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1. Introduction. Let R_n denote Euclidean n-space. Select a cartesian coordinate system in R_n. Let $X_1, X_2, \cdots, X_n$ be n linearly independent points of R_n and with the usual point-vector notation we arrive at the set Λ of all points

$$g_1X_1 + g_2X_2 + \cdots + g_nX_n$$

such that $g_1, g_2, \cdots, g_n$ are rational integers. The set Λ is called a lattice and $X_1, X_2, \cdots, X_n$ are said to form a basis of the lattice Λ. For a given lattice the basis may be chosen in an infinite number of distinct ways. But if in coordinates $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})$ for $i = 1, 2, \cdots, n$, then the absolute value of the determinant $\|x_{ij}\|$ is independent of the choice of basis. This number is called the determinant of the lattice Λ and is denoted by $d(\Lambda)$.

A convex body K of n dimensions is a closed, bounded, convex set in R_n with inner points. A lattice Λ is said to be K-admissible if no point of Λ other than the origin 0 is an inner point of K. The critical determinant $\Delta(K)$ of K is then defined to be the infimum of $d(\Lambda)$ extended over all K-admissible lattices Λ.

Denote by $\mu_1(\Lambda), \mu_2(\Lambda), \cdots, \mu_n(\Lambda)$ the least upper bounds respectively of real numbers $c_1, c_2, \cdots, c_n$ such that c_iK contains at most $i-1$ linearly independent points of Λ, for $i = 1, 2, \cdots, n$. The numbers $\mu_1(\Lambda), \mu_2(\Lambda), \cdots, \mu_n(\Lambda)$ are called the successive minima of Λ with respect to K. The question has been raised whether the inequality

$$\mu_1(\Lambda)\mu_2(\Lambda) \cdots \mu_n(\Lambda)\Delta(K) \leqq d(\Lambda) \tag{1}$$

is true for convex bodies K that are symmetric in the origin 0. This is known to hold for $n=2$ [1] and for $n=3$ [4] but the general case remains open. It is shown here that for $n=2$ the inequality (1) holds for convex bodies that are not necessarily symmetric in 0. This result is then applied to extend to such bodies a theorem of Mahler's [2] on two-dimensional convex bodies symmetric in 0.

2. Preliminary lemmas. Henceforth all considerations will be in R_2. Thus let K be a two-dimensional convex body. The following lemmas are needed for the proofs of our theorems.

LEMMA 1. *Given a lattice Λ there exists a lattice Λ^* such that*

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(i) $d(\Lambda^*)=d(\Lambda)$,

(ii) $\mu_1(\Lambda^*)\mu_2(\Lambda^*)\geqq\mu_1(\Lambda)\mu_2(\Lambda)$,

(iii) $\mu_2(\Lambda^*)<2\mu_1(\Lambda^*)$.

A proof of this result has been given by Rankin [3] for K symmetric in 0. However his proof makes no use of the symmetry of K and so we may refer the reader to it for a proof of the lemma. The next result is classical.

LEMMA 2. *If X_1, X_2 are two linearly independent points of a lattice Λ such that the triangle with vertices $X_1, X_2, 0$ contains no point of Λ apart from its vertices then X_1, X_2 form a basis of Λ.*

Let C be a convex set in R_2 and for any point X in R_2 define the shadow $S(C, X)$ of C in X to be the set of points Y such that the line segment YX produced past X meets C. That is to say, $S(C, X)$ is the set of Y such that $tX+(1-t)Y\in C$ for some $t>1$. Thus if X is not an inner point of a convex body K and $C\subset K$ then $S(C, X)$ does not contain an inner point of K. For assume that this assertion is false so that there is an inner point Z say of K which is also in $S(C, X)$. By definition of $S(C, X)$ the line segment ZX produced past X meets C and therefore also K. This implies that X is an inner point of K contrary to the hypothesis.

LEMMA 3. *Let K be a convex body containing the origin as an inner point. Let X_1, X_2 be a pair of linearly independent points of the boundary of K such that no one of the points $-X_1$, $-X_2$, $\pm X_1$, $\pm X_2$ is an inner point of K. Then the lattice generated by X_1, X_2 is K-admissible.*

Proof. Take coordinates such that X_1, X_2 are the points $(1, 0)$, $(0, 1)$ respectively. Let C be the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$. Then $C\subset K$ from which it follows that no one of the sets $S(C, X_1)$, $S(C, X_2)$, $S(C, -X_1)$, $S(C, -X_2)$, $S(C, X_1+X_2)$, $S(C, -X_1+X_2)$, $S(C, X_1-X_2)$, $S(C, -X_1-X_2)$ contains an inner point of K. But the union of these sets contains every point with integral coordinates other than 0, that is to say it contains every point of the lattice generated by X_1, X_2 other than 0. This completes the proof of the lemma.

3. On the successive minima.

In this section we prove the following.

THEOREM 1. *If K is any convex body in R_2 and Λ a lattice then*

$$\mu_1(\Lambda)\mu_2(\Lambda)\Delta(K)\leqq d(\Lambda)\ .$$

Proof. If K does not contain 0 as an inner point then $\Delta(K)=0$ and the theorem is trivial. We therefore assume from now on that 0 is an inner point of K. Now any convex body may be approximated arbitrarily closely by strictly convex bodies i.e. convex bodies such that their boundaries contain no line segment, so by an obvious continuity argument there is no loss of generality in assuming that K is strictly convex. Finally by Lemma 1 it is evident that we may also assume $\mu_2(\Lambda)<2\mu_1(\Lambda)$.

Let X_1, X_2 be two linearly independent points of Λ such that $X_1 \in \mu_1(\Lambda)K$, $X_2 \in \mu_2(\Lambda)K$. It follows from the definition of successive minima and the strict convexity of K that the triangle with vertices 0, X_1, X_2 contains no point of Λ apart from its vertices. By Lemma 2, the points X_1, X_2 form a basis of Λ.

By definition of the successive minima no point of the form gX_1 where g is a non-zero integer is in the interior of $\mu_1(\Lambda)K$ and no point of the form fX_1+hX_2 where f, h are integers with $h\neq 0$ is in the interior of $\mu_2(\Lambda)K$.

Put $c=\mu_1(\Lambda)^{-1}\mu_2(\Lambda)$ so that $1\leqq c<2$ and suppose that there exists a $\mu_2(\Lambda)K$-admissible lattice of determinant $cd(\Lambda)$. Then $\Delta(\mu_2(\Lambda)K)\leqq cd(\Lambda)$ or

$$\mu_1(\Lambda)\mu_2(\Lambda)\Delta(K)\leqq d(\Lambda)$$

and the theorem is true. It then remains to prove that there exists a $\mu_2(\Lambda)K$-admissible lattice of determinant $cd(\Lambda)$.

By a preceding remark no point of the form gcX_1 where g is a non-zero integer is in the interior of $\mu_2(\Lambda)K$. Denote by Λ^* the lattice generated by cX_1, X_2 of determinant $cd(\Lambda)$. If Λ^* is $\mu_2(\Lambda)K$-admissible then the theorem is true. Hence we assume from now on that Λ^* is not $\mu_2(\Lambda)K$-admissible. It is evident that cX_1, X_2 are boundary points of $\mu_2(\Lambda)K$. Applying Lemma 3 we see that one of the points $-cX_1$, $-X_2$, $\pm cX_1$, $\pm X_2$ is an inner point of $\mu_2(\Lambda)K$. Clearly $-cX_1$, $-X_2$ are not inner points of $\mu_2(\Lambda)K$. Moreover cX_1+X_2 is in the shadow $S(\{X_2\}, X_1+X_2)$ while $-cX_1+X_2$ is in the shadow $S(\{X_2\}, -X_1+X_2)$ and as $X_2 \in \mu_2(\Lambda)K$ whereas X_1+X_2, $-X_1+X_2$ are not inner points of $\mu_2(\Lambda)K$ so also cX_1+X_2, $-cX_1+X_2$ are not inner points of $\mu_2(\Lambda)K$. We conclude that either cX_1-X_2 or $-cX_1-X_2$ is an inner point of $\mu_2(\Lambda)K$. This is equivalent to saying that the line tX_1-X_2 with parameter t meets $\mu_2(\Lambda)K$ in a line segment with endpoints $a_1X_1-X_2$, $a_2X_1-X_2$ such that

$$\text{(i)}\quad 1\leqq|a_1|<c<|a_2|\leqq 2$$

and

$$\text{(ii)}\quad a_1,\ a_2 \text{ are of comparable sign.}$$

On the other hand the line tX_1+X_2 with parameter t meets $\mu_2(\Lambda)K$ in a line segment one endpoint of which is X_2 the other endpoint being of the form bX_1+X_2 where $0\leqq|b|\leqq 1$. We distinguish the four cases arising when the sign of a_1, a_2 and the sign of b are both taken into account.

(1) $$a_1>0\ ,\quad a_2>0\ ;\quad b\geqq 0\ .$$

The lattice generated by the points cX_1, $a_2X_1-X_2$ is of determinant $cd(\Lambda)$. We assert that this lattice is $\mu_2(\Lambda)K$-admissible for assume that this is false. Since cX_1, $a_2X_1-X_2$ are both on the boundary of $\mu_2(\Lambda)K$ it follows from Lemma 2 that one of the points $-cX_1$, $-a_2X_1+X_2$, $(c+a_2)X_1-X_2$, $(c-a_2)X_1+X_2$, $(a_2-c)X_1-X_2$, $-(c+a_2)X_1+X_2$ is an inner point of $\mu_2(\Lambda)K$. But by what has been said already this is impossible, hence the lattice is $\mu_2(\Lambda)K$-admissible from which the theorem follows.

(2) $$a_1>0\ ,\quad a_2>0\ ,\quad b<0\ .$$

The lattice generated by the points cX_1, $a_1X_1-X_2$ is of determinant $cd(\Lambda)$. We assert that this lattice is $\mu_2(\Lambda)K$-admissible for assume that this is false. Since cX_1, $a_1X_1-X_2$ are both on the boundary of $\mu_2(\Lambda)K$ it follows from Lemma 2 that one of the points $-cX_1$, $-a_1X_1+X_2$, $(c+a_1)X_1-X_2$, $(c-a_1)X_1+X_2$, $(a_1-c)X_1-X_2$, $-(c+a_1)X_1+X_2$ is an inner point of $\mu_2(\Lambda)K$. But by what has already been said this is impossible, hence the lattice is $\mu_2(\Lambda)K$-admissible from which the theorem follows.

(3) $$a_1<0\ ,\quad a_2<0\ ;\quad b\geqq 0\ .$$

The lattice generated by the points cX_1, $a_1X_1-X_2$ is of determinant $cd(\Lambda)$. We assume that this lattice is $\mu_2(\Lambda)K$-admissible for assume that this is false. Since cX_1, $a_1X_1-X_2$ are both on the boundary of $\mu_2(\Lambda)K$ it follows from Lemma 2 that one of the points $-cX_1$, $-a_1X_1+X_2$, $(c+a_1)X_1-X_2$, $(c-a_1)X_1+X_2$, $(a_1-c)X_1-X_2$, $-(c+a_1)X_1+X_2$ is an inner point of $\mu_2(\Lambda)K$. But by what has been said already this is impossible, hence the lattice is $\mu_2(\Lambda)K$-admissible from which the theorem follows.

(4) $$a_1<0\ ,\quad a_2<0\ ;\quad b<0\ .$$

The lattice generated by the points cX_1, $a_2X_1-X_2$ is of determinant $cd(\Lambda)$. We assert that this lattice is $\mu_2(\Lambda)K$-admissible for assume that this is false. Since cX_1, $a_2X_1-X_2$ are both on the boundary of $\mu_2(\Lambda)K$ it follows from Lemma 2 that one of the points $-cX_1$, $-a_2X_1+X_2$, $(c+a_2)X_1-X_2$, $(c-a_2)X_1+X_2$, $(a_2-c)X_1-X_2$, $-(c+a_2)X_1+X_2$ is an inner point of $\mu_2(\Lambda)K$. But by what has already been said this is impossible, hence the lattice is $\mu_2(\Lambda)K$-admissible from which the theorem follows.

The above four cases exhaust all possibilities and so the theorem is proved.

4. A decreasing function. In this section we apply Theorem 1 to prove for any two-dimensional convex body a theorem of Mahler's [2] on two-dimensional symmetric convex bodies. Thus let K again be any two-dimensional convex body and choose a coordinate system such that

(x_1, x_2) are the general coordinates of a point. Denote by $K(t)$ the set of points in K that satisfy the inequality $|x_2| \leqq t$. Mahler [2] has shown that if K is symmetric on the origin then $\Delta(K(t))/t$ is a decreasing function of t, for $t>0$. We will prove the following theorem.

THEOREM 2. *If K is any two-dimensional convex body then $\Delta(K(t))/t$ is a decreasing function of t for $t>0$.*

Proof. If K does not contain the origin as an inner point then $\Delta(K(t))=0$ for all $t>0$, and the theorem is trivial. So we assume from now on that 0 is an inner point of K. Further, appealing to a continuity argument similar to that employed in Theorem 1 it is evident that the truth of Theorem 2 for all strictly convex bodies K implies its truth for any convex body. Hence we assume from now on that K is strictly convex.

The theorem will be proved if for any given $t>0$ we can show $\Delta(K(s))/s \leqq \Delta(K(t))/t$ for all s greater than t and sufficiently close to t. Thus let $t>0$ be fixed. Denote by Λ a critical lattice of $K(t)$, that is Λ is a $K(t)$-admissible lattice such that $d(\Lambda)=\Delta(K(t))$. Let $N(\Lambda)$ be the number of points of Λ which are on the boundary of $K(t)$ but which are not on the boundary of K, under one proviso, namely that if a pair of points of the form $\pm X$ both satisfy the above condition then the pair is to be counted as one point. We distinguish cases.

(1) $$N(\Lambda)=0 .$$

Since any bounded region of the plane contains only a finite number of points of Λ it follows that there exists an $\varepsilon>0$ such that Λ is $K(t+\varepsilon)$-admissible and therefore also $K(s)$-admissible provided only that $t \leqq s \leqq t+\varepsilon$. But for such values of s, $K(t) \subset K(s)$ and thus $\Delta(K(t))=\Delta(K(s))$ whence

$$\Delta(K(t))/t \geqq \Delta(K(s))/s .$$

(2) $$N(\Lambda)=1 .$$

Denote by X a point of Λ on the boundary of $K(t)$ but not on the boundary of K. If there is another such point then it is necessarily-X. There exists an $\varepsilon>0$ such that the ray $0X$ produced meets the boundary of $K(t+\varepsilon)$ in an inner point of K and such that $K(t+\varepsilon)$ contains no point of Λ within its interior other then 0 and $\pm X$. Let s be such that $t \leqq s \leqq t+\varepsilon$ so that $K(s)$ also contains no point of Λ within its interior apart from 0 and $\pm X$. Let further $\mu_1(\Lambda)$, $\mu_2(\Lambda)$ denote the successive minima of Λ with respect to $K(s)$. Evidently $\mu_2(\Lambda) \geqq 1$ and moreover X lies on the boundary of $\mu_1(\Lambda)K(s)$. But X lies on the boundary of $K(t)$ and furthermore the ray $0X$ produced meets the boundary of $K(s)$ in an inner point of K. Therefore $\mu_1(\Lambda)=t/s$. By Theorem 1

$$\mu_1(\Lambda)\mu_2(\Lambda)\Delta(K(s))\leqq d(\Lambda)$$

hence

$$(t/s)\Delta(K(s))\leqq \Delta(K(t))$$

or

$$\Delta(K(s))/s\leqq \Delta(K(t))/t\ .$$

(3) $$N(\Lambda)\geqq 2\ .$$

There exist two linearly independent points X_1, X_2 say of Λ on the boundary of $K(t)$ but not on the boundary of K. This implies that their x_2-coordinates both satisfy the equality $|x_2|=t$. If their x_2-coordinates both have the same value we may assume X_1, X_2 to be chosen so that the line segment connecting the two points contains no further point of Λ. It then follows from the strict convexity of K and the fact that 0 is an inner point of K that the triangle with vertices $0X_1X_2$ contains no point of Λ apart from its vertices. By Lemma 2 the vectors X_1, X_2 form a basis of Λ. But either X_1, X_2 both lie on the line $x_2=t$, or one lies on this line while the other lies on the line $x_2=-t$, or else they both lie on the latter line. In all cases the points of Λ are confined to the lines $x_2=nt$, $n=0, \pm 1, \pm 2, \cdots$. For given $s\geqq t$ denote by $\Lambda(s)$ the set of points $(x_1, sx_2/t)$ where $(x_1, x_2)\in\Lambda$. Then $\Lambda(s)$ is a lattice of determinant $d(\Lambda(s))=(s/t)d(\Lambda)=(s/t)\Delta(K(t))$. Moreover $\Lambda(s)$ is $K(s)$-admissible since all points of $\Lambda(s)$ lie on the lines $x_2=ns$, $n=0, \pm 1, \pm 2, \cdots$ and those that lie on the line $x_2=0$ coincide with those points of Λ lying on this line and these points other than 0 are not inner points of K so also are not inner points of $K(s)$. Hence

$$\Delta(K(s))\leqq d(\Lambda(s))=(s/t)\Delta(K(t))$$

or

$$\Delta(K(s))/s\leqq \Delta(K(t))/t\ .$$

The above three cases exhaust all the possibilities and we conclude that the theorem is true.

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