INVERSION AND REPRESENTATION THEOREMS FOR A GENERALIZED LAPLACE INTEGRAL

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1. Varma [8] introduced a generalization of the Laplace integral

\[ \mathcal{F}(x) = \int_0^\infty e^{-zt} \phi(t) dt \]  

in the form

\[ F(x) = \int_0^\infty (xt)^{m-1/2} e^{-xt/2} W_{k,m}(xt) \phi(t) dt \]  

where \( \phi(t) \in L(0, \infty) \), \( m > -1/2 \) and \( x > 0 \). This generalization is a slight variant of an equivalent integral introduced earlier by Meijer [7] and reduces to (1) when \( k + m = 1/2 \). In a recent paper [1] Erdélyi has pointed out that the nucleus of (2) can be expressed as a fractional integral of \( e^{-xt} \) in terms of the operators of fractional integration introduced by Kober [6]. In this note two theorems have been given—one giving an inversion formula for the transform (2) and another giving necessary and sufficient conditions for the representation of a function as an integral of the form (2) by considering its nucleus as a fractional integral of \( e^{-xt} \).

2. The operators are defined as follows.

\[ I^+_{\eta, \alpha} \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} x^{-\eta-\alpha} \int_0^x (x - u)^{a-1} u^\eta \mathcal{F}(u) du \]

\[ K^\zeta_\xi, \alpha \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} x^{\zeta} \int_x^\infty (u - x)^{a-1} u^{-\zeta - \alpha} \mathcal{F}(u) du \]

where \( \mathcal{F}(x) \in L_p(0, \infty), \ 1/p + 1/q = 1 \) if \( 1 < p < \infty, \ 1/q = 0 \) if \( p = 1, \ \alpha > 0, \ \eta > -1/q, \ \zeta > -1/p \).

The Mellin transform \( \overline{M}_p \mathcal{F}(x) \) of a function \( \mathcal{F}(x) \in L_p(0, \infty) \) is defined as

\[ \overline{M}_p \mathcal{F}(x) = \int_0^\infty \mathcal{F}(x)x^{it} dx \quad (p = 1) \]

and

\[ = \text{index}_q \int_{X^{-\infty}}^X \mathcal{F}(x)x^{it-1/q} dx \quad (p > 1) \]

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The inverse Mellin transform $M^{-1}_q \phi(t)$ of a function $\phi(t) \in L_q(-\infty, \infty)$ is defined by

\begin{equation}
M^{-1}_q \phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(t) x^{-st} dt \quad (q = 1)
\end{equation}

and

\begin{equation}
= \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \phi(t) x^{-st-1/p} dt \quad (q > 1).
\end{equation}

If the Mellin transform is applied to Kober's operators and the orders of integration are interchanged we obtain, under certain conditions,

\begin{align*}
\mathcal{M}_t \{ I_{\eta,\alpha} \mathcal{F} (x) \} &= \frac{\Gamma \left( \eta + \frac{1}{q} - it \right)}{\Gamma \left[ \alpha + \left( \eta + \frac{1}{q} - it \right) \right]} \mathcal{M}_t \mathcal{F} (x) \\
\mathcal{M}_t \{ K_{\zeta,\alpha} \mathcal{F} (x) \} &= \frac{\Gamma \left( \zeta + \frac{1}{p} + it \right)}{\Gamma \left[ \alpha + \left( \zeta + \frac{1}{p} + it \right) \right]} \mathcal{M}_t \mathcal{F} (x) .
\end{align*}

But

\begin{equation}
\mathcal{M}_t (e^{-x}) = \int_0^\infty e^{-x} x^{it-1/q} dx = \Gamma \left( \frac{1}{p} + it \right) \text{ if } \frac{1}{p} > 0 .
\end{equation}

Therefore

\begin{align*}
\mathcal{M}_t \{ I_{\eta,\alpha} (e^{-x}) \} &= \frac{\Gamma \left( \eta + \frac{1}{q} - it \right) \Gamma \left( \frac{1}{p} + it \right)}{\Gamma \left[ \alpha + \left( \eta + \frac{1}{q} - it \right) \right]} \\
\mathcal{M}_t \{ K_{\zeta,\alpha} (e^{-x}) \} &= \frac{\Gamma \left( \zeta + \frac{1}{p} + it \right) \Gamma \left( \frac{1}{p} + it \right)}{\Gamma \left[ \alpha + \left( \zeta + \frac{1}{p} + it \right) \right]} .
\end{align*}

By (3)
\[ I_{\eta, \alpha}^+(e^{-x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\eta + \frac{1}{q} - it) \Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]} x^{-it-1/p} dt \]

and

\[ K_{\zeta, \alpha}^-(e^{-x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\zeta + \frac{1}{p} + it) \Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} x^{-it-1/p} dt \]

provided that \(1/p > 0\), \(\eta + 1/q > 0\) and \(\zeta + 1/p > 0\).

It has also been shown by Erdelyi [2] that if the integral in (4) is evaluated by the calculus of residues then it can be expressed in terms of a confluent hypergeometric function. In particular,

\[ K_{2m, (1/2) - m - k}^-(e^{-x}) = x^{m-1/2} e^{-x/2} W_{k, m}(x) \]

where \(x > 0\), \((1/2) - m - k > 0\).

3. THEOREM 1. Assume \(\phi(t) \in L_p(0, \infty)\), \(1 \leq p < \infty\), \(x > 0\). If \(2m > -1/q\) when \((1/2) - m - k > 0\) and \((1/2) + m - k > -1/q\) when \((1/2) - m - k > 0\), then \(K_{2m, (1/2) - m - k}[\mathcal{F}(x)]\) exists and is equal to

\[ \int_0^\infty K_{2m, (1/2) - m - k}^-(e^{-zt}) \phi(t) dt = F(x) \]

where \(\mathcal{F}(x)\) and \(F(x)\) are given by (1) and (2) respectively.

**Proof.** Case I \((1/2) - m - k > 0\), \(1 < p < \infty\).

If \(\phi(t) \in L_p(0, \infty)\), \(1 \leq p < \infty\) and \(x > 0\) it is easy to see that \(\mathcal{F}(x)\) exists. Therefore

\[
K_{2m, (1/2) - m - k}^-(\mathcal{F}(x)) = \frac{x^{2m}}{\Gamma((1/2) - m - k)}
\times \int_x^\infty (u - x)^{-(1/2) - m - k} u^{-(1/2) - m + k} \left\{ \int_0^\infty e^{-ut} \phi(t) dt \right\} du.
\]

But from a theorem of Hardy [5] we know that if \(\phi(t) \in L_p(0, \infty)\), \(1 < p < \infty\) then \(u^{-1/p} \mathcal{F}(u) \in L_p(0, \infty)\) and therefore \((u - x)^{\alpha} u^\beta \mathcal{F}(u) \in L_p(x, \infty)\) provided that \(\alpha + \beta = 1 - 2/p\) and \(\alpha p > -1\). Therefore the integral

\[
\int_x^\infty (u - x)^{-(1/2) - m - k} u^{-(1/2) - m + k} \mathcal{F}(u) du
\]

\[= \int_x^\infty \{(u - x)^{-(1/2) - m - k} u^{-(1/2) - m + k - \beta}\} \{(u - x)^{\alpha} u^\beta \mathcal{F}(u)\} du \]
will exist if the expressions within the brackets in the integrand belong to $L_p(x, \infty)$ and $L_q(x, \infty)$ respectively. The conditions for these are $-(1/2) - m - k - \alpha > -1$, $-(1 - 2m - \alpha - \beta)q < -1$ and $\alpha + \beta = 1 - 2/p$, $\alpha p > -1$ which reduce to $2m > -1/q$ and $(1/2) - m - k > 0$. Hence under these conditions the integral converges absolutely and we can change the order of integration. Therefore

$$K_{(1/2) - m - k}[\mathcal{F}(x)] = \frac{x^{m - k}}{I'((1/2) - m - k)} \int_0^\infty v^{-(1/2) - m - k}(x + v)^{-(1/2) - m + k} e^{-vt}$$

$$\times \left\{ \int_0^\infty e^{-xt}\phi(t)dt \right\} dv = \frac{x^{m - k}}{I'((1/2) - m - k)} \int_0^\infty e^{-xt}\phi(t)$$

$$\times \left\{ \int_0^\infty v^{-(1/2) - m - k}(x + v)^{-(1/2) - m + k} e^{-vt} dv \right\} dt$$

$$= \int_0^\infty (xt)^{m - (1/2)} e^{-(1/2)xt} W_{k, -m}(xt)\phi(t)dt = F(x)$$

as $W_{k, -m}(x) = W_{k, m}(x)$. If $p = 1$, it is similarly seen that the change in the order of integration is justified if $2m > 0$ and $(1/2) - m - K > 0$.

**Case II.** $(1/2) - m - k < 0$, $1 < p < \infty$.

If $\alpha < 0$ then the operator $K_{(1/2) + m - k}[\mathcal{F}(x)]$ is defined as the solution, if any, of the integral equation $\mathcal{F}(x) = K_{(1/2) + m - k}[g(x)]$. Now

$$K_{(1/2) + m - k}[F(x)] = \frac{x^{(1/2) + m - k}}{I'(-1/2 + m + k)} \int_0^\infty (u - x)^{-(1/2) + m + k} u^{-2m}$$

$$\times \left\{ \int_0^\infty (ut)^{m - (1/2)} e^{-(1/2)ut} W_{k, m}(ut)\phi(t)dt \right\} du .$$

Again from a result of Hardy [5] we know that if

$$F(x) = \int_0^\infty K(xy)\phi(y)dy$$

then

$$\int_0^\infty x^{p - 2} |F(x)|^p dx < \left\{ \phi\left( \frac{1}{q} \right) \right\}^p \int_0^\infty \phi(y)^p dy$$

where

$$\phi(s) = \int_0^\infty x^{s-1} K(x) dx .$$

If
\[ K(x) = |x^{m-(1/2)}e^{-x^{1/2}}W_{k,m}(x)| \]

then

\[ \psi(s) = \frac{\Gamma(2m + s)\Gamma(\beta)}{\Gamma(m - k + \frac{1}{2} + s)} \]

by Goldstein’s formula [4]. Therefore

\[
\int_0^\infty x^{p-2}\{F(x)\}^pdx < \left[ \frac{\Gamma(2m + \frac{1}{q})\Gamma(\frac{1}{q})}{\Gamma(m - k + \frac{1}{2} + \frac{1}{q})} \right]^p \int_0^\infty \{\phi(y)\}^pdy
\]

provided that \(2m > -1/q,\) or \(x^{-(2/p)}F(x) \in L_p(0, \infty)\) if \(\phi(y) \in L_p(0, \infty)\) \((p > 1)\). Hence \((u - x)^{\alpha\beta}F(u) \in L_p(x, \infty)\) if \(\alpha + \beta = 1 - (2/p)\) and \(\alpha > -1/p\). Also \((u - x)^{-(3/2)+m+k-\alpha}e^{-2m-\beta} \in L_p(x, \infty)\) if \((- (3/2) + m + k - \alpha)q + 1 > 0\) and \((- (3/2) + m + k - \alpha - \beta)q + 1 < 0\). These four conditions reduce to \(m + k - (1/2) > 0\) and \(m - k + (1/2) > -1/q\). So the integral \(\int_0^\infty (u - x)^{-(3/2)+m+k-2m}F(u)du\) exists under these conditions and

\[
K^+_{(1/2)+m-k, -(1/2)+m+k}[F(x)] = \frac{x^{(1/2)+m-k}}{I'(- (1/2) + m + k)} \int_0^\infty t^{m-(1/2)}\phi(t)dt \\
\times \int_0^\infty (u - x)^{m+k-(3/2)}u^{-m-(1/2)}e^{-(1/2)ut}W_{k,m}(ut)du
\]

on changing the order of integration which is permissible since the integral is absolutely convergent. But [4]

\[
\int_z^\infty u^{\lambda-1}(u - x)^{k-\lambda-1}e^{-u^{1/2}}W_{k,m}(u)du = \Gamma(k - \lambda)x^{k-1}e^{-x^{1/2}}W_{k,m}(x)
\]

where \(k > \lambda\) and \(x\) is positive. Therefore

\[
K^+_{(1/2)+m-k, -(1/2)+m+k}[F(x)] = \int_0^\infty (xt)^{m-(1/2)}e^{-(xt^{1/2})}W_{-m+(1/2), m}(xt)\phi(t)dt \\
= \int_0^\infty e^{-xt}\phi(t)dt
\]

under the conditions \(m + k - (1/2) > 0,\) \(m - k + (1/2) > -1/q,\) \(x > 0\).

If \(p = 1,\) the change in the order of integration is justified if \(m + K - (1/2) > 0\) and \((1/2) + m - k > 0.\) Hence \(K^+_{(1/2)+m-k, -(1/2)+m+k}[F(x)] = \mathcal{F}(x)\) and the theorem is proved.

**Theorem 2.** Under the conditions of Theorem 1 we have
This is a consequence of Theorem 2 of Erdélyi [3] and is proved similarly.

4. **We are now in a position to give inversion and representation theorems for the transform.**

We have seen that, under certain conditions,

\[ K_{(1/2)+m-k,-(1/2)+m+k}[F(x)] = \mathcal{F}(x). \]

Also \( \mathcal{F}(x) \) has derivatives of all orders for \( x \) sufficiently large and vanishes at infinity. So we can apply the Post-Widder operator \( L_{\lambda,u} \) defined by the relation

\[ L_{\lambda,u}[\mathcal{F}(x)] = \frac{(-1)^{\lambda}}{\lambda !} \mathcal{F}(\lambda) \left( \frac{\lambda}{u} \right) \left( \frac{1}{u} \right)^{\lambda+1} \]

(where \( \lambda \) is a positive integer and \( u \) a real positive number) to \( \mathcal{F}(x) \) and obtain an inversion theorem.

**Lemma.** If \( \phi(t) \in L_p \) in \( 0 \leq t < \infty \) and

\[ \psi(u) = \int_0^\infty |\phi(ut) - \phi(t)|^p dt \]

then

(i) \[ \left| \frac{u\psi(u)}{1 + u} \right| \leq \|\phi\|_p \text{ for } u \geq 0 \]

and

(ii) \[ \psi(u) \to 0 \text{ as } u \to 1 \]

where \( \|\mathcal{F}\|_p \) denotes the norm of the function \( \mathcal{F}(t) \in L_p(0, \infty) \), that is

\[ \|\mathcal{F}\|_p = \left\{ \int_0^\infty |\mathcal{F}(t)|^p dt \right\}^{(1/p)} . \]

**Proof.** We have

\[ |\psi(u)| \leq \int_0^\infty |\phi(ut)|^p dt + \int_0^\infty |\phi(t)|^p dt = \left( 1 + \frac{1}{u} \right) \int_0^\infty |\phi(t)|^p dt \]

which proves (i).

Also, by a change of variable,
\[ \psi(e^v) = \int_{-\infty}^{\infty} |\phi(e^{x+v}) - \phi(e^x)|^p e^x dx . \]

If \( \alpha(x) = e^{(x/p)}\phi(e^x) \) then
\[ \int_{-\infty}^{\infty} |\alpha(x)|^p dx = \int_{-\infty}^{\infty} |\phi(e^x)|^p e^x dx = \| \phi \|_p^p \]
and so \( \alpha(x) \in L_p(-\infty, \infty) \). Again
\[ \{\psi(e^v)\}^{1/p} = \left[ \int_{-\infty}^{\infty} \{\alpha(x + y)e^{-(y/p)} - \alpha(x)e^{-(y/p)}\} \right]^{1/p} \]
\[ + \{\alpha(x)e^{-(y/p)} - \alpha(x)\} \left[ \int_{-\infty}^{\infty} \alpha(x + y) - \alpha(x) \right]^{1/p} \]
\[ \leq e^{-(y/p)} \left[ \int_{-\infty}^{\infty} |\alpha(x + y) - \alpha(x)\right]^{1/p} \]
\[ + |e^{-(y/p)} - 1| \left[ \int_{-\infty}^{\infty} |\alpha(x)|^p dx \right]^{1/p} \]
by Minkowski's inequality. And \( \int_{-\infty}^{\infty} |\alpha(x + y) - \alpha(x)|^p dx \to 0 \) as \( y \to 0 \) if \( \alpha(x) \in L_p(-\infty, \infty) \) and so does \( |e^{-(y/p)} - 1| \). Therefore \( \psi(e^v) = o(1) \) as \( y \to 0 \) or \( \psi(u) \to 0 \) as \( u \to 1 \).

**Theorem 3.** Assume \( \phi(t) \in L_p \) (1 \( \leq p < \infty \)) in 0 \( \leq t \leq R \) for every positive \( R \). If the integral \( \mathcal{F}(x) \) converges for \( x > 0 \) and \( 2m > -1/q \) when \( (1/2) - m - k > 0 ; (1/2) + m - k > -1/q \) when \( (1/2) - m - k < 0 \), then, for almost all positive \( t \),
\[
\lim_{\lambda \to \infty} L_{\lambda, t}[K_{(1/2) + m - k, - (1/2) + m + k} \{F(x)\}] = \phi(t) .
\]

**Proof.** We have seen in the proof of Theorem 1 that, under the conditions of the theorem,
\[
K_{(1/2) + m - k, - (1/2) + m + k} \{F(x)\} = \mathcal{F}(x) .
\]

Therefore
\[
L_{\lambda, t} = L_{\lambda, t}[K_{(1/2) + m - k, - (1/2) + m + k} \{F(x)\}]
\]
\[
= \frac{1}{\lambda!} \left( \frac{\lambda}{t} \right)^{\lambda+1} \int_0^{\infty} e^{-(\lambda u/t)} u^\lambda \phi(u) du
\]
by simple computation and
\[
|L_{\lambda, t} - \phi(t)| \leq \frac{1}{\lambda!} \left( \frac{\lambda}{t} \right)^{\lambda+1} \int_0^{\infty} e^{-(\lambda u/t)} u^\lambda |\phi(u) - \phi(t)| du
\]
\[
= \frac{1}{\lambda!} \lambda^{\lambda+1} \int_0^{\infty} e^{-\lambda v} v^\lambda |\phi(v t) - \phi(t)| dv .
\]
Therefore
\[ |L_{\lambda,t} - \phi(t)|^p \leq \frac{\lambda^{\lambda+1}}{\lambda !} \int_0^\infty e^{-\lambda v^\lambda} |\phi(vt) - \phi(t)| dv \]
\[ \leq \frac{\lambda^{\lambda+1}}{\lambda !} \int_0^\infty e^{-\lambda v^\lambda} |\phi(vt) - \phi(t)| dv \left[ \frac{\lambda^{\lambda+1}}{\lambda !} \int_0^\infty e^{-\lambda v^\lambda} dv \right]^{p/q} \]
\[ \frac{\lambda^{\lambda+1}}{\lambda !} \int_0^\infty e^{-\lambda v^\lambda} |\phi(vt) - \phi(t)| dv . \]

Hence
\[ \int_0^\infty |L_{\lambda,t} - \phi(t)|^p dt \leq \frac{\lambda^{\lambda+1}}{\lambda !} \int_0^\infty dt \int_0^\infty e^{-\lambda v^\lambda} |\phi(vt) - \phi(t)| dv \]
\[ = \frac{\lambda^{\lambda+1}}{\lambda !} \int_0^\infty e^{-\lambda v^\lambda} \left\{ \int_0^\infty |\phi(vt) - \phi(t)|^p dt \right\} . \]

In changing the order of integration, this becomes
\[ (6) \]
\[ \frac{\lambda^{\lambda+1}}{\lambda !} \int_0^\infty e^{-\lambda v^\lambda} \psi(v) dv \]
where \( \psi(v) \) is defined as in the lemma. From the lemma it is easily seen that
\[ \psi(u) = 0(1) \quad (u \to \infty) \]
\[ = 0(u^{-1}) \quad (u \to 0^+) . \]

Therefore \( \int_0^\infty e^{-\lambda v^\lambda} \psi(v) dv \) converges for \( \lambda \geq 1 \) and the inversion of the order of integration is justified by Fubini's theorem. By a familiar result [9, Theorem 3c, p. 283] the integral (6) approaches \( \psi(1) \) as \( \lambda \to \infty \). But, by the lemma, \( \psi(u) = o(1) \) as \( u \to 1 \). Therefore \( L_{\lambda,t} \) converges in mean to \( \phi(t) \) with index \( p \) on \( 0 \leq t < \infty \) and the result is proved.

**Theorem 4.** The necessary and sufficient conditions for a function \( F(x) \) to have the representation (2) with \( \phi(t) \in L_p(0, \infty) \), \( p \geq 1 \), \( x > 1 \), and with \( 2m > -1/g \) when \( 1/2 - m - K > 0 \) and \( m - k + 1/2 > -1/g \) when \( 1/2 - m - k < 0 \) are

(i) \( K_{1/2+m-K,-1/2+m+k} \{F(x)\} = G(x) \) exists, has derivatives of all orders in \( 0 < x < \infty \) and vanishes at infinity and

(ii) there exist constants \( M \) and \( p \) (\( p \geq 1 \)) such that
\[ \int_0^\infty |L_{\lambda,t}[G(x)]|^p dt < M \quad (\lambda = 1, 2, \ldots) . \]

**Proof.** First let \( F(x) \) have the representation (2). Then, from Theorem 1,
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\[ G(x) \equiv K^{1/2+m-k, -1/2+m+k}_{m} \{ F(x) \} = \mathcal{F}(x) \]

and as in the proof of Widder [9, Theorem 15a, pp. 313-14] we see that the conditions are satisfied.

Conversely, let the conditions be satisfied. Then again, as in the proof of Widder’s theorem referred to before, we see that

\[ G(x) = \int_{0}^{\infty} e^{-xt} \phi(t) dt = \mathcal{F}(x). \]

Therefore [3, p. 300]

\[ F(x) = (K^{-(1/2)+m-k, -(1/2)+m+k}_{m})^{-1} \mathcal{F}(x) = K^{2m,1/2-m-k}_{2m,1/2-m-k} \{ \mathcal{F}(x) \} \]

\[ = \int_{0}^{\infty} (xt)^{m-1/2} e^{-xt/2} W_{m}(xt) \phi(t) dt \]

by Theorem 1; and the theorem is proved.

**COROLLARY.** If the fractional derivatives or integrals

\[ K^{-(1/2)+m-k+r, -(1/2)+m+k-r}_{m} \{ F(x) \} \]

exist for \( r = 0 \) and every positive integer, then the integral in the condition (ii) of Theorem 4 can be replaced by

\[ \int_{0}^{\infty} \frac{(-1)^{\lambda}}{\lambda!} \left( \frac{\lambda}{t} \right)^{\lambda} \sum_{r=0}^{\lambda} (-1)^{r} A_{r} K^{-(1/2)+m-k+r, -(1/2)+m+k-r}_{m} \{ F\left( \frac{\lambda}{t} \right) \} \] \[ dt \]

where

\[ A_{r} = \lambda C_{r}(m - k + (1/2))(m - k - (1/2)) \cdots (m - k - \lambda + (3/2) + r) \]

\( (r = 0, 1, \ldots, \lambda - 1), \quad A_{\lambda} = 1. \)

For [6]

\[ t^{\alpha} K^{\lambda}_{\xi,\alpha} \{ \mathcal{F}(t) \} = K^{\lambda}_{\xi+\alpha,\alpha} t^{\alpha} \mathcal{F}(t). \]

Therefore

\[ K^{\lambda}_{\xi,\alpha} \{ F(x) \} = x^{\alpha} K^{\lambda}_{0,\alpha} \{ x^{-\xi} F(x) \} \]

and

\[ \frac{d^{\lambda}}{d\alpha^{\lambda}} \left[ K^{\lambda}_{\xi,\alpha} \{ F(x) \} \right] = \frac{d^{\lambda}}{d\alpha^{\lambda}} \left( x^{\alpha} \right) \left[ K^{\lambda}_{0,\alpha} \{ x^{-\xi} F(x) \} \right] \]

\[ + \lambda C_{1} \frac{d^{\lambda-1}}{d\alpha^{\lambda-1}} \left( x^{\alpha} \right) \frac{d}{dx} \left[ K^{\lambda}_{0,\alpha} \{ x^{-\xi} F(x) \} \right] + \cdots \]

\[ + \lambda C_{\lambda-1} \frac{d}{dx} \left( x^{\alpha} \right) \frac{d^{\lambda-1}}{d\alpha^{\lambda-1}} \left[ K^{\lambda}_{0,\alpha} \{ x^{-\xi} F(x) \} \right] \]

\[ + x^{\alpha} \frac{d^{\lambda}}{d\alpha^{\lambda}} \left[ K^{\lambda}_{0,\alpha} \{ x^{-\xi} F(x) \} \right]. \]
By Leibnitz's theorem this becomes
\[\zeta(\zeta - 1) \cdots (\zeta - \lambda + 1) x^{\zeta - \lambda} [K_{0,\alpha \xi}[x^{-\xi}F(x)]] - \lambda C_{1}\zeta(\zeta - 1) \cdots (\zeta - \lambda + 2) x^{\zeta - \lambda + 1} [K_{0,\alpha - 1}[x^{-\xi - 1}F(x)]] + \cdots + (-1)^{\lambda} x^{\zeta} [K_{0,\alpha - \lambda}[x^{-\xi - \lambda}F(x)]] .\]

Therefore
\[
\frac{(-1)^{\lambda}}{\lambda!} x^{\lambda + 1} \frac{d^{\lambda}}{dx^{\lambda}} \left[ K_{\zeta,\alpha}[F(x)] \right] = \frac{(-1)^{\lambda}}{\lambda!} \sum_{r=0}^{\lambda} (-1)^{r} A_{r} x^{\zeta + r + 1} \left[ K_{0,\alpha - r}[x^{-\xi - r}F(x)] \right]
\]

where
\[A_{r} = \lambda C_{r} \zeta(\zeta - 1) \cdots (\zeta - \lambda + r + 1)\]
\[A_{\lambda} = 1, \quad (r = 0, 1, \ldots, \lambda - 1),\]

and
\[
L_{\lambda,t}[K_{\zeta,\alpha}[F(x)]] = \frac{(-1)^{\lambda}}{\lambda!} \sum_{r=0}^{\lambda} (-1)^{r} A_{r} \left( \frac{\lambda}{t} \right)^{\zeta + r + 1} \left[ K_{0,\alpha - r}[\left( \frac{\lambda}{t} \right)^{-\xi - r}F\left( \frac{\lambda}{t} \right)] \right] = \frac{(-1)^{\lambda}}{\lambda!} \left( \frac{\lambda}{t} \right)^{\zeta} \sum_{r=0}^{\lambda} (-1)^{r} A_{r} [K_{\zeta + r,\alpha - r}[F\left( \frac{\lambda}{t} \right)]] .
\]

Putting \(\zeta = m - k + 1/2\) and \(\alpha = m + k - 1/2\) we have the required result.

**Theorem 5a.** If \(F(x)\) has representation (2) with the conditions of Theorem 4 on \(\phi(t)\), \(x\), \(k\) and \(m\) satisfied and if the fractional derivatives or integrals \(K_{(1/2) + m - k + r, -(1/2) + m + k - r}[F(x)]\) exist for \(r = 0\) and every positive integer, than
\[
\lim_{\lambda \to \infty} \int_{0}^{\infty} \left[ \frac{(-1)^{\lambda}}{\lambda!} \left( \frac{\lambda}{t} \right) \sum_{r=0}^{\lambda} (-1)^{r} A_{r} \left[ K_{(1/2) + m - k + r, -(1/2) + m + k - r}[F\left( \frac{\lambda}{t} \right)] \right] \right]^{p} dt = \| \phi \|_{p}^{p} ,
\]
where the \(A_{r}\) 's have values as in the Corollary to Theorem 4.

**Proof.** The proof is similar to that of Widder [9, Theorem 15b, p. 314]

**Theorem 5b.** If the function \(F(x)\) has representation (2) with the conditions of Theorem 4 on \(\phi(t)\), \(x\), \(k\) and \(m\) satisfied, then
\[
\lim_{\lambda \to \infty} \int_{0}^{\infty} \left| L_{\lambda,t}[F(x)] \right|^{p} dt = \int_{0}^{\infty} \left| L_{2m, -(1/2) - m - k}[\phi(t)] \right|^{p} dt .
\]

**Proof.** If \(F(x)\) has the representation (2), then, by Theorem 2 we have
\[ F(x) = \int_0^\infty e^{-zt} I_{2m(1/2)-m-k}^+ \{\phi(t)\} \, dt. \]

Also if \( \phi(t) \in L_p(0, \infty) \) so does \( I_{2m(1/2)-m-k}^+ \{\phi(t)\} \) provided that \( 2m > -1/p \).

Therefore, as in Widder [9, Theorem 15b, p. 314], we can prove again that

\[
\lim_{\lambda \to \infty} \lambda^{-1} \int_0^\infty \left| L_{\lambda,t}^+ \{F(x)\} \right|^p \, dt = \int_0^\infty \left| I_{2m(1/2)-m-k}^+ \{\phi(t)\} \right|^p \, dt.
\]

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REFERENCES

Michael Israel Aissen, *A set function defined for convex plane domaines* . . . . 383
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