INVERSION AND REPRESENTATION THEOREMS FOR A GENERALIZED LAPLACE INTEGRAL

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1. Varma [8] introduced a generalization of the Laplace integral

\[ \mathcal{F}(x) = \int_0^\infty e^{-xt}\phi(t)dt \]

in the form

\[ F(x) = \int_0^\infty (xt)^{m-1/2}e^{-xt/2}W_{\kappa,m}(xt)\phi(t)dt \]

where \( \phi(t) \in L(0, \infty) \), \( m > -1/2 \) and \( x > 0 \). This generalization is a slight variant of an equivalent integral introduced earlier by Meijer [7] and reduces to (1) when \( k + m = 1/2 \). In a recent paper [1] Erdélyi has pointed out that the nucleus of (2) can be expressed as a fractional integral of \( e^{-xt} \) in terms of the operators of fractional integration introduced by Kober [6]. In this note two theorems have been given—one giving an inversion formula for the transform (2) and another giving necessary and sufficient conditions for the representation of a function as an integral of the form (2) by considering its nucleus as a fractional integral of \( e^{-xt} \).

2. The operators are defined as follows.

\[ I_{\kappa,\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}x^{-\eta-\alpha}\int_0^x(x-u)^{\alpha-1}u^\eta\mathcal{F}(u)du \]

\[ K_{\xi,\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}x^{\xi}\int_x^\infty(u-x)^{\alpha-1}u^{-\xi-\alpha}\mathcal{F}(u)du \]

where \( \mathcal{F}(x) \in L_p(0, \infty) \), \( 1/p + 1/q = 1 \) if \( 1 < p < \infty \), \( 1/q = 0 \) if \( p = 1 \), \( \alpha > 0 \), \( \eta > -1/q \), \( \xi > -1/p \).

The Mellin transform \( \mathcal{M}_p\mathcal{F}(x) \) of a function \( \mathcal{F}(x) \in L_p(0, \infty) \) is defined as

\[ \mathcal{M}_p\mathcal{F}(x) = \int_0^\infty \mathcal{F}(x)x^{p-1}dx \quad (p = 1) \]

and

\[ = \lim_{X \to \infty} \int_{1/X}^X \mathcal{F}(x)x^{p-1/2}dx \quad (p > 1) \]

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The inverse Mellin transform $M^{-1}_q\phi(t)$ of a function $\phi(t) \in L_q(-\infty, \infty)$ is defined by

$$M^{-1}_q\phi(t) = \frac{1}{2\pi j} \int_{c-i\infty}^{c+i\infty} \phi(t) x^{-it} \, dt \quad (q = 1)$$

and

$$M^{-1}_q\phi(t) = \frac{1}{2\pi} \text{I} \text{r} e^{i \pi p} \int_{c-i\infty}^{c+i\infty} \phi(t) x^{-it-\frac{1}{p}} \, dt \quad (q > 1).$$

If the Mellin transform is applied to Kober’s operators and the orders of integration are interchanged we obtain, under certain conditions,

$$\bar{M}_t\{I_{\alpha,q}\mathcal{F}(x)\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]} \bar{M}_t\mathcal{F}(x)$$

and

$$\bar{M}_t\{K_{\zeta,q}\mathcal{F}(x)\} = \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} \bar{M}_t\mathcal{F}(x).$$

But

$$\bar{M}_t(e^{-x}) = \int_0^\infty e^{-x} x^{-\frac{1}{p}} \, dx = \Gamma\left(\frac{1}{p} + it\right)$$

if $\frac{1}{p} > 0$.

Therefore

$$\bar{M}_t\{I_{\alpha,q}(e^{-x})\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]}$$

and

$$\bar{M}_t\{K_{\zeta,q}(e^{-x})\} = \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]}.$$
\[ I_{\gamma,\alpha}(e^{-x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma\left(\eta + 1 - it\right) \Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + (\eta + \frac{1}{q} - it)\right]} x^{-it-1/p} dt \]

and

\[ K_{\zeta,\alpha}(e^{-x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma\left(\zeta + 1 + it\right) \Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + (\zeta + \frac{1}{p} + it)\right]} x^{-it-1/p} dt \]

provided that \(1/p > 0, \eta + 1/q > 0\), \(\eta + 1/q > 0\), and \(\zeta + 1/p > 0\).

It has also been shown by Erdélyi [2] that if the integral in (4) is evaluated by the calculus of residues then it can be expressed in terms of a confluent hypergeometric function. In particular,

\[ K_{2m, (1/2) - m - k}(e^{-x}) = x^{m-1/2} e^{-x^{1/2}} W_{a, m}(x) \]

where \(x > 0, (1/2) - m - k > 0\).

3. **Theorem 1.** Assume \(\phi(t) \in L_\alpha(0, \infty), 1 \leq p < \infty, x > 0\). If \(2m > 1/q\) when \(1/2) - m - k > 0\) and \((1/2) + m - k > -1/q\) when \((1/2) - m - k > 0\), then \(K_{2m, (1/2) - m - k}(\mathcal{F}(x))\) exists and is equal to

\[ \int_0^\infty K_{2m, (1/2) - m - k}(e^{-xt}) \phi(t) dt = F(x) \]

where \(\mathcal{F}(x)\) and \(F(x)\) are given by (1) and (2) respectively.

**Proof.** Case I \((1/2) - m - k > 0, 1 < p < \infty\).

If \(\phi(t) \in L_\alpha(0, \infty), 1 \leq p < \infty\) and \(x > 0\) it is easy to see that \(\mathcal{F}(x)\) exists. Therefore

\[
K_{2m, (1/2) - m - k}(\mathcal{F}(x)) = \frac{x^{2m}}{\Gamma((1/2) - m - k)} \times \int_x^\infty (u - x)^{-(1/2) - m - k} u^{-(1/2) - m + k} \left\{ \int_0^\infty e^{-u} \phi(t) dt \right\} du .
\]

But from a theorem of Hardy [5] we know that if \(\phi(t) \in L_\alpha(0, \infty), 1 < p < \infty\) then \(u^{1-3/p}, \mathcal{F}(u) \in L_\beta(0, \infty)\) and therefore \((u - x)^{\alpha \beta}, \mathcal{F}(u) \in L_\beta(x, \infty)\) provided that \(\alpha + \beta = 1 - 2/p\) and \(\alpha p > -1\). Therefore the integral

\[
\int_x^\infty (u - x)^{-(1/2) - m - k} u^{-(1/2) - m + k} \mathcal{F}(u) du
\]

\[= \int_x^\infty ((u - x)^{-(1/2) - m - k} u^{-(1/2) - m + k - \beta}) \{(u - x)^{\alpha \beta} \mathcal{F}(u)\} du
\]
will exist if the expressions within the brackets in the integrand belong to $L_p(x, \infty)$ and $L_q(x, \infty)$ respectively. The conditions for these are

\[-(1/2) - m - k - \alpha < -1,\]

\[-1 - 2m - \alpha - \beta < -1\]

and

\[\alpha + \beta = 1 - 2/p,\]

\[\alpha \beta > -1\]

which reduce to $2m > -1/q$ and $(1/2) - m - k > 0$. Hence under these conditions the integral converges absolutely and we can change the order of integration. Therefore

\[K_{2m, (1/2) - m - k}[\mathcal{F}(x)] = \frac{x^{2m}}{\Gamma((1/2) - m - k)} \int_0^\infty v^{-(1/2) - m - k}(x + v)^{-(1/2) - m + k}e^{-vt} \frac{a^m}{\Gamma((1/2) - m - k)} \int_0^\infty e^{-xt} \phi(t) dt \times \left\{ \int_0^\infty v^{-(1/2) - m - k}(x + v)^{-(1/2) - m + k}e^{-vt} dv \right\} \int_0^\infty (xt)^{m - (1/2)} W_{k - m}(xt) \phi(t) dt = F(x)\]

as $W_{k - m}(x) = W_{k, m}(x)$.

If $\alpha > 0$, it is similarly seen that the change in the order of integration is justified if $2m > 0$ and $(1/2) - m - K > 0$.

**Case II.** $(1/2) - m - k < 0, 1 < p < \infty$.

If $\alpha < 0$ then the operator $K_{\alpha, -\alpha}[\mathcal{F}(x)]$ is defined as the solution, if any, of the integral equation $\mathcal{F}(x) = K_{\alpha, -\alpha}[g(x)]$. Now

\[K_{(1/2) + m - k, - (1/2) + m + k}[F(x)] = \frac{x^{(1/2) + m - k}}{\Gamma(- (1/2) + m + k)} \int_0^\infty (x - u)^{-(1/2) + m + k}u^{-2m} \times \left\{ \int_0^\infty (ut)^{m - (1/2)} e^{-(1/2)ut} W_{k, m}(ut) \phi(t) dt \right\} du .\]

Again from a result of Hardy [5] we know that if

\[F(x) = \int_0^\infty K(xy) \phi(y) dy\]

then

\[\int_0^\infty x^{p-2} \{F(x)\}^p dx < \left\{ \phi\left( \frac{1}{q} \right) \right\}^p \int_0^\infty \{\phi(y)\}^p dy\]

where

\[\psi(s) = \int_0^\infty x^{s-1} K(x) dx .\]
\[ K(x) = |x^{m-\frac{1}{2}}e^{-\frac{1}{2}x}W_{k,m}(x)| \]

then

\[ \varphi(s) = \frac{\Gamma(2m + s)\Gamma(s)}{\Gamma(m - k + \frac{1}{2} + s)} \]

by Goldstein's formula \[4\]. Therefore

\[ \int_{0}^{\infty} x^{p-1}\{F(x)\}^pdx \leq \left[ \frac{\Gamma(2m + \frac{1}{q})\Gamma(\frac{1}{q})}{\Gamma(m - k + \frac{1}{2} + \frac{1}{q})} \right]^p \int_{0}^{\infty} \{\phi(y)\}^pdy \]

provided that \(2m > -1/q\), or \(x^{-\frac{(3/2) + m - k}{p}}F(x) \in L_{p}(0, \infty)\) if \(\phi(y) \in L_{p}(0, \infty)\) \((p > 1)\). Hence \((u - x)^{m - k}F(u) \in L_{p}(x, \infty)\) if \(\alpha + \beta = 1 - (2/p)\) and \(\alpha > -1/p\). Also \((u - x)^{-(3/2) + m + k}W_{2m - \beta} \in L_{q}(x, \infty)\) if \(- (3/2) + m + k - \alpha)q + 1 > 0\) and \((- (3/2) - m + k - \alpha - \beta)q + 1 < 0\). These four conditions reduce to \(m + k - (1/2) > 0\) and \(m - k + (1/2) > -1/q\). So the integral \(\int_{x}^{\infty} (u - x)^{-(3/2) + m + k}u^{-2m}F(u)du\) exists under these conditions and

\[ K_{(1/2) + m - k, -(1/2) + m + k}[F(x)] = \frac{\varphi(t)dt}{\int_{0}^{\infty} t^{m-\frac{1}{2}}\phi(t)dt} \times \int_{x}^{\infty} (u - x)^{m + k - \frac{3}{2}}u^{-m - \frac{1}{2}}e^{-(1/2)ut}W_{k,m}(ut)du \]

on changing the order of integration which is permissible since the integral is absolutely convergent. But \[4\]

\[ \int_{x}^{\infty} u^{\lambda - 1}(u - x)^{\kappa - 1}e^{-\mu u}W_{k,m}(u)du = G(k - \lambda)x^{k - 1}e^{-x/2}W_{k,m}(x) \]

where \(k > \lambda\) and \(x\) is positive. Therefore

\[ K_{(1/2) + m - k, -(1/2) + m + k}[F(x)] = \int_{0}^{\infty} (xt)^{m - \frac{1}{2}}e^{-(x/t^{1/2})}W_{-m + (1/2), m}(xt)\phi(t)dt \]

\[ = \int_{0}^{\infty} e^{-xt}\phi(t)dt \]

under the conditions \(m + k - (1/2) > 0\), \(m - k + (1/2) > -1/q\), \(x > 0\). If \(p = 1\), the change in the order of integration is justified if \(m + K - (1/2) > 0\) and \((1/2) + m - k > 0\).

Hence \(K_{(1/2) + m - k, -(1/2) + m + k}[F(x)] = \mathcal{F}(x)\) and the theorem is proved.

**Theorem 2.** Under the conditions of Theorem 1 we have
\[(5) \quad \int_0^\infty e^{-xt}I_{2m,(1/2)-m-K}(\phi(t))\,dt = \int_0^\infty K_{2m,(1/2)-m-K}(e^{-xt})\phi(t)\,dt.\]

This is a consequence of Theorem 2 of Erdélyi [3] and is proved similarly.

4. We are now in a position to give inversion and representation theorems for the transform.

We have seen that, under certain conditions,

\[K_{(1/2)+m-\varepsilon,(1/2)+m-\varepsilon}[F(x)] = \mathcal{F}(x).\]

Also \(\mathcal{F}(x)\) has derivatives of all orders for \(x\) sufficiently large and vanishes at infinity. So we can apply the Post-Widder operator \(L_{\lambda,u}\) defined by the relation

\[L_{\lambda,u}[\mathcal{F}(x)] = \frac{(-1)^\lambda}{\lambda!} \mathcal{F}(\lambda \left(\frac{x}{u}\right))\left(\frac{x}{u}\right)^{\lambda+1}\]

(where \(\lambda\) is a positive integer and \(u\) a real positive number) to \(\mathcal{F}(x)\) and obtain an inversion theorem.

**Lemma.** If \(\phi(t) \in L_p\) in \((0, t < \infty)\) and

\[\psi(u) = \int_0^\infty |\phi(ut) - \phi(t)|^p\,dt\]

then

(i) \[
\left| \frac{u\psi(u)}{1 + u} \right| \leq \|\phi\|_p^p \quad \text{for } u \geq 0
\]

and

(ii) \[
\psi(u) \to 0 \quad \text{as } u \to 1
\]

where \(\|\mathcal{F}\|_p\) denotes the norm of the function \(\mathcal{F}(t) \in L_p(0, \infty)\), that is

\[\|\mathcal{F}\|_p = \left\{\int_0^\infty |\mathcal{F}(t)|^p\,dt\right\}^{1/p}.
\]

**Proof.** We have

\[|\phi(u)| \leq \int_0^\infty |\phi(ut)|^p\,dt + \int_0^\infty |\phi(t)|^p\,dt = \left(1 + \frac{1}{u}\right)\int_0^\infty |\phi(t)|^p\,dt\]

which proves (i).

Also, by a change of variable,
\[ \psi(e^y) = \int_{-\infty}^{\infty} \left| \phi(e^{x+y}) - \phi(e^x) \right|^p e^x dx. \]

If \( \alpha(x) = e^{(x/\rho)} \phi(e^x) \) then
\[
\int_{-\infty}^{\infty} \alpha(x)^p dx = \int_{-\infty}^{\infty} |\phi(e^x)|^p e^x dx = ||\phi||^p_p
\]
and so \( \alpha(x) \in L_p(-\infty, \infty) \). Again
\[
\{\psi(e^y)\}^{1/p} = \left[ \int_{-\infty}^{\infty} \left| \{\alpha(x + y) e^{-(y/\rho)} - \alpha(x) e^{-(y/\rho)} \} \right|^p dx \right]^{1/p}
+ \left[ \alpha(x) e^{-(y/\rho)} - \alpha(x) \right]^p dx \]
\[
\leq e^{-(y/\rho)} \left[ \int_{-\infty}^{\infty} \left| \alpha(x + y) - \alpha(x) \right|^p dx \right]^{1/p}
+ \left| e^{-(y/\rho)} - 1 \right| \left[ \int_{-\infty}^{\infty} \alpha(x)^p dx \right]^{1/p}
\]
by Minkowski's inequality. And \( \int_{-\infty}^{\infty} |\alpha(x + y) - \alpha(x)|^p dx \to 0 \) as \( y \to 0 \) if \( \alpha(x) \in L_p(-\infty, \infty) \) and so does \( |e^{-y/\rho} - 1| \). Therefore \( \psi(e^y) = o(1) \) as \( y \to 0 \) or \( \varphi(u) \to 0 \) as \( u \to 1 \).

**Theorem 3.** Assume \( \phi(t) \in L_p \) \( (1 \leq p < \infty) \) in \( 0 \leq t \leq R \) for every positive \( R \). If the integral \( \mathcal{F}^{-1}(x) \) converges for \( x > 0 \) and \( 2m > -1/q \) when \( (1/2) - m - k > 0 \); \( (1/2) + m - k > -1/q \) when \( (1/2) - m - k < 0 \), then, for almost all positive \( t \),
\[
\lim_{\lambda \to \infty} L^{\lambda}_{\lambda,t}[K^{-1}_{(1/2) + m - k, -(1/2) + m + k} \{F(x)\}] = \phi(t). \]

**Proof.** We have seen in the proof of Theorem 1 that, under the conditions of the theorem,
\[
K^{-1}_{(1/2) + m - k, -(1/2) + m + k} \{F(x)\} = \mathcal{F}^{-1}(x). \]
Therefore
\[
L^{\lambda}_{\lambda,t} = L^{\lambda}_{\lambda,t}[K^{-1}_{(1/2) + m - k, -(1/2) + m + k} \{F(x)\}]
= \frac{1}{\lambda !} \left( \frac{\lambda}{t} \right)^{\lambda+1} \int_0^{\infty} e^{-(\lambda u/t)} u^\lambda \phi(u) du
\]
by simple computation and
\[
|L^{\lambda}_{\lambda,t} - \phi(t)| \leq \frac{1}{\lambda !} \left( \frac{\lambda}{t} \right)^{\lambda+1} \int_0^{\infty} e^{-(\lambda u/t)} u^\lambda \phi(u) - \phi(t) |du
= \frac{1}{\lambda !} \lambda^{\lambda+1} \int_0^{\infty} e^{-\lambda v} v^\lambda \phi(vt) - \phi(t) |dv. \]
Therefore

\[ |L_{\lambda,t} - \phi(t)|^p \leq \left| \frac{\lambda^{t+1}}{\lambda!} \int_0^\infty e^{-\lambda v^\lambda} |\phi(vt) - \phi(t)| dv \right|^p \]

\[ \leq \left[ \frac{\lambda^{t+1}}{\lambda!} \int_0^\infty e^{-\lambda v^\lambda} |\phi(vt) - \phi(t)| dv \right]^p q^{p/q} \]

\[ \frac{\lambda^{t+1}}{\lambda!} \int_0^\infty e^{-\lambda v^\lambda} |\phi(vt) - \phi(t)| dv . \]

Hence

\[ \int_0^\infty |L_{\lambda,t} - \phi(t)|^p dt \leq \frac{\lambda^{t+1}}{\lambda!} \int_0^\infty dt \int_0^\infty e^{-\lambda v^\lambda} |\phi(vt) - \phi(t)|^p dv \]

\[ = \frac{\lambda^{t+1}}{\lambda!} \int_0^\infty e^{-\lambda v^\lambda} dv \left\{ \int_0^\infty |\phi(vt) - \phi(t)|^p dt \right\} . \]

In changing the order of integration, this becomes

(6)

\[ \frac{\lambda^{t+1}}{\lambda!} \int_0^\infty \int_0^\infty e^{-\lambda v^\lambda} v \phi(v) dv \]

where \( \phi(v) \) is defined as in the lemma. From the lemma it is easily seen that

\[ \phi(u) = 0(1) \quad (u \to \infty) \]

\[ = 0(u^{-1}) \quad (u \to 0+) . \]

Therefore \( \int_0^\infty e^{-\lambda v^\lambda} \phi(v) dv \) converges for \( \lambda \geq 1 \) and the inversion of the order of integration is justified by Fubini's theorem. By a familiar result [9, Theorem 3c, p. 283] the integral (6) approaches \( \phi(1) \) as \( \lambda \to \infty \). But, by the lemma, \( \phi(u) = 0(1) \) as \( u \to 1 \). Therefore \( L_{\lambda,t} \) converges in mean to \( \phi(t) \) with index \( p \) on \( 0 \leq t < \infty \) and the result is proved.

**Theorem 4.** The necessary and sufficient conditions for a function \( F(x) \) to have the representation (2) with \( \phi(t) \in L_p(0, \infty) \), \( p \geq 1 \), \( x > 1 \), and with \( 2m > -1/q \) when \( 1/2 - m - K > 0 \) and \( m - k + 1/2 > -1/q \) when \( 1/2 - m - k < 0 \) are

(i) \( K_{1/2+m-K,-1/2+m-K} \{ F(x) \} \equiv G(x) \) exists, has derivatives of all orders in \( 0 < x < \infty \) and vanishes at infinity and

(ii) there exist constants \( M \) and \( p \) (\( p \geq 1 \)) such that

\[ \int_0^\infty |L_{\lambda,t}[G(x)]|^p dt < M \quad (\lambda = 1, 2, \ldots) . \]

**Proof.** First let \( F(x) \) have the representation (2). Then, from Theorem 1,
\[ G(x) \equiv K_{\frac{1}{2} + m - k, (1/2) + m + k} \{ F(x) \} = \mathcal{F}(x) \]

and as in the proof of Widder [9, Theorem 15a, pp. 313-14] we see that the conditions are satisfied.

Conversely, let the conditions be satisfied. Then again, as in the proof of Widder's theorem referred to before, we see that

\[ G(x) = \int_0^a e^{-\phi(t)} dt = \mathcal{F}(x) . \]

Therefore [3, p. 300]

\[ F(x) = (K_{\frac{1}{2} + m - k, (1/2) + m + k})^{-1} \mathcal{F}(x) = K_{2m,1/2-m-k} \{ \mathcal{F}(x) \} \]

\[ = \int_0^a (xt)^{\frac{m}{2} - \frac{1}{2}} e^{-\frac{\pi^2 t^2}{4W}} W_{K,m}(xt) \phi(t) dt \]

by Theorem 1; and the theorem is proved.

**COROLLARY.** If the fractional derivatives or integrals

\[ K_{\frac{1}{2} + m - k + r, (1/2) + m + k - r} \{ F(x) \} \]

exist for \( r = 0 \) and every positive integer, then the integral in the condition (ii) of Theorem 4 can be replaced by

\[ \int_0^a \left( \frac{-1}{t} \right)^t \left( \frac{\lambda}{t} \right) \sum_{r=0}^\lambda (-1)^r A_r K_{\frac{1}{2} + m - k + r, (1/2) + m + k - r} \{ F\left( \frac{\lambda}{t} \right) \} dt \]

where

\[ A_r = \lambda C_\lambda (m - k + (1/2))(m - k - (1/2)) \cdots (m - k - \lambda + (3/2) + r) \]

\( (r = 0, 1, \ldots, \lambda - 1), \quad A_\lambda = 1 . \)

For [6]

\[ t^\lambda K_{\xi,a} \{ \mathcal{F}(t) \} = K_{\xi+a, a} \{ t^\lambda \mathcal{F}(t) \} . \]

Therefore

\[ K_{\xi,a} \{ F(x) \} = x^\lambda K_{\xi+\lambda, a} \{ x^{-\lambda} F(x) \} \]

and

\[
\frac{d^\lambda}{dx^\lambda} \left[ K_{\xi,a} \{ F(x) \} \right] = \frac{d^\lambda}{dx^\lambda} \left( x^\lambda \right) \left[ K_{\xi+\lambda, a} \{ x^{-\lambda} F(x) \} \right] \\
+ \lambda C_\lambda \frac{d^{\lambda-1}}{dx^{\lambda-1}} \left( x^\lambda \right) \frac{d}{dx} \left[ K_{\xi,a} \{ x^{-\lambda} F(x) \} \right] + \cdots \\
+ \lambda C_{\lambda-1} \frac{d}{dx} \left( x^\lambda \right) \frac{d^{\lambda-1}}{dx^{\lambda-1}} \left[ K_{\xi,a} \{ x^{-\lambda} F(x) \} \right] \\
+ x^\lambda \frac{d^\lambda}{dx^\lambda} \left[ K_{\xi+\lambda, a} \{ x^{-\lambda} F(x) \} \right].
\]
By Leibnitz's theorem this becomes
\[ \zeta(\zeta - 1) \cdots (\zeta - \lambda + 1) x^{\zeta - \lambda} \left[ K_{\zeta - \lambda} \{ x^{-\xi} F(x) \} \right] \]
\[ - \lambda \zeta(\zeta - 1) \cdots (\zeta - \lambda + 2) x^{\zeta - \lambda + 1} \left[ K_{\zeta - \lambda - 1} \{ x^{-\xi - 1} F(x) \} \right] \]
\[ + \cdots + (-1)^{\lambda} x^{\zeta} \left[ K_{\zeta - \lambda} \{ x^{-\xi - \lambda} F(x) \} \right]. \]

Therefore
\[ \frac{(-1)^{\lambda}}{\lambda!} \frac{d^\lambda}{dx^\lambda} \left[ K_{\zeta - \lambda} \{ F(x) \} \right] \]
\[ = \frac{(-1)^{\lambda}}{\lambda!} \sum_{r=0}^{\lambda} (-1)^r A_r x^{\zeta + r} \left[ K_{\zeta - \lambda - r} \{ x^{-\xi - \lambda - r} F(x) \} \right] \]

where
\[ A_r = \zeta \cdots (\zeta - \lambda + r + 1) \]
\[ A_\lambda = 1, \quad (r = 0, 1, \ldots, \lambda - 1), \]
and
\[ L_{\lambda,t} \left[ K_{\zeta - \lambda} \{ F(x) \} \right] = \frac{(-1)^{\lambda}}{\lambda!} \sum_{r=0}^{\lambda} (-1)^r A_r \left[ K_{\zeta - \lambda - r} \left\{ \left( \frac{\lambda}{t} \right)^{-\xi - \lambda - r} F \left( \frac{\lambda}{t} \right) \right\} \right] \]
\[ = \frac{(-1)^{\lambda}}{\lambda!} \left( \frac{\lambda}{t} \right) \sum_{r=0}^{\lambda} (-1)^r A_r \left[ K_{\zeta + r - \lambda} \left\{ F \left( \frac{\lambda}{t} \right) \right\} \right]. \]

Putting \( \zeta = m - k + 1/2 \) and \( \alpha = m + k - 1/2 \) we have the required result.

**Theorem 5a.** If \( F(x) \) has representation (2) with the conditions of Theorem 4 on \( \phi(t) \), \( x \), \( k \) and \( m \) satisfied and if the fractional derivatives or integrals \( K_{(1/2)^+ + m - k + r} \{ F(x) \} \) exist for \( r = 0 \) and every positive integer, than

\[ \lim_{\lambda \to \infty} \int_0^\infty \left( \frac{-1)^{\lambda}}{\lambda!} \left( \frac{\lambda}{t} \right) \sum_{r=0}^{\lambda} (-1)^r A_r \left[ K_{(1/2)^+ + m - k + r} \{ F \left( \frac{\lambda}{t} \right) \} \right] \right)^p dt = \left\| \phi \right\|^p . \]

where the \( A_r \) 's have values as in the Corollary to Theorem 4.

**Proof.** The proof is similar to that of Widder [9, Theorem 15b, p. 314]

**Theorem 5b.** If the function \( F(x) \) has representation (2) with the conditions of Theorem 4 on \( \phi(t) \), \( x \), \( k \) and \( m \) satisfied, then

\[ \lim_{\lambda \to \infty} \int_0^\infty \left| L_{\lambda,t} \{ F(x) \} \right|^p dt = \int_0^\infty \left| I_{m,(1/2)^+ - m - k} \{ \phi(t) \} \right|^p dt . \]

**Proof.** If \( F(x) \) has the representation (2), then, by Theorem 2 we have
\[ F(x) = \int_0^\infty e^{-xt} I_{2\text{im},(1/2)-m+ke} \{\phi(t)\} dt. \]

Also if \( \phi(t) \in L^p(0,\infty) \) so does \( I_{2\text{im},(1/2)-m+ke} \{\phi(t)\} \) provided that \( 2m > -1/q \).

Therefore, as in Widder [9, Theorem 15b, p. 314], we can prove again that

\[ \lim_{\lambda \to \infty} \int_0^\infty |L_{\lambda,t} \{F(x)\}|^p dt = \int_0^\infty |I_{2\text{im},(1/2)-m+ke} \{\phi(t)\}|^p dt. \]

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REFERENCES

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