Pacific Journal of Mathematics

MULTIPLICITY FREE REPRESENTATIONS OF FINITE GROUPS

GEORGE W. MACKEY

Vol. 8, No. 3 May 1958

MULTIPLICITY FREE REPRESENTATIONS OF FINITE GROUPS

GEORGE W. MACKEY

Introduction. Wigner in [5] has defined a finite group $\mathscr{G}$ to be simply reducible if (a) every conjugate class in $\mathscr{G}$ is self inverse and (b) for each two irreducible representations L and M of $\mathscr{G}$ the Kronecker product $L \otimes M$ is a direct sum of inequivalent irreducible representations. The principal result of [5] is a curious purely group theoretical characterization of simply reducible groups. For each $x \in \mathscr{G}$ let $v(x)$ denote the number of elements of $\mathscr{G}$ which commute with x and let $\zeta(x)$ denote the number of solutions of the equation $y^2 = x$. Then $\mathscr{G}$ is simply reducible if and only if $\sum_{x \in G} v(x)^2 = \sum_{x \in G} \zeta(x)^3$. As the author has shown in [3] this result may be "explained" as follows. Let $\tilde{\mathscr{G}}_3$ be the diagonal subgroup of $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$, that is the set of all x, y, z with $x = y = z$. Then it is easily seen that the number of $\tilde{\mathscr{G}}_3 : \tilde{\mathscr{G}}_3$ double cosets in $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ is equal to $\sum_{x \in G} v(x)^2$ while the number of self inverse $\tilde{\mathscr{G}}_3 : \tilde{\mathscr{G}}_3$ double cosets in $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ is equal to $\sum_{x \in G} \zeta(x)^3$. Thus Wigner's condition is equivalent to the condition that every $\tilde{\mathscr{G}}_3 : \tilde{\mathscr{G}}_3$ double coset be self inverse. On the other hand if H is an arbitrary subgroup of the finite group $\mathscr{G}$ and U^H is the corresponding permutation representation of $\mathscr{G}$ one can prove that every $H : H$ double coset is self inverse if and only if each irreducible component M_j of U^H occurs with multiplicity one and is such that the intertwining operators of M_j with $\bar{M}_j$ are symmetric. This result is a corollary of a general theorem on anti-symmetric intertwining numbers for induced representations and certain elementary lemmas. It leads easily to Wigner's theorem when applied to $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ and its diagonal subgroup.

Now of the two conditions in the definition of simple reducibility (b) is much the more interesting. Moreover, as we shall see, there are examples of groups which satisfy (b) and not (a). This suggests looking for a generalization of Wigner's theorem in which (a) is dropped or weakened. The way to such a generalization is suggested by the considerations of [3] and a simple observation which plays a vital role in Gelfand's work [1] on "spherical functions" on Lie groups. Slightly generalized[1] and then applied to finite groups this observation is the following. Let $x \to x^a$ be an involutory anti-automorphism of the finite group $\mathscr{G}$. Let H be a subgroup of $\mathscr{G}$ such that the $H : H$ double cosets are invariant under $x \to x^a$. Let $\mathscr{A}_H$ be the subalgebra of the

Received March 21, 1958.

[1] Compare Mautner [4].

group algebra of $\mathscr{G}$ consisting of all functions on $\mathscr{G}$ which are constant on each $H:H$ double coset. Then $\mathscr{A}_H$ is commutative. Since $\mathscr{A}_H$ is the commuting algebra of the permutation representation U^I of $\mathscr{G}$ defined by H, it follows that each irreducible component of U^I occurs with multiplicity one. Confronting this result with the theorem from [3] on U^I cited above, we are led at once to consider the possibility of rewriting [3] with x^{-1} replaced in appropriate places by x^a thus obtaining the indicated generalization of Wigner's theorem as well as a converse for the Gelfand observation.

It is the purpose of the present note to show that this rewriting can be done. It turns out that the necessary arguments differ but little from their counterparts in [3]. Accordingly the emphasis will be on the formulation of definitions and results and insofar as possible the reader will be referred to [3] for detailed proofs. We shall make no attempt to generalize Theorems 1 and 2 or §§ 5 and 6 of [3].

1. Symmetric and anti-symmetric intertwining numbers. Let $\mathscr{G}$ be a finite group and let a denote a fixed involutory anti-automorphism of $\mathscr{G}$. Let $x\to U_x$ be an arbitrary representation of $\mathscr{G}$ by linear transformations in a finite dimensional vector space $\mathscr{H}(U)$ over a field $\mathscr{F}$ whose characteristic is not two. Then $x\to(U_{x^a})^*$ is a representation of $\mathscr{G}$ whose space is the dual $\overline{\mathscr{H}(U)}$ of $\mathscr{H}(U)$. We shall denote this representation by U^a. It is clear that $U^{aa}=U$. Let T be an intertwining operator for U^a and U; that is a linear operator from $\mathscr{H}(U^a)=\overline{\mathscr{H}(U)}$ to $\mathscr{H}(U)$ such that $TU^a_x=U_xT$ for all x in $\mathscr{G}$. Then $T^*U_x{}^*=(U^a_x)^*T^*=U_{x^a}T^*$ for all x. Hence $T^*U_{x^a}^*=U_xT^*$ for all x. Hence T^* is also an intertwining operator for U^a and U. Setting $T=(T+T^*)/2+(T-T^*)/2$ we see that every intertwining operator T for U^a and U is uniquely of the form T_1+T_2 where $T_1{}^*=T_1$ and $T_2{}^*=-T_2$. Hence if we denote the dimension of the space $\mathscr{R}(U^a, U)$ of all intertwining operators for U^a and U by $\mathscr{J}(U^a, U)$ we have $\mathscr{J}(U^a, U)=\mathscr{J}_S(U^a, U)+\mathscr{J}_A(U^a, U)$ where $\mathscr{J}_S(U^a, U)$ is the dimension of the space of all intertwining operators T such that $T^*=T$ and $\mathscr{J}_A(U^a, U)$ is the dimension of the space of all intertwining operators T such that $T^*=-T$. These two dimensions will be referred to respectively as the symmetric and anti-symmetric intertwining numbers of U^a with U. Their sum as usual will be call the intertwining number of U^a with U.

LEMMA 1. *If U and V are representations of $\mathscr{G}$ and $U+V$ denotes their direct sum, then*

$$\mathscr{J}_S((U+V)^a, U+V)=\mathscr{J}_S(U^a, U)+\mathscr{J}_S(V^a, V)+\mathscr{J}(U^a, V)$$

$$\mathscr{J}_A((U+V)^a, U+V)=\mathscr{J}_A(U^a, U)+\mathscr{J}_A(V^a, V)+\mathscr{J}(U^a, V)$$

Proof. The corresponding proof in [3] proceeds through symmetric and anti-symmetric Kronecker products and hence does not apply here. However, it is readily converted into a direct proof which generalizes immediately to the situation at hand.

COROLLARY. *If* $C^a(U)=\mathscr{J}_S(U^a, U)-\mathscr{J}_A(U^a, U)$, *then* $C^a(U+V)=C^a(U)+C^a(V)$.

LEMMA 2. *Let* $\mathscr{F}$ *be algebraically closed and let* $\mathscr{G}=\mathscr{G}_1\times\mathscr{G}_2$ *where* $(\mathscr{G}_1\times e)^a=\mathscr{G}_1\times e$, $(e\times\mathscr{G}_2)^a=e\times\mathscr{G}_2$. *Let* U *and* V *be irreducible representations of* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *respectively. Then* $C^a(L\times M)=C^a(L)C^a(M)$.

Proof. See proof of Lemma 2 of [3].

COROLLARY. *The conclusion continues to hold if* U *and* V *are direct sums of irreducible representations.*

LEMMA 3. *Let* $\mathscr{F}$ *be algebraically closed, let* U *be a direct sum of irreducible representations, and let* b *denote an involutory anti-automorphism of* $\mathscr{G}$ *which commutes with* a. *Then* $C^a(U^b)=C^a(U)$.

Proof. Clearly we need only consider the case in which U is irreducible. If U and U^a are not equivalent then U^b and $U^{ab}=U^{ba}$ are not equivalent. Hence $C^a(U)=0$ and $C^a(U^b)=0$. If U and U^a are equivalent let T set up the equivalence. Then $TU_x=U^a_xT$ for all x. Hence for all x we have

$$U^*_{x^b}T^*=T^*U^{a*}_{x^b} \text{ or } U^b_xT^*=T^*U^{ab}_x=T^*U^{ba}_x .$$

Thus T^* sets up the equivalence between U^b and U^{ba}. Since $T=\pm T^*$ if and only if $(T^*)^*=\pm T^*$ the truth of the lemma follows at once.

LEMMA 4. *Let* $\mathscr{F}$ *be algebraically closed and let* G *be a subgroup of* $\mathscr{G}$ *such that* $G^a=G$. *Let* L *be an irreducible representation of* $\mathscr{G}$ *and suppose that the restriction* M *of* L *to* G *is a direct sum of inequivalent representations* M_j. *Then* $C^a(M_j)=C^a(L)$ *for all* j.

Proof. See proof of Lemma 4 of [3].

2. Multiplicity-free permutation representations. Let I be the one-dimensional identity representation of the subgroup G of the finite group $\mathscr{G}$. As in [3] we shall denote by U^I the representation of $\mathscr{G}$ induced by I, that is, the permutation representation of $\mathscr{G}$ defined by G.

LEMMA 5. *Let I and G be as just described. Let n_1 denote the number of $G:G^a$ double cosets which are invariant under a. Let n_2 denote the number of $G:G^a$ double cosets which are not invariant under a. Then $\mathscr{J}_S((U^I)^a U^I)=n_1+\frac{1}{2}n_2$ and $\mathscr{J}_A((U^I)^a, U^I)=n_2/2$.*

Proof. The proof is an obvious adaptation of the proof of Theorem 2′ of [3].

LEMMA 6. *Let $M=M_1+M_2+\cdots M_n$ where the M_j are irreducible representations of $\mathscr{G}$. Then $\mathscr{J}_A(M^a, M)=0$ if and only if for each j one of the two following conditions holds.*

(a) *$C^a(M_j)=1$ and M_j is not equivalent to M_k for any $k\neq j$.*

(b) *$C^a(M_j)=0$ and M_j^a is not equivalent to any M_k.*

Proof. See proof of Lemma 5 of [3].

We shall suppose henceforth that $\mathscr{F}$ is algebraically closed and that the characteristic of $\mathscr{F}$ does not divide the order of $\mathscr{G}$. Hence in particular every representation U of $\mathscr{G}$ will be a direct sum of irreducible representations. When these irreducible components are mutually inequivalent, we shall say that U is multiplicity-free.

THEOREM 1. *Let G be a subgroup of $\mathscr{G}$ such that $G^2=G$. Let I be the identity representation of G. Then the following statements are equivalent:*

(a) *$\mathscr{J}_A((U^I)^a, U^I)=0$.*

(b) *Every $G:G$ double coset is invariant under a.*

(c) *U^I is multiplicity-free and each irreducible component M of U^I is such that $C^a(M)=1$.*

Proof. The equivalence of (a) and (b) follows at once from Lemma 5. On the other hand, it is easy to verify that U^I and $(U^I)^a$ are equivalent. Hence, when we apply Lemma 6 to U^I, alternative (b) is impossible, and the equivalence of (a) and (c) follows at once.

COROLLARY (of proof). *The theorem remains true if the hypothesis that $G^a=G$ is replaced by the hypothesis that $(U^I)^a$ and U^I are equivalent and in* (b) *$G:G$ is replaced by $G:G^a$.*

THEOREM 2. *Let $\tilde{\mathscr{G}}_3$ denote the subgroup of $\mathscr{G}\times\mathscr{G}\times\mathscr{G}$ consisting of all x, y, z, with $x=y=z$. Then the following two statements are equivalent:*

(a) *For each pair L, M of irreducible representations of $\mathscr{G}$ the Kronecker product $L\otimes M$ is multiplicity-free and L^a and L are equivalent.*

(b) *Every* $\tilde{\mathscr{G}}_3 : \tilde{\mathscr{G}}_3$ *double coset in* $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ *is invariant under* a.

Proof. We apply Theorem 1 with $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ playing the role of $\mathscr{G}$ and $\tilde{\mathscr{G}}_3$ that of G. Then condition (b) of the present theorem becomes exactly condition (b) of Theorem 1. Now the most general irreducible representation of $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ is $L \times M \times N$ where L, M, and N are irreducible representations of $\mathscr{G}$. Moreover by the Frobenius reciprocity theorem U^I contains $L \times M \times N$ just as often as $L \otimes M \otimes N$ contains the identity; that is, just as often as $L \otimes M$ contains $\overline{N}$. ($\overline{N}$ is N^b where $b(x) = x^{-1}$). Thus U^I is multiplicity-free if and only if each $L \otimes M$ is multiplicity-free. In other words the first part of condition (c) of Theorem 1 is equivalent to the first part of condition (a) of the present theorem. On the other hand, $C^a(L \times M \times N) = C^a(L)C^a(M)C^a(N)$ by Lemma 2. Thus the second part of condition (c) is equivalent to the condition that $C^a(L)C^a(M)C^a(N) = 1$ whenever $\overline{N}$ is a component of $L \otimes M$. But if the first part of the condition is satisfied, then, by Lemma 4, $C^a(L)C^a(M) = C^a(L \times M) = C^a(\overline{N})$ for all $\overline{N}$ in the decomposition of $L \otimes M$. Moreover, by Lemma 3, $C^a(\overline{N}) = C^a(N)$. Thus, in the presence of the first part, the second part of condition (c) is equivalent to the condition that $C^a(N)^2 = 1$ for all N occurring in the decomposition of $L \otimes M$. But $C^a(N)^2 = 1$ if and only if $C^a(N) = \pm 1$; that is, if and only if $N \cong N^a$. The truth of Theorem 2 follows at once.

3. Generalizations of Wigner's condition. We define $v(x)$ for $x \in \mathscr{G}$, just as in [3], [5] and the introduction to the present paper. We replace the function ζ however by a function ζ_a which we define as follows. For each $x \in \mathscr{G}$, $\zeta_a(x)$ is the number of elements z in $\mathscr{G}$ for which $z(z^a)^{-1} = x$. Theorem 5 of [3] relating $\sum_{x \in \mathscr{G}} v(x)^n$ to the number of $\tilde{\mathscr{G}}_{n+1} : \tilde{\mathscr{G}}_{n+1}$ double cosets in $\mathscr{G}_{n+1}$ can be used just as it stands but we need a generalization of Theorem 6 giving us information about $\sum_{x \in \mathscr{G}} \zeta_a(x)^{n+1}$. Here $\mathscr{G}_{n+1}$ is the direct product of $\mathscr{G}$ with itself $n+1$ times and $\tilde{\mathscr{G}}_{n+1}$ is the "diagonal" subgroup of $\mathscr{G}_{n+1}$.

THEOREM 3. *Let* $\mathscr{G}$ *be of order* h *and let* $\mathscr{G}_{n+1}$, $\tilde{\mathscr{G}}_{n+1}$ *and* ζ_a *be defined as above. Then the following three numbers are all equal.*

(a) $(1/h)\sum_{x \in \mathscr{G}} \zeta_a(x)^{n+1}$

(b) *The number of* $\tilde{\mathscr{G}}_{n+1} : \tilde{\mathscr{G}}_{n+1}$ *double cosets in* $\mathscr{G}_{n+1}$ *which are invariant under* a.

(c) *The number of* a *invariant orbits in* $\mathscr{G}_n$ *under the group of inner automorphisms defined by members of* $\tilde{\mathscr{G}}_n$.

Proof. We verify at once that if $x_1, x_2, \cdots, x_{n+1}$ and $y_1, y_2, \cdots, y_{n+1}$ are in the same $\tilde{\mathscr{G}}_{n+1}:\tilde{\mathscr{G}}_{n+1}$ double coset then $x_1x_{n+1}^{-1}$, $x_nx_{n+1}^{-1}$, and $y_1y_{n+1}^{-1}$, $\cdots$, $y_ny_{n+1}^{-1}$ are in the same orbit in $\mathscr{G}_n$. The mapping so defined is easily seen to be one-to-one and onto from double cosets to orbits and to carry the a invariant double cosets onto the a invariant orbits. Thus (b) and (c) are equal. We now apply Lemma 6 of [2] with $\mathscr{G}=\mathscr{G}_n$, $y(x_1, x_2, \cdots, x_n)=y^{-1}x_1y, y^{-1}x_2y, \cdots, y^{-1}x_ny$, and $T(x_1 x_2, \cdots, x_n)=x_1^a, x_2^a, \cdots, x_n^a$. (In the statement of the lemma, it is assumed that T commutes with y for all y and this condition does not hold here. However, the proof continues to hold under the weaker hypothesis that T takes each orbit in S into itself and that condition does hold here. We remark that the proof in question contains a typographical error. In the second line from the bottom on page 399, $T(s)$ should be followed by "is not" rather than "is".) Here $p(y)$ is the number of $x_1, x_2, \cdots, x_n$ such that $y^{-1}x_jy=x_j^a$ for $j=1, 2, \cdots, n$. Hence $p(y)=p_1(y)^n$ where $p_1(y)$ is the number of x in $\mathscr{G}$ with $y^{-1}xy=x^a$. But $y^{-1}xy=x^a$ if and only if $xy((xy)^a)^{-1}=y(y^a)^{-1}$. Moreover for fixed y the number of x such that $xy((xy)^a)^{-1}=y(^a)^{-1}$ is equal to the number of z such that $z(z^a)^{-1}=y(y^a)^{-1}$. Thus $p_1(y)=\zeta_a(y(y^a)^{-1})$ and $p(y)=\zeta_a(y(y^a)^{-1})^n$. Hence (c) is equal to

$$1/h\sum_{y\in\mathscr{G}}\zeta_a(y(y^a)^{-1})^n=1/h\sum_{z\in\tilde{\mathscr{G}}}(\zeta_a(z)^n(\text{no. of } y \text{ with } y(y^a)^{-1}=z)$$

$$=1/h\sum_{z\in\mathscr{G}}(\zeta_a(z))^{n+1}.$$

Thus (a)=(c) and the theorem is proved.

As an immediate consequence of the theorem just proved and Theorem 5 of [3], we obtain the following.

THEOREM 4. *Let* $\mathscr{G}$, a, v, ζ_a, $\mathscr{G}_n$, $\tilde{\mathscr{G}}_n$ *be as above. Then for all* $x=1, 2, \cdots$, *we have*

$$\sum_{x\in\mathscr{G}}\zeta_a(x)^{n+1}\leqq\sum_{x\in\mathscr{G}}v(x)^n\ .$$

Equality holds if and only if every $\tilde{\mathscr{G}}_{n+1}:\tilde{\mathscr{G}}_{n+1}$ double coset in $\mathscr{G}_{n+1}$ is invariant under a.

COROLLARY. *Condition* (a) *of Theorem* 2 *is satisfied if and only if* $\sum_{x\in\mathscr{G}}\zeta_a(x)^3=\sum_{x\in\mathscr{G}}v(x)^2$. *Theorem* 2 *of Wigner's paper* [5] *is this corollary with* $x^a=x^{-1}$. *If we take* $n=1$ *in Theorem* 3 *we conclude that* $1/h\sum_{x\in\mathscr{G}}\zeta_a(x)^2$ *is the number of conjugate classes in* $\mathscr{G}$ *which are invariant under* a. *When* $x^a=x^{-1}$, *this reduces to Theorem* 1 *of* [5].

Obvious adaptations of the proofs of Theorems 8 and 9 of [3] yield proofs of the two following theorems.

THEOREM 5. *If $\mathscr{G}$, a, ζ_a, and v are as above, then the following conditions are equivalent.*

(a) $\sum_{x\in\mathscr{G}}\zeta_a(x)^2=\sum_{x\in\mathscr{G}}v(x)$.

(b) *Every class in $\mathscr{G}$ is invariant under $x\to x^a$.*

(c) *For every representation L of $\mathscr{G}$, L and L^a are equivalent.*

THEOREM 6. *If $\mathscr{G}, a, \zeta_a$, and v are as above, then the following conditions are equivalent.*

(a) *For some integer $n\geq 3$, $\sum_{x\in\mathscr{G}}\zeta_a(x)^{n+1}=\sum_{x\in\mathscr{G}}v(x)^n$.*

(b) *For every positive integer n, $\sum_{x\in\mathscr{G}}\zeta_a(x)^{n+1}=\sum_{x\in\mathscr{G}}v(x)^n$*

(c) *$\mathscr{G}$ is commutative and a is the identity.*

4. Some examples. If $\mathscr{G}$ is abelian and not every element is of order two, then the Kronecker products of irreducible representations of $\mathscr{G}$ are trivially multiplicity-free but Wigner's theorem does not apply since the classes in $\mathscr{G}$ are not self inverse. On the other hand, taking a to be the identity we find that Corollary 2 of Theorem 4 of the present paper does apply. A slightly less trivial example may be constructed by letting $\mathscr{G}$ be the direct product $H\times K$ where H is abelian, K is a group for which the equivalent conditions of Wigner's theorem are satisfied, and $(\xi, x)^a=\xi, x^{-1}$. Still less trivial examples may be found by applying the following theorem:

THEOREM 7. *Let $\mathscr{G}$ have a commutative normal subgroup N such that $\mathscr{G}/N$ is of order 2. Then the Kronecker product of any two irreducible representations of $\mathscr{G}$ is multiplicity-free.*

Proof. The regular representation of $\mathscr{G}$ is a direct sum of the representations U^χ where U^χ denote the representation of $\mathscr{G}$ induced by the character χ of N. Since the U^χ are all two-dimensional, every irreducible representation of $\mathscr{G}$ is either a U^χ or is one-dimensional. Thus to prove the theorem it will suffice to show that $U^{\chi_1}\otimes U^{\chi_2}$ is multiplicity-free whenever U^{χ_1} and U^{χ_2} are irreducible. Let β denote the unique involutory automorphism of N such that $\beta(x)=yxy^{-1}$ for all $x\in N$ and all $y\in\mathscr{G}-N$. Let α denote the involutory automorphism of the character group $\hat{N}$ of N defined by β. Then (see [2]) the intertwining of U^{χ_1} and U^{χ_2} is equal to one, zero, or two, according as χ_1 is equal

to one, neither, or both of χ_2 and $\alpha(\chi_2)$. Thus U^χ is irreducible if and only if $\chi \neq \alpha(\chi)$ and when U^χ is reducible, it is multiplicity-free. Moreover if U^{χ_1} and U^{χ_2} are both reducible, then $U^{\chi_1}+U^{\chi_2}$ is multiplicity-free whenever $\chi_1 \neq \chi_2$. Again by a result of [2], $U^{\chi_1} \otimes U^{\chi_2} \cong U^{\chi_1\chi_2}+U^{\chi_1\alpha(\chi_2)}$. But if U^{χ_1} and U^{χ_2} are irreducible, then $\chi_1 \neq \alpha(\chi_1)$, $\chi_2 \neq \alpha(\chi_2)$. Hence $\chi_1\chi_2 \neq \chi_1\alpha(\chi_2)$ and $\chi_1\chi_2 \neq \alpha(\chi_1\alpha(\chi_2)) = \alpha(\chi_1)\chi_2$. Hence by the foregoing, $U^{\chi_1} \otimes U^{\chi_2}$ is multiplicity-free and the proof is complete.

Let $\mathscr{G}$ be the semi-direct product of a cyclic group N of order n and a cyclic group K of order 2 where $x\xi x^{-1} = \xi^{-1}$ for $\xi \in N$ and x the non identity element of K; that is, let $\mathscr{G}$ be the dihedral group of order $2n$. It is readily verified that the classes in $\mathscr{G}$ are all self inverse. Hence, by Theorem 7, $\mathscr{G}$ satisfies the equivalent conditions of Wigner's theorem. Now let N be a cyclic group of order $2n$, let K be as before, and let $\mathscr{G}$ be the set of all pairs ξ, x where $\xi \in N$ and $x \in K$. Let θ be a generator of N. We define a multiplication in $\mathscr{G}$ by setting

$$(\xi_1, x_1)(\xi_2, x_2) = \xi_1 x_1(\xi_2) h(x_1, x_2), x_1x_2$$

where $x_1(\xi_2) = \xi_2$ or ξ_2^{-1} according as $x_1 = e$ (the identity) or not, and $h(x_1, x_2) = \theta^n$ or e depending upon whether $x_1 = x_2 = e$ or not. It is readily verified that $\mathscr{G}$ is a group with respect to this multiplication. It is the well known dicyclic group of order $4n$. Let the elements of K be z and e. Then $(e, z)^{-1} = \theta^n, z$ but the class containing (e, z) is easily seen to consist just of the elements of the form ξ^2, z where $\xi \in N$. Hence if every class in $\mathscr{G}$ is self inverse, $a^n = a^{2k}$ for some k. Hence $n-2k$ is a multiple of $2n$ and n is even. Thus if n is odd, the dicyclic group of order $4n$ has a non self inverse class. On the other hand let $(\xi, x)^a = x(\xi), x$. Then it is easy to check that a is an involutory anti-automorphism of $\mathscr{G}$ which takes every class into itself. Hence $\mathscr{G}$ does not satisfy the equivalent conditions of Wigner's theorem but by Theorem 5 and 7 does satisfy the equivalent conditions of Theorem 2.

REFERENCES

1. I. M. Gelfand, *Spherical functions on symmetric Riemann spaces*, Dokl. Akad. Nauk. SSSR (N. S.), **70** (1950), 5-8.

2. G. W. Mackey, *On induced representations of groups*, Amer. J. Math., **73** (1951), 576-592.

3. G. W. Mackey, *Symmetric and anti symmetric Kronecker squares of induced representations of finite groups*, Amer. J. Math., **75** (1953), 387-405.

4. F. I. Mautner, *Fourier analysis and symmetric spaces*, Proc. Nat. Acad. Sci. USA, **37** (1951), 529-533.

5. E. P. Wigner, *On representations of certain finite groups*, Amer. J. Math., **63** (1941), 57-63.

HARVARD UNIVERSITY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics
Vol. 8, No. 3 May, 1958