

Pacific Journal of Mathematics

TWO THEOREMS OF GAUSS

DANIEL SHANKS

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The purpose of this note is to show that two famous theorems of Gauss can be derived from a common source. The theorems alluded to are the following:

THEOREM 1. (The triangular-exponent identity)

$$(1) \quad \prod_{s=1}^{\infty} \frac{1-x^{2s}}{1-x^{2s-1}} = \sum_{s=1}^{\infty} x^{s(s-1)/2}.$$

THEOREM 2. (The evaluation of Gauss sums)

$$(2) \quad \sum_{s=0}^{m-1} e^{2\pi i s^2/m} = \begin{cases} \sqrt{m} & \text{for } m \equiv 1 \pmod{4} \\ i\sqrt{m} & \text{for } m \equiv 3 \pmod{4}. \end{cases}$$

Both these results will be obtained as consequences of the following identity previously stated by the author [2] without proof.

A finite identity.

THEOREM. *If* $P_0 = 1$ *and*

$$P_n = \prod_{s=1}^n \left(\frac{1-x^{2s}}{1-x^{2s-1}} \right)$$

for $n = 1, 2, \dots$, *then*

$$(3) \quad A_n = \sum_{s=0}^{n-1} \frac{P_n}{P_s} x^{s(2n+1)} = \sum_{s=1}^{2n} x^{s(s-1)/2} = S_n$$

and

$$(3') \quad A'_n = \sum_{s=1}^n \frac{P_n}{P_s} x^{s(2n+1)} = \sum_{s=1}^{2n+1} x^{s(s-1)/2} = S'_n.$$

Proof. We readily verify that

$$(1-x^{2n})x^{s(2n+1)} = (1-x^{2n-1})x^{s(2n-1)} + (1-x^{2s+1})x^{(s+1)(2n-1)} \\ - (1-x^{2s})x^{s(2n-1)},$$

and by multiplying by $\frac{P_{n-1}}{P_s(1-x^{2n-1})}$ we find

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$$(4) \quad \frac{P_n}{P_s} x^{s(2n+1)} = \frac{P_{n-1}}{P_s} x^{s(\lfloor n-1 \rfloor + 1)} + \alpha_{s,n} - \beta_{s,n}$$

where

$$\alpha_{s,n} = \frac{1-x^{2s+1}}{1-x^{2n-1}} \frac{P_{n-1}}{P_s} x^{(s+1)(2n-1)}$$

and

$$\beta_{s,n} = \frac{1-x^{2s}}{1-x^{2n-1}} \frac{P_{n-1}}{P_s} x^{s(2n-1)}.$$

Now

$$\beta_{s+1,n} = \alpha_{s,n} \quad (\text{for } s = 0, 1, \dots, n-2)$$

and since, further,

$$\beta_{0,n} = 0 \text{ and } \alpha_{n-1,n} = x^{n(2n-1)},$$

by summing (4) from $s = 0$ to $s = n-1$ we obtain:

$$A_n = A_{n-1} + x^{(n-1)(2n-1)} + x^{n(2n-1)}.$$

But this may be written $A_n - A_{n-1} = S_n - S_{n-1}$, and by induction

$$A_n - S_n = A_1 - S_1 = \frac{1-x^2}{1-x} - (1+x) = 0.$$

This proves (3) and by adding $x^{n(2n+1)}$ to both sides we verify (3').

Gauss' triangular exponent Theorem (1), now follows at once from (3).

Proof of Theorem 1. The leading term in A_n (that is $s = 0$ in the left side of eq. 3) is P_n . Since the remaining terms ($s = 1, 2, \dots, n-1$) are of order x^{2n+1} and higher, the power series of the function $P_n(x)$ must agree with that of $S_n(x)$ at least to terms of order x^{2n} . By induction the function P_∞ must have the power series S_∞ and this proves (1).

Proof of Theorem 2. The magnitude of

$$G = \sum_{s=0}^{m-1} e^{2\pi i s^2/m}$$

for any odd integer m is given by $|G| = \sqrt{m}$. This is easily shown, [1, p. 163], by multiplying G by its complex conjugate. The real difficulty in Theorem (2) is to show that G is positive real or positive imaginary according as $m \equiv 1$ or $m \equiv 3 \pmod{4}$. But the identity (3) enables us to do this without undue computation.

First we write $x = v^2$ where $v = e^{i\theta}$. Then

$$P_n = v^n \prod_{s=1}^n \left(\frac{v^{2s} - v^{-2s}}{v^{2s-1} - v^{1-2s}} \right)$$

and if

$$Q_0 = 1, \quad Q_n = \prod_{s=1}^n \left(\frac{\sin 2s\theta}{\sin (2s-1)\theta} \right),$$

we may write

$$(5) \quad S_n = \sum_{s=1}^{2n} v^{s(s-1)} = \sum_{s=1}^{n-1} v^{n+s(4n+1)} \frac{Q_n}{Q_s}$$

$$(5') \quad S'_n = \sum_{s=1}^{2n+1} v^{s(s-1)} = \sum_{s=0}^n v^{n+s(4n+1)} \frac{Q_n}{Q_s}.$$

Now for any odd integer, $m = 2N + 1$, if $\theta = 2\pi/m$ we have

$$v^{2N} = v^{-1}$$

and thus

$$\begin{aligned} G &= \sum_{s=0}^{m-1} v^{s^2} = \sum_{s=-N}^{+N} v^{(N+s)^2} = v^{N^2} \sum_{s=-N}^{+N} v^{s(s-1)} \\ &= v^{N^2} \left[\sum_{s=1}^N v^{s(s-1)} + \sum_{s=1}^{N+1} v^{s(s-1)} \right]. \end{aligned}$$

Therefore if $m = 4n + 1$ and $N = 2n$,

$$(6) \quad G = \sum_{s=0}^{n-1} v^{(n+s)(4n+1)} \frac{Q_n}{Q_s} + \sum_{s=0}^n v^{(n+s)(4n+1)} \frac{Q_n}{Q_s}.$$

But $v^{4n+1} = 1$ and $Q_s =$ positive real for $s = 0, 1, \dots, n$ so that

$$G = +\sqrt{m} \quad m \equiv 1 \pmod{4}.$$

And, if $m = 4n + 3$ and $N = 2n + 1$, then

$$\begin{aligned} G &= v^{N^2} \left[\sum_{s=0}^n v^{n+s(4n+1)} \frac{Q_n}{Q_s} + \sum_{s=0}^n v^{n+1+s(4n+5)} \frac{Q_{n+1}}{Q_s} \right] \\ &= \sum_{s=0}^n v^{(4n+3)(n+s)} \left[v^{2n+1-2s} \frac{Q_n}{Q_s} + v^{2n+2+2s} \frac{Q_{n+1}}{Q_s} \right]. \end{aligned}$$

But now $v^{4n+3} = 1$ and

$$Q_{n+1} = \frac{\sin [(2n+2)\theta]}{\sin [(2n+1)\theta]} Q_n = -Q_n$$

and thus

$$(7) \quad G = 2i \sum_{s=0}^n \sin [(2n + 1 - 2s)\theta] \frac{Q_n}{Q_s}$$

which is positive imaginary. Therefore

$$G = +i\sqrt{m} \quad m \equiv 3 \pmod{4}.$$

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2. D. Shanks, *A short proof on an identity of Euler*, Proc. Amer. Math. Soc. **2** (1951), 749.

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