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1. Introduction. P. M. Whitman [4] defined an ordering of the set of lattice polynomials generated by a set of unrelated elements. R. P. Dilworth [3] generalized this ordering to apply to the case of lattice polynomials generated by an arbitrary partly ordered set $P$. Dilworth proved that this ordering gives a lattice isomorphic to the free lattice, $\text{FL}(P)$, which is generated by $P$ and which preserves bounds of pairs of elements of $P$. R. A. Dean [2] considered the ordering of lattice polynomials which preserves order of pairs of elements in $P$ and which leads to the completely free lattice $\text{CF}(P)$. He shows that $\text{CF}(P)$ and $\text{FL}(P)$ are identical in the case in which $P$ is a set of unrelated chains.

This article is a further study of $\text{FL}(P)$ where $P$ is a set of unrelated chains. An arbitrary element of $P$ will be denoted by $p$ or $q$. The set of chains consisting of

$$a_{11} < a_{12} < \cdots < a_{1n_1}; \quad a_{21} < a_{22} < \cdots < a_{2n_2}; \quad \cdots; \quad a_{m1} < a_{m2} < \cdots < a_{mn_m};$$

where $a_{ij}$ and $a_{kl}$ are unrelated when $i \neq k$, will be denoted by $n_1 + n_2 + \cdots + n_m$.

**DEFINITION 1.** Lattice polynomials over $P$ are defined inductively as follows.

(1) The elements $p, q, \cdots$, of $P$ are lattice polynomials over $P$.

(2) If $A$ and $B$ are lattice polynomials over $P$, then so are $A \cup B$ and $A \cap B$.

**DEFINITION 2.** The rank, $r(A)$, of a lattice polynomial $A$ is defined inductively as follows.

(1) $r(A) = 0$ if and only if $A$ is in $P$.

(2) $r(A \cup B) = r(A \cap B) = r(A) + r(B) + 1$.

**DEFINITION 3.** The dual polynomial, $A'$, of a polynomial $A$ of $\text{FL}(n_1 + n_2 + \cdots + n_m)$ is defined inductively as follows.

(1) If $A = a_{ij}$, then $A' = a_{ij}(n_i - j + 1)$.

(2) If $A = A_i \cup A_j$, then $A' = A_i' \cap A_j'$.

(3) If $A = A_i \cap A_j$, then $A' = A_i' \cup A_j'$.
From Definition 2, Lemma 1, and Lemma 2 of [2], and the fact that
FL(P) and CF(P) are identical in the case under consideration, we have
the following theorem.

**THEOREM 1.** Let P be a partly ordered set consisting of a set of un-
related chains. In FL(P), each relation $A \supseteq B$ is one of six types. These
types and necessary and sufficient conditions which apply to each case are
the following.

(A) $p \supseteq q$ if and only if $p \supseteq q$ in $P$.
(B) $p \supseteq B_1 \cap B_2$ if and only if $p \supseteq B_1$ or $p \supseteq B_2$.
(C) $A_1 \cup A_2 \supseteq p$ if and only if $A_1 \supseteq p$ or $A_2 \supseteq p$.
(D) $A \supseteq B_1 \cup B_2$ if and only if $A \supseteq B_1$ and $A \supseteq B_2$.
(E) $A_1 \cap A_2 \supseteq B$ if and only if $A_1 \supseteq B$ and $A_2 \supseteq B$.
(F) $A_1 \cup A_2 \supseteq B_1 \cap B_2$ if and only if $A_1 \supseteq B_1 \cap B_2$ or $A_2 \supseteq B_1 \cap B_2$ or
$A_1 \cup A_2 \supseteq B_1$ or $A_1 \cup A_2 \supseteq B_2$.

2. **FL(2+2).** Let $a_1 < a_2$ and $b_1 < b_2$ be the generators of FL(2+2).
The notation of the elements of FL(2+2) is defined in the following
recursive manner.

$$A_1 = a_2, \quad B_1 = b_2$$

and, for $n>1$,

$$A_n = a_2 \cap (a_1 \cup B_{n-1}), \quad B_n = b_2 \cap (b_1 \cup A_{n-1}).$$

$$C_n = a_1 \cup B_n.$$

$$D_n = b_1 \cup A_n.$$

$$P_n = A_n \cup B_n.$$

$$Q_n = C_n \cap D_n.$$

$$M_1 = a_1 \cup b_1.$$

$$M_2 = (a_2 \cap b_2) \cup a_1 \cup b_1.$$

$$V_1 = b_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1).$$

$$V_2 = (a_2 \cap b_2) \cup (b_2 \cap (a_1 \cup b_1)).$$

$$V_3 = b_2 \cap (a_1 \cup b_1).$$

$$W_1 = a_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1).$$

$$W_2 = (a_2 \cap b_2) \cup (a_2 \cap (a_1 \cup b_1)).$$

$$W_3 = a_2 \cap (a_1 \cup b_1).$$

These elements and their dual elements are all the elements of
FL(2+2). This is shown by considering the $\cap$ and $\cup$ tables of the
above elements and their dual elements. Since the generators of FL(2+2)
are among these elements and their duals, in order to show that these
are all the elements of FL(2+2) it is sufficient to show that this set is closed under \( \cup \) and \( \cap \). Actually, it is sufficient to show the set consisting of the above elements and their duals is closed under \( \cup \) (or \( \cap \)). This follows from the fact that the intersection of two elements, \( A \cap B \), can be expressed as \( A \cap B = (A')' \cap (B')' = (A' \cup B')' \), the latter being found from the \( \cup \) table. The diagram of FL(2+2) as shown in Figure 1 is obtained from the relations found in the \( \cup (\cap) \) table. Rather than give the entire \( \cup \) table, the diagram of FL(2+2) is given and a typical element, \( A_1 \cup B_1 \), of the \( \cup \) table is considered. The other parts of the \( \cup \) table are obtained in a similar manner. First, we consider the following theorem.

**Theorem 2.** In FL(2+2) we have \( A_1 > A_2 > \ldots, B_1 > B_2 > \ldots, C_1 > C_2 > \ldots, \) and \( D_1 > D_2 > \ldots \).

**Proof.** The proof of this theorem is similar to the proof, in §4 of [2], that FL(2+2) contains four infinite chains. In [2] the symbols, \( A_n, B_n, C_n, \) and \( D_n \) represent the same elements as \( A_{2n-1}, D_{2n-1}, B_{2n}, \) and \( C_{2n} \), respectively, of this paper. Thus we conclude from the results of [2] that \( A_1 > A_3 > A_5 > \ldots, B_2 > B_4 > B_6 > \ldots, C_3 > C_5 > C_7 > \ldots, \) and \( D_1 > D_3 > D_5 > \ldots \). The conclusion of this theorem follows in a similar manner.

We now show that

\[
A_1 \cup B_1 = \begin{cases} P_i, & i=j, \\ D_i, & i<j, \\ C_j, & j<i. \end{cases}
\]

\( A_1 \cup B_1 = P_i \) by definition. Since \( B_i \geq b_i \), it follows by (D) of Theorem 1 that \( A_1 \cup B_j \geq b_i \cup A_i = D_i \). Now consider the relation

![Diagram of FL(2+2)](attachment:diagram.png)
Since $A_n \geq A_m$, $n \leq m$ we have $b_i \cup A_i \geq b_i \cup A_{j-1}$, $i < j$. Hence by (F) of Theorem 1, $b_i \cup A_i \geq B_j$. Since $b_i \cup A_i \geq A_i$, it follows by (D) of Theorem 1 that $b_i \cup A_i \geq B_j \cup A$. This completes the proof that $D_i = b_i \cup A_i = B_j \cup A_i$, $i < j$.

It follows in a similar manner that $A_i \cup B_j = C_j$, $j < i$.

3. Order-convergence.

**Definition 4.** In a lattice, \{b_i\} is said to order-converge to $b$ if sequences \{u_i\} and \{v_i\} exist such that

$$v_i \geq v_{i+1} \geq b_{i+1} \geq u_{i+1} \geq u_i$$

for all $i$, and lub \{u_i\} = glb \{v_i\} = b.

As seen from Figure 1, or as can be shown directly using Theorem 1, it is clear that $A_n \geq W_1$ for each $n$, $B_n \geq V_1$ for each $n$, $C_n \geq M_2$ and $D_n \geq M_i$ for each $n$. Thus we conclude that $W_1$ is a lower bound to the set \{A_n\}, $V_1$ is a lower bound to the set \{B_n\}, and $M_2$ is a lower bound to each of the sets \{C_n\} and \{D_n\}.

**Theorem 3.** In FL(2+2) $W_1$ is glb \{A_n\}, $V_1$ is glb \{B_n\}, and $M_2$ is glb \{C_n\} and glb \{D_n\}.

**Proof.** Since each of $W_1$, $V_1$, and $M_2$ is a lower bound to the indicated sets, in order to prove the theorem it is sufficient to prove the following four statements.

1. If $A_n \geq K$ for each $n$, then $W_1 \geq K$.
2. If $B_n \geq K$ for each $n$, then $V_1 \geq K$.
3. If $C_n \geq K$ for each $n$, then $M_2 \geq K$.
4. If $D_n \geq K$ for each $n$, then $M_2 \geq K$.

The proof is as follows. Let $r(K) = 0$. If $A_n \geq K$ for each $n$, then $K = a_i$. In this case $W_1 \geq K$. Similarly if $B_n \geq K$ for each $n$, then $K = b_i$ and hence $V_1 \geq K$. If $C_n \geq K$ for each $n$, then $K = a_i$ or $K = b_i$. In either case $M_2 \geq K$. Similarly if $D_n \geq K$ for each $n$, then $M_2 \geq K$.

Proceeding by induction, we assume, when $r(K) < k$, that the four conditions (1), (2), (3), and (4) each hold. We now consider the cases when $r(K) = k$ and $K = K_1 \cap K_2$ or $K = K_1 \cup K_2$. First, let $K = K_1 \cap K$. If $A_n \geq K$ for each $n$, then $a_i \geq K$ and $a_i \cup B_{n-1} \geq K$ for each $n > 1$. The latter is true if and only if one of the following holds.

(a) $a_i \geq K_1 \cap K_2$,
(b) $B_{n-1} \geq K_1 \cap K_2$,
(c) $C_{n-1} \subseteq K_1$
(d) $C_{n-1} \subseteq K_2$

If (a) holds, then $W_i \subseteq K_i \cap K_j$. We now show that if (a) does not hold, then one of (b), (c), or (d) must hold for each $n>1$. Since $B_m > B_{m+1}$, if $B_m \not\subseteq K_1 \cap K_2$, then $B_n \not\subseteq K_1 \cap K_2$, $n > m$. Otherwise $B_m > B_n \subseteq K_1 \cap K_2$. Similarly $C_m \not\subseteq K$ implies $C_n \not\subseteq K$ when $n > m$. Thus if (b), (c), and (d) fail to hold for some $n = i, j$, or $k$, respectively, then (b), (c), and (d) fail to hold for $n = \max(i, j, k)$. This result with the assumption that (a) is false contradicts the fact that $a \subseteq B \subseteq K$ for each $n>1$. Thus one of (b), (c), or (d) holds for each $n>1$ if (a) fails to hold. If (b) is true, then $B_n \subseteq K_1 \cap K_2$. This, with $a \subseteq K_1 \cap K_2$, implies $W_i \subseteq K_1 \cap K_2$. By the induction hypothesis, (c) or (d) implies $M_2 \subseteq K_1$ or $K_2$, thus $M_2 \subseteq K_1 \cap K_2$. This, with $a \subseteq K_1 \cap K_2$, implies $W_i \subseteq K_1 \cap K_2$.

Thus we conclude that $A_n \subseteq K_1 \cap K_2$ for each $n$ and $r(K_1 \cap K_2) = k$ imply $W_i \subseteq K_1 \cap K_2$. Similarly $K \subseteq K_1 \cup K_2$ and $A_n \subseteq K_1 \cup K_2$ for each $n$ imply $W_i \subseteq K_1 \cup K_2$. It is shown in a similar manner that $D_n \subseteq K$ for each $n$ implies $M_2 \subseteq K$; $B_n \subseteq K$ for each $n$ implies $V_i \subseteq K$; and $C_n \subseteq K$ for each $n$ implies $M_2 \subseteq K$ where $r(K) = k$ in each case. Thus, by induction, the proof of the theorem is complete.

**Corollary.** In FL$(2+2)$ the sequence $\{A_n\}$ order-converges to $W_i$, $\{B_n\}$ order-converges to $V_i$, $\{C_n\}$ and $\{D_n\}$ each order-converge to $M_2$.

**Proof.** In the case of $\{A_n\}$ we let $u_n = W_i$ and $v_n = A_n$. Then each of the conditions of Definition 4 is satisfied where $\lub \{u_n\} = \glb \{v_n\} = W_i$. Thus $\{A_n\}$ order-converges to $W_i$. The other conclusions of the corollary follow in like manner.

We may generalize these results in the following manner. Let $n_1 + n_2 + \cdots + n_m$ be a set of chains in which two chains each have two or more elements. From each of these two chains take the least elements, $a_{i_1}, a_{i_2}$ and $a_{j_1}, a_{j_2}$. If we replace $a_r$ with $a_{i_r}$ and $b_r$ with $a_{j_r}$, $r = 1, 2$, in $A_n, B_n, C_n, D_n, W_i, V_1$, and $M_2$, the resulting elements will be mutually related in the same manner as $A_n, B_n, C_n, D_n, W_i, V_1$, and $M_2$ since the set $a_{i_1}, a_{i_2}, a_{j_1}, a_{j_2}$ is isomorphic to $2+2$.

If we substitute $a_{i_r}$ and $a_{j_r}$ in $A_n, B_n$, etc. as indicated above, and if we designate the resulting elements by the same symbols as the symbols from which they are obtained, we obtain the following theorem.

**Theorem 4.** In FL$(n_1+n_2+\cdots+n_m)$, where $n_i \geq 2$ and $n_j \geq 2$ for some unequal $i, j$, the set $\{A_n\}$ order-converges to $W_i$; $\{B_n\}$ order-converges to $V_1$; $\{C_n\}$ and $\{D_n\}$ each order-converge to $M_2$.

**4. FL$(4+1)$.** The notation for the elements of FL$(4+1)$ is defined recursively in the following manner.
A_1 = a_3, \ B_1 = a_1 \cup b, \ A_2 = a_3 \cap (a_2 \cup B_1), \ B_2 = (a_1 \cup b) \cap a_1,

and for \( n > 2 \),

\[
A_n = a_3 \cap (a_2 \cup B_{n-1}), \quad B_n = B_1 \cap ((a_1 \cap b) \cup A_{n-2}).
\]

\[
C_n = (a_1 \cap b) \cup A_n.
\]

\[
D_n = a_2 \cup B_n.
\]

\[
F_n = A_n \cup B_n.
\]

\[
G_n = A_n \cup B_{n+1}.
\]

\[
H_i = a_i \cap D_1 \text{ and, for } n > 1, \ H_n = C_{n-1} \cap D_n.
\]

\[
E_i = a_i \text{ and, for } n > 1, \ E_n = C_{n-1} \cap D_{n-1}.
\]

\[
S_n = A_n \cup H_n.
\]

\[
T_n = D_n \cap G_n.
\]

\[
P_i = a_i \cap F_1 \text{ and, for } n > 1, \ P_n = C_{n-1} \cap F_n.
\]

\[
Q_n = B_n \cup E_n.
\]

\[
V_i = a_3 \cap ((a_3 \cap (a_1 \cup b)) \cup a_2 \cup (a_1 \cap b)).
\]

\[
W_i = (a_3 \cap (a_1 \cup b)) \cup (a_2 \cap (a_1 \cap b)).
\]

\[
W_1 = (a_1 \cup b) \cap ((a_3 \cap (a_1 \cup b)) \cup a_2 \cup (a_1 \cap b)).
\]

\[
W_2 = (a_3 \cap (a_1 \cup b)) \cup ((a_1 \cup b) \cap (a_2 \cup (a_1 \cap b))).
\]

As in the case of \( FL(2+2) \), that these elements and their dual elements are all the elements of \( FL(4+1) \) follows from the fact that they include the generators of \( FL(4+1) \) and are closed under \( \cup \) and \( \cap \). The relations between the elements of \( FL(4+1) \) as shown by the diagram in Figure 2 are proved similar to the way the relations of the elements of \( FL(2+2) \) are proved. The following results are stated without proof since the proofs are similar to the proofs of the corresponding statements regarding \( FL(2+2) \).

**Theorem 5.** \( FL(4+1) \) contains the infinite chains \( A_1 > A_2 > \cdots, \ B_1 > B_2 > \cdots, \ C_1 > C_2 > \cdots, \) and \( D_1 > D_2 > \cdots. \)

**Theorem 6.** In \( FL(4+1) \), \( \{A_n\} \) order-converges to \( V_1 \), \( \{B_n\} \) order-converges to \( W_1 \), \( \{C_n\} \) and \( \{D_n\} \) each order-converge to \( M_1 \).

Theorem 6 can be generalized in the same manner as was the
corollary to Theorem 3. Let \( n_1 + n_2 + \cdots + n_m \) be a set of two or more chains in which one chain contains four or more elements. From the chain containing four elements, take the four least elements \( a_{i1}, a_{i2}, a_{i3}, \) and \( a_{i4} \). From another chain, take the least element \( a_{im} \). If we substitute \( a_{ir} \) for \( a_{r} \), \( r = 1, 2, 3, 4 \), and \( a_{im} \) for \( b \) in \( A_n, B_n, C_n, D_n, W_n, \) and \( V_n \), and if we designate the resulting elements by the same symbols as the symbols from which they are obtained then we get the following corollary in the same way as Theorem 4 was obtained.

**COROLLARY.** In \( FL(n_1 + n_2 + \cdots + n_m) \), where \( n_i \geq 4 \) for some \( i \) and \( m \geq 2 \), \( \{A_n\} \) order-converges to \( V_n \), \( \{B_n\} \) order-converges to \( W_n \), \( \{C_n\} \) and \( \{D_n\} \) each order-converge to \( M_n \).

5. \( FL(1+1+1) \) as a sublattice of \( FL(n_1 + n_2) \), \( n_1 \geq 3 \) and \( n_2 \geq 2 \), or \( n_1 \geq 5 \) and \( n_2 \geq 1 \).

From Theorem 4 and Theorem 6 of [2] we have the following theorem.

**THEOREM 7.** Let \( U \) be a subset of \( FL(n_1 + n_2 + \cdots + n_m) \) and let \( U = \{u_{ij}\} \) be isomorphic to \( n_1 + n_2 + \cdots + n_m \). Let \( u_{ij} \leq u_{pq} \) if and only if \( i = p \) and \( j \leq q \). \( FL(U) \) is isomorphic to \( FL(n_1 + n_2 + \cdots + n_m) \) if and only if \( \cup u_{ij} \leq u_{ab} \) implies \( i = a \) and \( j \geq b \) for some \( i, j \), and dually.

**THEOREM 8.** \( FL(n_1 + n_2) \), \( n_1 \geq 3 \) and \( n_2 \geq 2 \), contains a sublattice isomorphic to \( FL(1+1+1) \).

*Proof.* In \( FL(n_1 + n_2) \) let \( u_{i1} = a_{i2}, u_{i2} = a_{i} \cap (a_1 \cup b_1), \) and \( u_{31} = a_1 \cup (a_2 \cap b_2) \). In order to show that the sublattice generated by \( u_{i1}, u_{i2}, \) and \( u_{31} \) is isomorphic to \( FL(1+1+1) \) it is sufficient to show, by Theorem 7, that the \( u_{ij} \) are unrelated and that \( u_{ij} \cup u_{j1} \not\leq u_{k1} \) and \( u_{ij} \not\leq u_{j1} \cap u_{k1} \) when \( i, j, \) and \( k \) are all different.
A direct application of Theorem 1 shows $u_{i_1} \not\leq u_{j_1}$ and $u_{j_1} \not\leq u_{i_1}$ when $i \neq j$, thus the $u_i$ form an unrelated set. A straightforward application of Theorem 1 also shows that $u_{i_1} \cup u_{j_1} \not\leq u_{k_1}$ and $u_{i_1} \not\leq u_{j_1} \cap u_{k_1}$ when $i$, $j$, and $k$ are all different. Hence $\text{FL}(u_{i_1}, u_{j_1}, u_{k_1})$ is isomorphic to $\text{FL}(1+1+1)$.

**Theorem 9.** $\text{FL}(n_1+n_2)$, $n_1 \geq 5$ and $n_2 \geq 1$, contains a sublattice isomorphic to $\text{FL}(1+1+1)$.

**Proof.** A proof similar to the proof of Theorem 8 shows that the sublattice of $\text{FL}(n_1+n_2)$ generated by $u_{i_1}=a_i$, $u_{i_2}=a_i \cap (a_i \cup (a_i \cap b))$, and $u_{i_3}=a_i \cup (a_i \cap (a_i \cup b))$ is isomorphic to $\text{FL}(1+1+1)$.

6. $\text{FL}(n_1+n_2+\cdots+n_m)$ as a sublattice of $\text{FL}(1+1+1)$. In $\text{FL}(1+1+1)$, with generators $x_1$, $x_2$, $x_3$, define $u_i=x_i$, and for $n \geq 1$,

$$u_n=x_1 \cup (x_2 \cap (x_3 \cap (x_1 \cup (x_2 \cup u_{n-1})))) .$$

Whitman has shown [5, p. 112] that $u_0<u_1<u_2<\cdots$ (In his notation $u_i \equiv t_{n+1}$).

**Theorem 10.** The free lattice generated by $3m$ unrelated elements, $\text{FL}(1+1+\cdots+1)$, contains a sublattice isomorphic to $\text{FL}(n_1+n_2+\cdots+n_m)$.

**Proof.** Denote the generators of $\text{FL}(1+1+\cdots+1)$ by $x_1$, $x_2$, $x_3$, $x_{3m}$ and choose $m$ sets of elements of $\text{FL}(n_1+n_2+\cdots+n_m)$ in the following manner. For each $i$, $i=1, 2, \cdots, m$, let $u_{i_1}=x_{3i-2}$, and for $j \geq 1$,

$$u_{i_1}=x_{3i-2} \cup (x_{3i} \cap (x_{3i-1} \cap (x_{3i-2} \cap (x_{3i} \cup u_{i_{j-1}}))))) .$$

We note that the polynomials of each set $u_{i_1}$, $i$ fixed and $j=0, 1, 2, \cdots, n_i$, are the same, except for the subscripts of the $x$'s, as the polynomials $u_{i_1}$ defined immediately before this theorem. Since the $x$'s are unrelated, the reasoning that led to the conclusion $u_0<u_1<u_2<\cdots$ applies to the $u_{i_1}$. We then conclude that $u_{i_0}<u_{i_1}<u_{i_2}<\cdots$, $i=1, 2, \cdots, m$.

Since $x_{3i-3} \not\leq x_{3i-2}$ and $x_{3i} \not\leq x_{3i-3}$ when $i \neq p$, $u_{i_1} \not\leq u_{i_2}$ when $i \neq p$. Similarly $u_{i_2} \not\leq u_{i_1}$ when $i \neq p$. Thus $u_{i_1}$ is unrelated to $u_{i_2}$ when $i \neq p$. Letting $U$ denote the set of polynomials $u_{i_1}$, $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, n_i$ for each $i$, we see that $U$ is isomorphic to $n_1+n_2+\cdots+n_m$. By means of Theorem 7, we shall show that the sublattice generated by $U$ is isomorphic to $\text{FL}(n_1+n_2+\cdots+n_m)$.

If $u_{i_1} \not\geq u_{i_2}$, then it is necessary that one of the following holds.

1. $x_{3i-3} \not\geq x_{3i-2}$,
2. $x_{3i} \cap (x_{3i-1} \cap (x_{3i} \cup u_{i_{j-1}}))) \not\geq u_{i_1}$,
3. $u_{i_0} \geq u_{i_1}$ for some $i$, $j$.
Condition (1) is true if and only if \( x_{3a-2} \geq \text{some } u_{ij} \). Since \( j \neq 0 \), this is false; hence (1) cannot hold. Similarly (2) is false since \( x_{3a} \not\geq \text{some } u_{ij} \). Hence (3) must hold, but this is true if and only if \( a = i \) and \( b \geq j \). Thus \( u_{ab} \geq u_{ij} \) implies \( a = i \) and \( b \geq j \).

If \( \cup u_{ij} \geq u_{ab} \), then it is necessary that
\[
\cup u_{ij} \geq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab})))).
\]
This is true if and only if one of the following holds.

(a) \( \cup u_{ij} \geq x_{3a} \),
(b) \( \cup u_{ij} \geq x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab})))) \),
(c) some \( u_{ij} \geq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab})))) \).

Conditions (a) and (b) are false, respectively, since neither \( u_{ij} \geq x_{3a} \) nor \( u_{ij} \geq x_{3a-1} \) is ever true. Thus (c) must hold. If \( i = a \) and \( j < b \), since \( u_{aj} \geq x_{3a-2} \), it follows that (c) must be false, otherwise it implies \( u_{aj} \geq u_{ab} \) when \( j < b \) contrary to the known relationship \( u_{ab} > u_{aj} \), \( j < b \). If \( a \neq i \), (c) implies at least one of the following.

1. \( x_{3a-2} \geq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab})))) \),
2. \( x_{3a} \geq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab})))) \),
3. \( u_{ij} \leq x_{3a} \),
4. \( u_{ij} \geq x_{3a-1} \).

Since \( i \neq a \), each of these four conditions is false. Thus \( i \neq a \) contradicts (c). We then conclude that if (c) is true, \( i = a \) and \( j \geq b \). Furthermore, we conclude that \( \cap u_{ij} \geq u_{ab} \) implies that \( i = a \) and \( j \geq b \) for some \( i, j \), and dually. By Theorem 7, it follows that the sublattice generated by \( U \) is isomorphic to \( FL(n_1 + n_2 + \cdots + n_m) \).

**Corollary 1.** \( FL(1 + 1 + 1) \) contains a sublattice isomorphic to \( FL(n_1 + n_2 + \cdots + n_m) \).

**Proof.** \( FL(1 + 1 + 1) \) contains a sublattice isomorphic to \( FL(M) \), where \( M \) is a set of \( 3m \) unrelated elements, \( [5, \text{Theorem } 6] \), and \( FL(M) \) in turn contains a sublattice isomorphic to \( FL(n_1 + n_2 + \cdots + n_m) \).

**Corollary 2.** \( FL(m_1 + m_2) \), \( m_1 \geq 3 \) and \( m_2 \geq 2 \), or \( m_1 \geq 5 \) and \( m_2 \geq 1 \), contains a sublattice isomorphic to \( FL(n_1 + n_2 + \cdots + n_m) \).

**Proof.** By Theorems 8 and 9, \( FL(m_1 + m_2) \) contains a sublattice isomorphic to \( FL(1 + 1 + 1) \). In turn, Corollary 1 implies that \( FL(1 + 1 + 1) \) contains a sublattice isomorphic to \( FL(n_1 + n_2 + \cdots + n_m) \).

We note that the reasoning in the proof of Theorem 10 is valid if \( m \) is any cardinal number and each chain contains a finite or countable
number of elements. In the corollaries to Theorem 10 \( m \) must be countable since \( \text{FL}(1+1+1) \) contains only a countable number of elements.

7. Order-convergence in \( \text{FL}(n_1+n_2+\cdots+n_m) \). By Theorem 4 and the corollary to Theorem 6, we see that \( \text{FL}(n_1+n_2+\cdots+n_m) \), where \( n_i \geq 2 \) and \( n_j \geq 2 \) for some distinct \( i, j \), or some \( n_i \geq 4 \) and \( m \geq 2 \), contains an infinite subset that order-converges. We now show that in case \( m \geq 3 \) there exists an infinite subset that order-converges. We summarize this in the following theorem and prove the case \( m \geq 3 \) immediately following.

**Theorem 11.** \( \text{FL}(n_1+n_2+\cdots+n_m) \), where \( n_i \geq 2 \) and \( n_j \geq 2 \) for some distinct \( i \) and \( j \); or some \( n_i \geq 4 \) and \( m \geq 2 \); or \( m \geq 3 \), contains an infinite subset that order-converges.

**Proof.** Denote the least elements of three different chains of \( \text{FL}(n_1+n_2+\cdots+n_m) \) by \( x_1, x_2, x_3 \) and define

\[
\begin{align*}
u_0 &= x_1, \\
\nu_n &= x_2 \cup (x_3 \cap (x_2 \cap (x_3 \cup (x_2 \cap u_{n-1})))) , & n \geq 1 , \\
v_0 &= x_1 , \\
v_n &= x_2 \cup (x_3 \cap (x_2 \cap (x_3 \cup (x_2 \cap v_{n-1})))) , & n \geq 1 .
\end{align*}
\]

As mentioned previously, Whitman has shown \( u_0 < u_1 < u_2 < \cdots \) [5]. Similarly \( v_0 < v_1 < v_2 < \cdots \).

We now define the following elements in \( \text{FL}(n_1+n_2+\cdots+n_m) \).

\[
\begin{align*}
a_n &= (x_2 \cap (x_1 \cap x_3)) \cap u_n , & n = 1, 2 . \\
b_n &= (x_3 \cup (x_1 \cap x_2)) \cup v_n , & n = 1, 2 . \\
A_n &= a_2 \cap (a_1 \cup B_{n-1}) , & B_n = b_2 \cap (b_1 \cup A_{n-1}) . \\
C_n &= a_1 \cup B_n . \\
D_n &= b_1 \cup A_n . \\
W_n &= a_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1) . \\
V_n &= b_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1) . \\
M_n &= (a_2 \cap b_2) \cup a_1 \cup b_1 .
\end{align*}
\]

These elements correspond to the elements of \( \text{FL}(2+2) \) designated by the same symbols. By means of Theorem 1, in a rather tedious but straightforward manner, it is shown that the above elements are related in the same manner as their corresponding elements in \( \text{FL}(2+2) \). Thus
A proof similar to the proof of Theorem 3, although more tedious, shows that \( W_1 = \text{glb} \{A_n\} \), \( V_1 = \text{glb} \{B_n\} \), and \( M_2 = \text{glb} \{C_n\} \) and \( \{D_n\} \). The first step of the induction is vacuously true. If \( A_n \geq K \) for each \( n \) where \( r(K) = 0 \), then it is necessary that \( a \geq K \). In turn this implies \( x_1 \cup (x_1 \cap x_3) \geq K \) and \( u_2 \geq K \). Since \( K \) is an element of \( n_1 + n_2 + \cdots + n_m \), it follows from these two relations and Theorem 1, that at least two of \( x_1, x_2, x_3 \) must be \( \geq K \). Since each \( x_i \) is from a different unrelated chain, this is false. Hence \( A_n \geq K \), for each \( n \) and \( r(K) = 0 \), vacuously implies \( W_1 \geq K \). Similarly, statements (2), (3), and (4) at the beginning of the proof of Theorem 3 are vacuously true when \( r(K) = 0 \). The remainder of the proof is similar to the proof of Theorem 3.

We have answered, in the affirmative, the question posed by Whitman [5], "Does some infinite set in \( \text{FL}(1+1+\cdots+1) \) order-converge?" Theorem 11 states that each infinite free lattice generated by a set of chains contains an infinite subset that order-converges.

**References**


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