(0)}(t) = \alpha$ will be defined, where $x^{(m)}(t)$ is obtained by adding to $x^{(m-1)}(t)$ a vector correction $\Delta x^{(m-1)}(t)$ which is equal to a certain constant, ρ , times the vector sum of corrections parallel to the gradients of the $f_j(x; \beta)$ at $x = \alpha$. The vector $\Delta x^{(m-1)}(t)$, for a fixed t , is a special case of the corresponding correction of an iterative process for solving a general system $g_j(x) = 0$, ($j = 1, \dots, k$), $k \geq n$, introduced by the authors in a previous article [2].

For a particular system (1.1), the method of the present paper would be applicable to obtaining values of the $x_i(t)$ by use of a digital computing machine for any t sufficiently near $t = \beta$. Section 6 in [2] describes a related small arc method with the same objective; the two methods differ in the values of the arguments used in fundamental matrices which appear with similar roles in [2] and below. The method of [2] might be superior computationally to the method of the present paper. However, in § 6 in [2], Theorem 2.1 below was employed as a starting point. Thus the present paper shows that the composite gradient method is effective to establish the supporting Theorem 2.1 as well as the related small arc method of [2] for computing values of the implicit functions.

In connection with the present article, it is pertinent to mention the proof of Theorem 2.1 by E. Goursat, [1], extended by William L. Hart, [3] and [4], to various infinite systems. In the Goursat method for (1.1), a system

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$$(1.2) \quad x_j = \Phi_j(x; t) \quad (j = 1, 2, \dots, n)$$

equivalent to (1.1) is constructed by use of the inverse of the matrix¹ $A = (a_{ij})$, $a_{ij} = \partial f_j(\alpha; \beta) / \partial x_i$; then a solution $x(t)$ of (1.2) is defined by applying the method of successive substitutions to (1.2). In contrast, under the same hypotheses as those of Goursat, § 2 employs a system (1.2) constructed by direct use of A , without forming its inverse. This feature might be of computational advantage. In case $n = 1$, the present method with $\rho = 1$ is identical with the Goursat method.

Either Goursat's method or the present method can also be regarded as a constructive algorithm solving the problem of elimination of $n - 1$ variables x_1, \dots, x_{n-1} from n equations $f_j(x; t) = 0$ leading to a relation (such as $x_n = x_n(t)$) between the remaining variables (x_n, t_1, \dots, t_p) .

The problem of solving $y = F(x; u)$, $F = (F_1, \dots, F_n)$ and $y = (y_1, \dots, y_n)$, by $x = \phi(y; u)$ (inversion with and without parameters), for nonzero Jacobian F_x , is only apparently more general than the solution of (1.1) (to subsume it set $t = (y, u)$, $f = F - y$) and thus is equally amenable to our iterative procedure.

2. Construction of a system (1.2) equivalent to (1.1). We shall consider (1.1) subject to the following hypotheses:

$$(2.1) \quad \left\{ \begin{array}{l} \text{The } f_j \text{ are continuous, and all derivatives } \partial f_j / \partial x_i \text{ exist and} \\ \text{are continuous in some open neighborhood } \Omega \text{ of } (x = \alpha; t = \beta). \end{array} \right.$$

$$(2.2) \quad f_j(\alpha; \beta) = 0 \quad (j = 1, 2, \dots, n).$$

$$(2.3) \quad \text{The matrix } A = (a_{ij}), a_{ij} = \partial f_j(\alpha; \beta) / \partial x_i, \text{ is nonsingular.}$$

In x -space, let the positive gradient of $f_j(x; \beta)$ at $x = \alpha$ be defined as having the magnitude $(\sum_{i=1}^n a_{ij}^2)^{1/2}$, nonzero because of (2.3), and the direction angles Ψ_{ij} specified by

$$(2.4) \quad \cos \Psi_{ij} = a_{ij} w_j^{-1}; \quad w_j = \left(\sum_{i=1}^n a_{ij}^2 \right)^{\frac{1}{2}}.$$

For any $(x; t)$ and each j define, formally, a vector correction $\Delta_j x$ for x , where x is considered an approximation to a solution of $f_j(x; t) = 0$, by specifying the i th component $\Delta_j x_i$ of $\Delta_j x$ as follows:

$$(2.5) \quad \Delta_j x_i = -\rho f_j(x; t) w_j^{-1} \cos \Psi_{ij},$$

with a constant $\rho > 0$ to be restricted later. Then define the composite vector correction Δx for x , considered now as an approximation to a solution of (1.1), by specifying for Δx the i th component

¹ Capital italic letters represent n by n matrices. The transpose of a matrix A is denoted by A' . We treat x as a one-rowed matrix.

$$(2.6) \quad \Delta x_i = \sum_{j=1}^n \Delta_j x_i ; \text{ or, } \Delta x = \sum_{j=1}^n \Delta_j x .$$

By use of (2.6) we introduce, formally, a sequence $x^{(m)}(t)$ of approximations to a solution of (1.1) :

$$(2.7) \quad x^{(0)}(t) = \alpha ; x^{(m)}(t) = x^{(m-1)}(t) + \Delta x^{(m-1)}(t), \quad m > 0 .$$

From (2.5), the i th coordinate $x_i^{(m)}(t)$ is given by

$$(2.8) \quad x_i^{(m)}(t) = x_i^{(m-1)}(t) - \rho \sum_{j=1}^n a_{ij} w_j^{-2} f_j(x^{(m-1)}(t) ; t) .$$

Let the components Φ_i of a vector $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n)$ be defined by

$$(2.9) \quad \Phi_i(x ; t) = x_i - \rho \sum_{j=1}^n a_{ij} w_j^{-2} f_j(x ; t) .$$

Then (2.7) is the sequence of approximations $x^{(m)}(t)$ arising if the method of successive substitutions, with $x^{(0)}(t) = \alpha$, is used to seek a solution of the system

$$(2.10) \quad x = \Phi(x, t) .$$

By use of (2.1) and (2.3) we find that (1.1) and (2.10) are equivalent systems.

We remark that, in x -space, Δx of (2.6) is invariant under an orthogonal transformation of coordinates and under an alteration of $f_j(x ; t)$ to $h_j f_j(x ; t)$ if $h_j \neq 0$. Thus, before considering the existence and convergence of sequence (2.7), we may assume that (1.1) has been altered by dividing $f_j(x ; t)$ by w_j of (2.4). Then, without change of notation, from (2.4) and (2.5) for all j we obtain

$$(2.11) \quad w_j = 1 ; \Delta_j x_i = -\rho a_{ij} f_j(x ; t) ; \sum_{i=1}^n a_{ij}^2 = 1 .$$

Note that AA' is symmetric and positive definite Hence there exist positive characteristic constants $\lambda_1, \dots, \lambda_n$ and an orthogonal matrix $S = (s_{pq})$ such that

$$(2.12) \quad SAA'S' = (\delta_{ij}\lambda_i) = D, \text{ where } \delta_{ii} = 1 \text{ and } \delta_{ij} = 0 \text{ if } i \neq j .$$

Now, in (1.1), let the coordinates be changed from (x_1, \dots, x_n) to $(z_1, \dots, z_n) = z$ by the orthogonal transformation $x = zS$. Then, with $g_j(z ; t) = f_j(x ; t)$ when $x = zS$, and $\alpha = \gamma S$ or $\gamma = \alpha S'$, (1.1) becomes

$$(2.13) \quad g_j(z ; t) = 0, \text{ where } g_j(\gamma ; \beta) = 0, \quad (j = 1, 2, \dots, n) .$$

If we let $b_{ij} = \partial g_j(\gamma ; \beta) / \partial z_i$ and $B = (b_{ij})$, we have

$$(2.14) \quad B = SA ; BB' = SAA'S' = D ; B'B = A'A .$$

From $B'B = A'A$ on comparing main diagonal terms we obtain $\sum_{i=1}^n b_{ij}^2 = \sum_{i=1}^n a_{ij}^2 = 1$, for all j . Hence, if sequence (2.7) is formed for (2.13) by use of the analogue of (2.11) in the z -coordinates, from (2.8) we arrive at

$$(2.15) \quad \begin{cases} z^{(0)}(t) = \gamma ; z^{(m)}(t) = z^{(m-1)}(t) + \Delta^{(m-1)}(t), m > 0 ; \\ \Delta z_i^{(m-1)}(t) = -\rho \sum_{j=1}^n b_{ij} g_j(z^{(m-1)}(t); t) . \end{cases}$$

On account of the invariant features which were mentioned concerning the gradient corrections $\Delta x^{(m-1)}(t)$ of (2.7) for (1.1), it follows that the existence of all $z^{(m)}(t)$ for any t is equivalent to the existence of all $x^{(m)}(t)$ and that $x^{(m)}(t)$ and $z^{(m)}(t)$ represent the same point. We shall find it convenient to discuss $z^{(m)}(t)$ instead of $x^{(m)}(t)$.

We introduce the functions

$$(2.16) \quad \phi_h(z; t) = z_h - \rho \sum_{j=1}^n b_{hj} g_j(z; t) \quad (h = 1, \dots, n) ,$$

and consider the following system, obtained as in (2.10), which is equivalent to (2.13):

$$(2.17) \quad z_h = \phi_h(z; t) \quad (h = 1, \dots, n) .$$

In (2.17) the ϕ_h and all derivatives $\partial \phi_h / \partial z_i$ are continuous when $(z; t)$ is in Ω , now defined with coordinates $(z; t)$. With $\phi = (\phi_1, \dots, \phi_n)$, sequence (2.15) can be written

$$(2.18) \quad z^{(0)}(t) = \gamma ; z^{(m)}(t) = \phi(z^{(m-1)}(t); t) .$$

From (2.16) and $BB' = D$ we obtain $\frac{\partial \phi_h}{\partial z_i} = \delta_{hi} - \rho \sum_{j=1}^n b_{hj} \frac{\partial g_j}{\partial z_i}$;

$$(2.19) \quad \frac{\partial \phi_h(\gamma; \beta)}{\partial z_h} = 1 - \rho \sum_{j=1}^n b_{hj} b_{hj} = 1 - \rho \lambda_h ;$$

$$(2.20) \quad \frac{\partial \phi_h(\gamma; \beta)}{\partial z_i} = 0 , \quad \text{if } h \neq i .$$

Let $\mu_i = 1 - \rho \lambda_i$ and $\sigma_\rho = \max_{i \leq n} |\mu_i|$. Since $(\lambda_1, \dots, \lambda_n)$ are the characteristic constants of AA' and $\sum_{i=1}^n a_{ij}^2 = 1$ for all j , we have

$$\sum_{i=1}^n \lambda_i = n ; \quad 0 < \lambda_i < n , \text{ if } n > 1 .$$

Then the following lemma can be proved easily as in [2]².

LEMMA 2.1. *In order that $\sigma_\rho < 1$, it is necessary that $0 < \rho < 2$, and it is sufficient that*

² See formulas (4.16)-(4.18) in [2] with $r = \omega = n$.

$$0 < \rho \leq 2/n \qquad (0 < \rho < 2/n \text{ if } n = 1) .$$

Moreover, the minimum value of σ_ρ occurs for a single value $\rho = \rho_0$ where

$$\frac{2}{n} < \rho_0 < \frac{2(n-1)}{n} \quad \text{if } n > 2, \text{ and } \rho_0 = 1 \text{ if } n \leq 2 .$$

For each t , and $i = 1, 2, \dots, n$, let $(z = \xi^{(i)} ; t)$ be a point in Ω and define

$$(2.21) \qquad v_{hi}(t) = \frac{\partial \phi_i(\xi^{(i)} ; t)}{\partial z_h} - \delta_{hi}(1 - \rho \lambda_i) .$$

Let $V(t) = (v_{hi}(t))$, and introduce the following matrices :

$$(2.22) \qquad B_\rho = I - \rho D = (\delta_{hi}(1 - \rho \lambda_i)) ;$$

$$(2.23) \qquad U(t) = B_\rho V'(t) + V(t)B_\rho + V(t)V'(t) = (u_{ij}(t)) .$$

Note that $u_{ij}(t)$ is a polynomial with each term of degree 1 or 2 in the elements $v_{ij}(t)$ of $V(t)$. Let $H(t) = [\sum_{i,j=1}^n u_{ij}^2(t)]^{1/2}$.

LEMMA 2.2. *Select $\rho > 0$ so that $\sigma_\rho < 1$, and choose $\theta > 0$ with $\sigma_\rho < \theta < 1$. Then there exist $\varepsilon > 0$ and $\delta \leq \varepsilon, \delta > 0$, such that, if³ $\|t - \beta\| \leq \delta, \|z - \gamma\| \leq \varepsilon$, and all $\|\xi^{(i)} - \gamma\| \leq \varepsilon$ in (2.21), then the functions $g_j(z ; t), \partial g_j(z ; t)/\partial z_i$, and $z^{(1)}(t)$ exist and are continuous, and*

$$(2.24) \qquad \|z^{(1)}(t) - \gamma\| \leq \varepsilon(1 - \theta) ;$$

$$(2.25) \qquad 0 \leq H(t) \leq \theta^2 - \sigma_\rho^2 .$$

To establish Lemma 2.2 first notice that, if $t = \beta$ and all $\xi^{(i)} = \gamma$ in (2.21), then all $v_{ij}(t) = 0$ and thus all $u_{ij}(t) = 0$. Hence $\varepsilon > 0$ exists so that the specified conditions are satisfied by the $g_j, \partial g_j/\partial z_i$, and $H(t)$ if $\|z - \gamma\| \leq \varepsilon, \|t - \beta\| \leq \varepsilon$, and all $\|\xi^{(i)} - \gamma\| \leq \varepsilon$ in (2.21). From (2.18), $z^{(1)}(t) = \phi(\gamma ; t)$ and thus $z^{(1)}(\beta) = \gamma$. Hence, if δ is sufficiently small and $0 < \delta \leq \varepsilon$, we have (2.24) when $\|t - \beta\| \leq \delta$. This completes the proof of Lemma 2.2.

THEOREM 2.1. *Suppose that $\rho > 0$ and is such that $\sigma_\rho < 1$. Assume that (2.1), (2.2), and (2.3) are satisfied. Then there exist $\varepsilon > 0$ and $\delta > 0, \delta \leq \varepsilon$, such that, if $\|t - \beta\| \leq \delta$, all $x^{(m)}(t)$ of (2.7) exist, are continuous, and satisfy $\|x^{(m)}(t) - \alpha\| \leq \varepsilon$. Also there exists, uniformly for $\|t - \beta\| \leq \delta$,*

³ For any vector z we use $\|z\|$ for the length. Thus, $\|z\| = (\sum_{i=1}^n z_i^2)^{1/2}$.

$$\lim_{m \rightarrow \infty} x^{(m)}(t) = x(t),$$

where $x = x(t)$ satisfies (1.1). Moreover, if a point $(x; t)$ with $\|x - \alpha\| \leq \varepsilon$ and $\|t - \beta\| \leq \delta$ satisfies (1.1), then $x = x(t)$.

To establish Theorem 2.1, we shall prove the corresponding facts for the sequence $z^{(m)}(t)$ of (2.15) and system (2.17). Let $\rho, \theta, \varepsilon$, and δ be determined by Lemma 2.2 and, hereafter, assume that $\|t - \beta\| \leq \delta$. Then $z^{(0)}(t)$ and $z^{(1)}(t)$ exist in the region $\|z - \gamma\| \leq \varepsilon$; by (2.24), since $z^{(0)}(t) = \gamma$, the following inequalities are true when $k = 1$:

$$(2.26) \quad \|z^{(k)}(t) - \gamma\| \leq \varepsilon; \|z^{(k)}(t) - z^{(k-1)}(t)\| \leq \varepsilon \theta^{k-1} (1 - \theta).$$

Assume now, for $m > 1$, that $z^{(k)}(t)$ has been proved to exist, to be continuous, and to satisfy (2.26) when $k = 1, 2, \dots, (m-1)$. Then $z^{(m)}(t)$ exists and is continuous; also, by the mean value theorem with respect to (z_1, \dots, z_n) for fixed t ,

$$(2.27) \quad z_i^{(m)}(t) - z_i^{(m-1)}(t) = \phi_i(z^{(m-1)}(t); t) - \phi_i(z^{(m-2)}(t); t) \\ = \sum_{h=1}^n \frac{\partial \phi_i(\xi^{(m,i)}(t); t)}{\partial z_h} [z_h^{(m-1)}(t) - z_h^{(m-2)}(t)],$$

where $\xi^{(m,i)}(t)$ is a properly chosen point in z -space on the line segment joining $z^{(m-2)}(t)$ and $z^{(m-1)}(t)$. With $\xi^{(i)} = \xi^{(m,i)}(t)$ in (2.21), let $V(t)$ be the matrix with elements $v_{hi}(t)$, and let $U(t)$ be defined by (2.23). Note that $\|\xi^{(m,i)}(t) - \gamma\| \leq \varepsilon$. Then, from (2.27),

$$(2.28) \quad \begin{cases} z^{(m)}(t) - z^{(m-1)}(t) = (z^{(m-1)}(t) - z^{(m-2)}(t))(B_\rho + V(t)); \\ \|z^{(m)}(t) - z^{(m-1)}(t)\|^2 = (z^{(m)}(t) - z^{(m-1)}(t))(z^{(m)}(t) - z^{(m-1)}(t))' \\ = (z^{(m-1)}(t) - z^{(m-2)}(t))(B_\rho^2 + U(t))(z^{(m-1)}(t) - z^{(m-2)}(t))'. \end{cases}$$

On applying the Cauchy inequality twice⁴ to the term involving $U(t)$ in (2.28), we find

$$(2.29) \quad \|z^{(m)}(t) - z^{(m-1)}(t)\|^2 \leq \|z^{(m-1)}(t) - z^{(m-2)}(t)\|^2 [\sigma_\rho^2 + H(t)] \\ \leq \theta^2 \|z^{(m-1)}(t) - z^{(m-2)}(t)\|^2.$$

From (2.26) for $k = 1, 2, \dots, (m-1)$ and (2.29), we obtain (2.26) for $k = m$. Thus, by induction, all $z^{(m)}(t)$ are defined and satisfy (2.26) if $\|t - \beta\| \leq \delta$. From (2.26), the series $\sum_{m=1}^{\infty} [z_i^{(m)}(t) - z_i^{(m-1)}(t)]$ is termwise dominated by the series $\sum_{m=1}^{\infty} \varepsilon (1 - \theta)^{m-1}$, and hence converges uniformly. Thus the sequence $z^{(m)}(t)$ approaches a limit, $z(t)$, uniformly for $\|t - \beta\| \leq \delta$. Since all $z^{(m)}(t)$ are continuous, $z(t)$ is continuous. It follows from $z^{(m)}(t) = \phi(z^{(m-1)}(t); t)$ that $z = z(t)$ satisfies $z = \phi(z; t)$.

⁴ As follows: $[\sum_{i=1}^n a_i \sum_{j=1}^n u_{ij} a_j]^2 \leq (\sum_{i=1}^n a_i^2) (\sum_{i=1}^n \sum_{j=1}^n u_{ij} a_j)^2 \leq (\sum_{i=1}^n a_i^2)^2 \sum_{i,j=1}^n u_{ij}^2$.

To prove that $z(t)$ is the unique solution of (2.13), suppose that $(\hat{z}; t)$ satisfies (2.13) for $\|\hat{z} - \gamma\| \leq \epsilon$ and $\|t - \beta\| \leq \delta$ and assume that $\hat{z} \neq z(t)$. Then, from $\hat{z} = \phi(\hat{z}; t)$ and $z(t) = \phi(z(t); t)$, by details duplicating the proof of (2.29), we have

$$\|\hat{z} - z(t)\| \leq \theta \|\hat{z} - z(t)\| < \|\hat{z} - z(t)\|,$$

which contradicts the assumption that $\hat{z} \neq z(t)$. Hence the proof of Theorem 2.1 is complete, because the point $z^{(m)}(t)$ in n -space is the same point as $x^{(m)}(t)$, and the region $\|x - \alpha\| \leq \epsilon$ is the same as the region $\|z - \gamma\| \leq \epsilon$.

Note 2.1. With a different arrangement of details, we could arrive at Theorem 2.1 with rectangular neighborhoods $\{|t_i - \beta_i| \leq \delta \text{ for all } i\}$ and $\{|x_i - \alpha_i| \leq \epsilon \text{ for all } i\}$ replacing the spherical neighborhoods $\|t - \beta\| \leq \delta$ and $\|x - \alpha\| \leq \epsilon$.

Note 2.2. In use of the sequence $\{x^{(m)}(t)\}$ in any particular case to obtain approximate values of $x(t)$, flexibility is introduced through the presence of the somewhat arbitrary constant ρ . Greater flexibility could be introduced (as in § 5 of [2]) by permitting suitably restricted variation in ρ , with $\rho = \rho^{(m)}$ at the m th iteration; revised details would establish Theorem 2.1. with this change.

Note 2.3. Suppose that $(\alpha; \beta)$ is not a solution of (1.1.). With only (2.1) and (2.3) as hypotheses, there exists $\epsilon > 0$ so that the region $(\|x - \alpha\| \leq \epsilon, \|t - \beta\| \leq \epsilon)$ is in Ω and (2.25) is true when $\|t - \beta\| \leq \epsilon$ and all $\xi^{(i)}$ of (2.21) satisfy $\|\xi^{(i)} - \gamma\| \leq \epsilon$, as in Lemma 2.2. Now assume that

$$(2.30) \quad \|\phi(\alpha; \beta) - \alpha\| < \epsilon(1 - \theta).$$

Then, with $x^{(0)}(t) = \alpha$, there exists $\delta \leq \epsilon, \delta > 0$, such that, if $\|t - \beta\| \leq \delta$, $x^{(1)}(t)$ exists and

$$(2.31) \quad \|z^{(1)}(t) - \gamma\| = \|x^{(1)}(t) - \alpha\| = \|\phi(\alpha; t) - \alpha\| \leq \epsilon(1 - \theta),$$

which is (2.24). Thus, with hypothesis (2.30) replacing (2.2) and δ defined as above, $\{x^{(m)}(t)\}$ converges as specified in Theorem 2.1 even when $(\alpha; \beta)$ is not a solution of (1.1)

REFERENCES

1. É. Goursat, *Sur la théorie des fonctions implicites*, Bull. Soc. Math. France **31** (1903), 184-192. Also, cf. Gilbert A. Bliss, Princeton Colloquium Lectures on Mathematics, American Mathematical Society (1913), 16-18.
2. William L. Hart and Theodore S. Motzkin, *A composite Newton-Raphson gradient method*

for the solution of systems of equations, Pacific J. Math. **6** (1956), 691-707.

3. William L. Hart, *Differential equations and implicit functions in infinitely many variables*, Trans. Amer. Math. Soc. **18** (1917), 125-160.

4. -----, *Functions of infinitely many variables in Hilbert space*, Trans. Amer. Math. Soc. **22** (1922), 30-50.

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