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**INVERSION AND REPRESENTATION THEOREMS FOR A
GENERALIZED LAPLACE INTEGRAL**

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1. Varma [8] introduced a generalization of the Laplace integral

$$(1) \quad \mathcal{F}(x) = \int_0^{\infty} e^{-xt} \phi(t) dt$$

in the form

$$(2) \quad F(x) = \int_0^{\infty} (xt)^{m-1/2} e^{-xt/2} W_{k,m}(xt) \phi(t) dt$$

where $\phi(t) \in L(0, \infty)$, $m > -1/2$ and $x > 0$. This generalization is a slight variant of an equivalent integral introduced earlier by Meijer [7] and reduces to (1) when $k + m = 1/2$. In a recent paper [1] Erdélyi has pointed out that the nucleus of (2) can be expressed as a fractional integral of e^{-xt} in terms of the operators of fractional integration introduced by Kober [6]. In this note two theorems have been given—one giving an inversion formula for the transform (2) and another giving necessary and sufficient conditions for the representation of a function as an integral of the form (2) by considering its nucleus as a fractional integral of e^{-xt} .

2. The operators are defined as follows.

$$I_{\eta, \alpha}^+ \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} x^{-\eta-\alpha} \int_0^x (x-u)^{\alpha-1} u^{\eta} \mathcal{F}(u) du$$

$$K_{\zeta, \alpha}^- \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} x^{\zeta} \int_x^{\infty} (u-x)^{\alpha-1} u^{-\zeta-\alpha} \mathcal{F}(u) du$$

where $\mathcal{F}(x) \in L_p(0, \infty)$, $1/p + 1/q = 1$ if $1 < p < \infty$, $1/q = 0$ if $p = 1$, $\alpha > 0$, $\eta > -1/q$, $\zeta > -1/p$.

The Mellin transform $\bar{M}_t \mathcal{F}(x)$ of a function $\mathcal{F}(x) \in L_p(0, \infty)$ is defined as

$$\bar{M}_t \mathcal{F}(x) = \int_0^{\infty} \mathcal{F}(x) x^{it} dx \quad (p = 1)$$

and

$$= \lim_{X \rightarrow \infty} \int_{1/X}^X \mathcal{F}(x) x^{it-1/q} dx \quad (p > 1)$$

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The inverse Mellin transform $\bar{M}^{-1}\phi(t)$ of a function $\phi(t) \in L_q(-\infty, \infty)$ is defined by

$$(3) \quad \bar{M}^{-1}\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t)x^{-it} dt \quad (q = 1)$$

and

$$= \frac{1}{2\pi} \underset{T \rightarrow \infty}{\text{l.i.m}}^{\text{index } p} \int_{-T}^T \phi(t)x^{-it-1/p} dt \quad (q > 1).$$

If the Mellin transform is applied to Kober's operators and the orders of integration are interchanged we obtain, under certain conditions,

$$\bar{M}_t \{I_{\eta, \alpha}^+ \mathcal{F}(x)\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]} \bar{M}_t \mathcal{F}(x)$$

and

$$\bar{M}_t \{K_{\zeta, \alpha}^- \mathcal{F}(x)\} = \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} \bar{M}_t \mathcal{F}(x).$$

But

$$\bar{M}_t(e^{-x}) = \int_0^{\infty} e^{-x} x^{it-1/q} dx = \Gamma\left(\frac{1}{p} + it\right) \text{ if } \frac{1}{p} > 0.$$

Therefore

$$\bar{M}_t \{I_{\eta, \alpha}^+(e^{-x})\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]}$$

and

$$\bar{M}_t \{K_{\zeta, \alpha}^-(e^{-x})\} = \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]}.$$

By (3)

$$I_{\eta, \alpha}^+(e^{-x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]} x^{-it-1/p} dt$$

and

$$(4) \quad K_{\zeta, \alpha}^-(e^{-x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} x^{-it-1/p} dt$$

provided that $1/p > 0$, $\eta + 1/q > 0$ and $\zeta + 1/p > 0$.

It has also been shown by Erdélyi [2] that if the integral in (4) is evaluated by the calculus of residues then it can be expressed in terms of a confluent hypergeometric function. In particular,

$$K_{2m, (1/2)-m-k}^-(e^{-x}) = x^{m-1/2} e^{-x/2} W_{k, m}(x)$$

where $x > 0$, $(1/2) - m - k > 0$.

3. THEOREM 1. Assume $\phi(t) \in L_p(0, \infty)$, $1 \leq p < \infty$, $x > 0$. If $2m > -1/q$ when $(1/2) - m - k > 0$ and $(1/2) + m - k > -1/q$ when $(1/2) - m - k > 0$, then $K_{2m, (1/2)-m-k}^-[\mathcal{F}(x)]$ exists and is equal to

$$\int_0^\infty K_{2m, (1/2)-m-k}^-(e^{-xt}) \phi(t) dt = F(x)$$

where $\mathcal{F}(x)$ and $F(x)$ are given by (1) and (2) respectively.

Proof. Case I $(1/2) - m - k > 0$, $1 < p < \infty$.

If $\phi(t) \in L_p(0, \infty)$, $1 \leq p < \infty$ and $x > 0$ it is easy to see that $\mathcal{F}(x)$ exists. Therefore

$$K_{2m, (1/2)-m-k}^-[\mathcal{F}(x)] = \frac{x^{2m}}{\Gamma((1/2) - m - k)} \times \int_x^\infty (u - x)^{-(1/2)-m-k} u^{-(1/2)-m+k} \left\{ \int_0^\infty e^{-ut} \phi(t) dt \right\} du.$$

But from a theorem of Hardy [5] we know that if $\phi(t) \in L_p(0, \infty)$, $1 < p < \infty$ then $u^{1-2/p} \mathcal{F}(u) \in L_p(0, \infty)$ and therefore $(u - x)^{\alpha} u^{\beta} \mathcal{F}(u) \in L_p(x, \infty)$ provided that $\alpha + \beta = 1 - 2/p$ and $\alpha p > -1$. Therefore the integral

$$\begin{aligned} & \int_x^\infty (u - x)^{-(1/2)-m-k} u^{-(1/2)-m+k} \mathcal{F}(u) du \\ &= \int_x^\infty \left\{ (u - x)^{-(1/2)-m-k-\alpha} u^{-(1/2)-m+k-\beta} \right\} \left\{ (u - x)^{\alpha} u^{\beta} \mathcal{F}(u) \right\} du \end{aligned}$$

will exist if the expressions within the brackets in the integrand belong to $L_p(x, \infty)$ and $L_q(x, \infty)$ respectively. The conditions for these are $(- (1/2) - m - k - \alpha)q > -1$, $(-1 - 2m - \alpha - \beta)q < -1$ and $\alpha + \beta = 1 - 2/p$, $\alpha p > -1$ which reduce to $2m > -1/q$ and $(1/2) - m - k > 0$. Hence under these conditions the integral converges absolutely and we can change the order of integration. Therefore

$$\begin{aligned} K_{2m, (1/2) - m - k}^-[\mathcal{F}(x)] &= \frac{x^{2m}}{\Gamma((1/2) - m - k)} \int_0^\infty v^{-(1/2) - m - k} (x + v)^{-(1/2) - m + k} e^{-vt} \\ &\times \left\{ \int_0^\infty e^{-xt} \phi(t) dt \right\} dv = \frac{x^{2m}}{\Gamma((1/2) - m - k)} \int_0^\infty e^{-xt} \phi(t) \\ &\times \left\{ \int_0^\infty v^{-(1/2) - m - k} (x + v)^{-(1/2) - m + k} e^{-vt} dv \right\} dt \\ &= \int_0^\infty (xt)^{m - (1/2)} e^{-(1/2)xt} W_{k, -m}(xt) \phi(t) dt = F(x) \end{aligned}$$

as $W_{k, -m}(x) = W_{k, m}(x)$.

If $p = 1$, it is similarly seen that the change in the order of integration is justified if $2m > 0$ and $(1/2) - m - K > 0$.

Case II. $(1/2) - m - k < 0$, $1 < p < \infty$.

If $\alpha < 0$ then the operator $K_{\eta, \alpha}^- \{ \mathcal{F}(x) \}$ is defined as the solution, if any, of the integral equation $\mathcal{F}(x) = K_{\eta + \alpha, -\alpha}^- \{ g(x) \}$. Now

$$\begin{aligned} K_{(1/2) + m - k, -(1/2) + m + k}^- [F(x)] \\ &= \frac{x^{(1/2) + m - k}}{\Gamma(-(1/2) + m + k)} \int_0^\infty (u - x)^{-(3/2) + m + k} u^{-2m} \\ &\times \left\{ \int_0^\infty (ut)^{m - (1/2)} e^{-(1/2)ut} W_{k, m}(ut) \phi(t) dt \right\} du. \end{aligned}$$

Again from a result of Hardy [5] we know that if

$$F(x) = \int_0^\infty K(xy) \phi(y) dy$$

then

$$\int_0^\infty x^{p-2} \{F(x)\}^p dx < \left\{ \psi\left(\frac{1}{q}\right) \right\}^p \int_0^\infty \{\phi(y)\}^p dy$$

where

$$\psi(s) = \int_0^\infty x^{s-1} K(x) dx.$$

If

$$K(x) = |x^{m-(1/2)} e^{-(1/2)x} W_{k,m}(x)|$$

then

$$\phi(s) = \frac{\Gamma(2m + s)\Gamma(s)}{\Gamma\left(m - k + \frac{1}{2} + s\right)}$$

by Goldstein's formula [4]. Therefore

$$\int_0^\infty x^{p-2} \{F(x)\}^p dx < \left\{ \frac{\Gamma\left(2m + \frac{1}{q}\right)\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(m - k + \frac{1}{2} + \frac{1}{q}\right)} \right\}^p \int_0^\infty \{\phi(y)\}^p dy$$

provided that $2m > -1/q$, or $x^{1-(2/p)}F(x) \in L_p(0, \infty)$ if $\phi(y) \in L_p(0, \infty)$ ($p > 1$). Hence $(u - x)^\alpha u^\beta F(u) \in L_p(x, \infty)$ if $\alpha + \beta = 1 - (2/p)$ and $\alpha > -1/p$. Also $(u - x)^{-(3/2) + m + k - \alpha} u^{-2m - \beta} \in L_q(x, \infty)$ if $(- (3/2) + m + k - \alpha)q + 1 > 0$ and $(- (3/2) - m + k - \alpha - \beta)q + 1 < 0$. These four conditions reduce to $m + k - (1/2) > 0$ and $m - k + (1/2) > -1/q$. So the integral $\int_x^\infty (u - x)^{-(3/2) + m + k} u^{-2m} F(u) du$ exists under these conditions and

$$\begin{aligned} &K_{\bar{(\frac{1}{2}) + m - k, -(\frac{1}{2}) + m + k}}[F(x)] \\ &= \frac{x^{(1/2) + m - K}}{\Gamma(- (1/2) + m + k)} \int_0^\infty t^{m - (1/2)} \phi(t) dt \\ &\times \int_x^\infty (u - x)^{m + k - (3/2)} u^{-m - (1/2)} e^{-(1/2)ut} W_{k,m}(ut) du \end{aligned}$$

on changing the order of integration which is permissible since the integral is absolutely convergent. But [4]

$$\int_x^\infty u^{\lambda-1} (u - x)^{k-\lambda-1} e^{-u/2} W_{k,m}(u) du = \Gamma(k - \lambda) x^{k-1} e^{-x/2} W_{\lambda,m}(x)$$

where $k > \lambda$ and x is positive. Therefore

$$\begin{aligned} K_{\bar{(\frac{1}{2}) + m - k, -(\frac{1}{2}) + m + k}}[F(x)] &= \int_0^\infty (xt)^{m - (1/2)} e^{-(xt/2)} W_{-m + (1/2), m}(xt) \phi(t) dt \\ &= \int_0^\infty e^{-xt} \phi(t) dt \end{aligned}$$

under the conditions $m + k - (1/2) > 0$, $m - k + (1/2) > -1/q$, $x > 0$.

If $p = 1$, the change in the order of integration is justified if $m + K - (1/2) > 0$ and $(1/2) + m - k > 0$.

Hence $K_{\bar{(\frac{1}{2}) + m - k, -(\frac{1}{2}) + m + k}}[F(x)] = \mathcal{F}(x)$ and the theorem is proved.

THEOREM 2. *Under the conditions of Theorem 1 we have*

$$(5) \quad \int_0^{\infty} e^{-xt} I_{2m, (1/2) - m - \varepsilon}^+ \{\phi(t)\} dt = \int_0^{\infty} K_{2m, (1/2) - m - \varepsilon}^- (e^{-xt}) \phi(t) dt.$$

This is a consequence of Theorem 2 of Erdélyi [3] and is proved similarly.

4. We are now in a position to give inversion and representation theorems for the transform.

We have seen that, under certain conditions,

$$K_{(1/2) + m - k, -(1/2) + m + k}^- [F(x)] = \mathcal{F}(x).$$

Also $\mathcal{F}(x)$ has derivatives of all orders for x sufficiently large and vanishes at infinity. So we can apply the Post-Widder operator $L_{\lambda, u}$ defined by the relation

$$L_{\lambda, u}[\mathcal{F}(x)] = \frac{(-1)^\lambda}{\lambda!} \mathcal{F}^{(\lambda)}\left(\frac{\lambda}{u}\right) \left(\frac{\lambda}{u}\right)^{\lambda+1}$$

(where λ is a positive integer and u a real positive number) to $\mathcal{F}(x)$ and obtain an inversion theorem.

LEMMA. If $\phi(t) \in L_p$ in $(0 \leq t < \infty)$ and

$$\psi(u) = \int_0^{\infty} |\phi(ut) - \phi(t)|^p dt$$

then

$$(i) \quad \left| \frac{u\psi(u)}{1+u} \right| \leq \|\phi\|_p^p \text{ for } u \geq 0$$

and

$$(ii) \quad \psi(u) \rightarrow 0 \text{ as } u \rightarrow 1$$

where $\|\mathcal{F}\|_p$ denotes the norm of the function $\mathcal{F}(t) \in L_p(0, \infty)$, that is

$$\|\mathcal{F}\|_p = \left\{ \int_0^{\infty} |\mathcal{F}(t)|^p dt \right\}^{(1/p)}.$$

Proof. We have

$$|\psi(u)| \leq \int_0^{\infty} |\phi(ut)|^p dt + \int_0^{\infty} |\phi(t)|^p dt = \left(1 + \frac{1}{u}\right) \int_0^{\infty} |\phi(t)|^p dt$$

which proves (i).

Also, by a change of variable,

$$\psi(e^y) = \int_{-\infty}^{\infty} |\phi(e^{x+y}) - \phi(e^x)|^p e^x dx .$$

If $\alpha(x) = e^{(x/p)}\phi(e^x)$ then

$$\int_{-\infty}^{\infty} |\alpha(x)|^p dx = \int_{-\infty}^{\infty} |\phi(e^x)|^p e^x dx = \|\phi\|_p^p$$

and so $\alpha(x) \in L_p(-\infty, \infty)$. Again

$$\begin{aligned} \{\psi(e^y)\}^{1/p} &= \left[\int_{-\infty}^{\infty} \{|\alpha(x+y)e^{-(y/p)} - \alpha(x)e^{-(y/p)}\} \right. \\ &\quad \left. + \{|\alpha(x)e^{-(y/p)} - \alpha(x)|\}^p dx \right]^{1/p} \\ &\leq e^{-(y/p)} \left[\int_{-\infty}^{\infty} |\alpha(x+y) - \alpha(x)|^p dx \right]^{1/p} \\ &\quad + |e^{-(y/p)} - 1| \left[\int_{-\infty}^{\infty} |\alpha(x)|^p dx \right]^{1/p} \end{aligned}$$

by Minkowski's inequality. And $\int_{-\infty}^{\infty} |\alpha(x+y) - \alpha(x)|^p dx \rightarrow 0$ as $y \rightarrow 0$ if $\alpha(x) \in L_p(-\infty, \infty)$ and so does $|e^{-y/p} - 1|$. Therefore $\psi(e^y) = o(1)$ as $y \rightarrow 0$ or $\psi(u) \rightarrow 0$ as $u \rightarrow 1$.

THEOREM 3. Assume $\phi(t) \in L_p$ ($1 \leq p < \infty$) in $0 \leq t \leq R$ for every positive R . If the integral $\mathcal{F}(x)$ converges for $x > 0$ and $2m > -1/q$ when $(1/2) - m - k > 0$; $(1/2) + m - k > -1/q$ when $(1/2) - m - k < 0$, then, for almost all positive t ,

$$\text{l.i.m.}_{\lambda \rightarrow \infty}^{\text{index } p} L_{\lambda, t} [K_{(1/2)+m-k, -(1/2)+m+k}^- \{F(x)\}] = \phi(t) .$$

Proof. We have seen in the proof of Theorem 1 that, under the conditions of the theorem,

$$K_{(1/2)+m-k, -(1/2)+m+k}^- \{F(x)\} = \mathcal{F}(x) .$$

Therefore

$$\begin{aligned} L_{\lambda, t} &\equiv L_{\lambda, t} [K_{(1/2)+m-k, -(1/2)+m+k}^- \{F(x)\}] \\ &= \frac{1}{\lambda!} \left(\frac{\lambda}{t}\right)^{\lambda+1} \int_0^\infty e^{-(\lambda u/t)} u^\lambda \phi(u) du \end{aligned}$$

by simple computation and

$$\begin{aligned} |L_{\lambda, t} - \phi(t)| &\leq \frac{1}{\lambda!} \left(\frac{\lambda}{t}\right)^{\lambda+1} \int_0^\infty e^{-(\lambda u/t)} u^\lambda |\phi(u) - \phi(t)| du \\ &= \frac{1}{\lambda!} \lambda^{\lambda+1} \int_0^\infty e^{-\lambda v} v^\lambda |\phi(vt) - \phi(t)| dv . \end{aligned}$$

Therefore

$$\begin{aligned}
 |L_{\lambda,t} - \phi(t)|^p &\leq \left| \frac{\lambda^{\lambda+1}}{\lambda!} \int_0^\infty e^{-\lambda v} v^\lambda |\phi(vt) - \phi(t)|^p dv \right|^p \\
 &\leq \left[\frac{\lambda^{\lambda+1}}{\lambda!} \int_0^\infty e^{-\lambda v} v^\lambda |\phi(vt) - \phi(t)|^p dv \right] \left[\frac{\lambda^{\lambda+1}}{\lambda!} \int_0^\infty e^{-\lambda v} v^\lambda dv \right]^{p/q} \\
 &\quad \frac{\lambda^{\lambda+1}}{\lambda!} \int_0^\infty e^{-\lambda v} v^\lambda |\phi(vt) - \phi(t)|^p dv .
 \end{aligned}$$

Hence

$$\begin{aligned}
 \int_0^\infty |L_{\lambda,t} - \phi(t)|^p dt &\leq \frac{\lambda^{\lambda+1}}{\lambda!} \int_0^\infty dt \int_0^\infty e^{-\lambda v} v^\lambda |\phi(vt) - \phi(t)|^p dv \\
 &= \frac{\lambda^{\lambda+1}}{\lambda!} \int_0^\infty e^{-\lambda v} v^\lambda dv \left\{ \int_0^\infty |\phi(vt) - \phi(t)|^p dt \right\} .
 \end{aligned}$$

In changing the order of integration, this becomes

$$(6) \quad \frac{\lambda^{\lambda+1}}{\lambda!} \int_0^\infty e^{-\lambda v} v^\lambda \psi(v) dv$$

where $\psi(v)$ is defined as in the lemma. From the lemma it is easily seen that

$$\begin{aligned}
 \psi(u) &= o(1) \quad (u \rightarrow \infty) \\
 &= o(u^{-1}) \quad (u \rightarrow 0+) .
 \end{aligned}$$

Therefore $\int_0^\infty e^{-\lambda v} v^\lambda \psi(v) dv$ converges for $\lambda \geq 1$ and the inversion of the order of integration is justified by Fubini's theorem. By a familiar result [9, Theorem 3c, p. 283] the integral (6) approaches $\psi(1)$ as $\lambda \rightarrow \infty$. But, by the lemma, $\psi(u) = o(1)$ as $u \rightarrow 1$. Therefore $L_{\lambda,t}$ converges in mean to $\phi(t)$ with index p on $0 \leq t < \infty$ and the result is proved.

THEOREM 4. *The necessary and sufficient conditions for a function $F(x)$ to have the representation (2) with $\phi(t) \in L_p(0, \infty)$, $p \geq 1$, $x > 1$, and with $2m > -1/q$ when $1/2 - m - K > 0$ and $m - k + 1/2 > -1/q$ when $1/2 - m - k < 0$ are*

- (i) $K_{1/2+m-K, -1/2+m+K}^- \{F(x)\} \equiv G(x)$ exists, has derivatives of all orders in $0 < x < \infty$ and vanishes at infinity and
- (ii) there exist constants M and p ($p \geq 1$) such that

$$\int_0^\infty |L_{\lambda,t}[G(x)]|^p dt < M \quad (\lambda = 1, 2, \dots) .$$

Proof. First let $F(x)$ have the representation (2). Then, from Theorem 1,

$$G(x) \equiv K_{1/2+m-k, -1/2+m+k}^- \{F(x)\} = \mathcal{F}(x)$$

and as in the proof of Widder [9, Theorem 15a, pp. 313-14] we see that the conditions are satisfied.

Conversely, let the conditions be satisfied. Then again, as in the proof of Widder's theorem referred to before, we see that

$$G(x) = \int_0^\infty e^{-xt} \phi(t) dt = \mathcal{F}(x) .$$

Therefore [3, p. 300]

$$\begin{aligned} F(x) &= (K_{(1/2)+m-k, -(1/2)+m+k}^-)^{-1} \mathcal{F}(x) = K_{2m, 1/2-m-k}^- \{ \mathcal{F}(x) \} \\ &= \int_0^\infty (xt)^{m-1/2} e^{-xt/2} W_{K, m}(xt) \phi(t) dt \end{aligned}$$

by Theorem 1; and the theorem is proved.

COROLLARY. *If the fractional derivatives or integrals*

$$K_{(1/2)+m-k+r, -(1/2)+m+k-r}^- \{F(x)\}$$

exist for $r = 0$ and every positive integer, then the integral in the condition (ii) of Theorem 4 can be replaced by

$$\int_0^\infty \left| \frac{(-1)^\lambda}{\lambda!} \left(\frac{\lambda}{t} \right) \sum_{r=0}^\lambda (-1)^r A_r K_{(1/2)+m-k+r, (1/2)+m+k-r}^- \left\{ F\left(\frac{\lambda}{t} \right) \right\}^p \right| dt$$

where

$$\begin{aligned} A_r &= {}^\lambda C_r (m - k + (1/2))(m - k - (1/2)) \cdots (m - k - \lambda + (3/2) + r) \\ &(r = 0, 1, \dots, \lambda - 1), \quad A_\lambda = 1 . \end{aligned}$$

For [6]

$$t^a K_{\xi, a}^- \{ \mathcal{F}(t) \} = K_{\xi+a, a}^- \{ t^a \mathcal{F}(t) \} .$$

Therefore

$$K_{\xi, a}^- \{ F(x) \} = x^\xi K_{0, a}^- \{ x^{-\xi} F(x) \}$$

and

$$\begin{aligned} \frac{d^\lambda}{dx^\lambda} \left[K_{\xi, a}^- \{ F(x) \} \right] &= \frac{d^\lambda}{dx^\lambda} (x^\xi) \left[K_{0, a}^- \{ x^{-\xi} F(x) \} \right] \\ &+ {}^\lambda C_1 \frac{d^{\lambda-1}}{dx^{\lambda-1}} (x^\xi) \frac{d}{dx} \left[K_{0, a}^- \{ x^{-\xi} F(x) \} \right] + \dots \\ &+ {}^\lambda C_{\lambda-1} \frac{d}{dx} (x^\xi) \frac{d^{\lambda-1}}{dx^{\lambda-1}} \left[K_{0, a}^- \{ x^{-\xi} F(x) \} \right] \\ &+ x^\xi \frac{d^\lambda}{dx^\lambda} \left[K_{0, a}^- \{ x^{-\xi} F(x) \} \right] . \end{aligned}$$

By Leibnitz's theorem this becomes

$$\begin{aligned}
 &= \zeta(\zeta - 1) \cdots (\zeta - \lambda + 1)x^{\zeta-\lambda}[K_{0,\alpha}^-\{x^{-\zeta}F(x)\}] \\
 &- {}^\lambda C_1 \zeta(\zeta - 1) \cdots (\zeta - \lambda + 2)x^{\zeta-\lambda+1}[K_{0,\alpha-1}^-\{x^{-\zeta-1}F(x)\}] \\
 &+ \cdots + (-1)^\lambda x^\zeta [K_{0,\alpha-\lambda}^-\{x^{-\zeta-\lambda}F(x)\}].
 \end{aligned}$$

Therefore

$$\begin{aligned}
 &\frac{(-1)^\lambda}{\lambda!} x^{\lambda+1} \frac{d^\lambda}{dx^\lambda} [K_{\xi,\alpha}^-\{F(x)\}] \\
 &= \frac{(-1)^\lambda}{\lambda!} \sum_{r=0}^\lambda (-1)^r A_r x^{\zeta+r+1} [K_{0,\alpha-r}^-\{x^{-\zeta-r}F(x)\}]
 \end{aligned}$$

where

$$\begin{aligned}
 A_r &= {}^\lambda C_r \zeta(\zeta - 1) \cdots (\zeta - \lambda + r + 1) \\
 A_\lambda &= 1, \qquad (r = 0, 1, \dots, \lambda - 1),
 \end{aligned}$$

and

$$\begin{aligned}
 L_{\lambda,t} [K_{\xi,\alpha}^-\{F(x)\}] &= \frac{(-1)^\lambda}{\lambda!} \sum_{r=0}^\lambda (-1)^r A_r \left(\frac{\lambda}{t}\right)^{\zeta+r+1} [K_{0,\alpha-r}^-\left\{\left(\frac{\lambda}{t}\right)^{-\zeta-r} F\left(\frac{\lambda}{t}\right)\right\}] \\
 &= \frac{(-1)^\lambda}{\lambda!} \left(\frac{\lambda}{t}\right) \sum_{r=0}^\lambda (-1)^r A_r [K_{\xi+r,\alpha-r}^-\left\{F\left(\frac{\lambda}{t}\right)\right\}].
 \end{aligned}$$

Putting $\zeta = m - k + 1/2$ and $\alpha = m + k - 1/2$ we have the required result.

THEOREM 5a. *If $F(x)$ has representation (2) with the conditions of Theorem 4 on $\phi(t)$, x , k and m satisfied and if the fractional derivatives or integrals $K_{(1/2)+m-k+r, -(1/2)+m+k-r}^-\{F(x)\}$ exist for $r = 0$ and every positive integer, then*

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty \left| \frac{(-1)^\lambda}{\lambda!} \left(\frac{\lambda}{t}\right) \sum_{r=0}^\lambda (-1)^r A_r [K_{(1/2)+m-k+r, -(1/2)+m+k-r}^-\left\{F\left(\frac{\lambda}{t}\right)\right\}] \right|^p dt = \left\| \phi \right\|_p^p.$$

where the A_r 's have values as in the Corollary to Theorem 4.

Proof. The proof is similar to that of Widder [9, Theorem 15b, p. 314]

THEOREM 5b. *If the function $F(x)$ has representation (2) with the conditions of Theorem 4 on $\phi(t)$, x , k and m satisfied, then*

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty |L_{\lambda,t}\{F(x)\}|^p dt = \int_0^\infty |I_{2m, (1/2)-m-k}^+\{\phi(t)\}|^p dt.$$

Proof. If $F(x)$ has the representation (2), then, by Theorem 2 we have

$$F(x) = \int_0^{\infty} e^{-xt} I_{2m, (1/2) - m - k}^+ \{\phi(t)\} dt .$$

Also if $\phi(t) \in L_p(0, \infty)$ so does $I_{2m, (1/2) - m - k}^+ \{\phi(t)\}$ provided that $2m > -1/q$.

Therefore, as in Widder [9, Theorem 15b, p. 314], we can prove again that

$$\lim_{\lambda \rightarrow \infty} \int_0^{\infty} |L_{\lambda, t} \{F(x)\}|^p dt = \int_0^{\infty} |I_{2m, (1/2) - m - k}^+ \{\phi(t)\}|^p dt .$$

I am deeply grateful to Professor A. Erdélyi for many helpful suggestions.

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