In the theory of isometric embedding in metric spaces the following theorem is proved: Let $M$ be a metric space every $n + 3$ points of which can be mapped isometrically into Euclidean $n$-space, then there exists an isometry from all of $M$ into Euclidean $n$-space. Because of this theorem Euclidean $n$-space is said to have *congruence order* $n + 3$. [1].

L. M. Blumenthal has raised the question as to whether a notion analogous to that of congruence order could be developed for algebraic systems. In this paper a definition of *isomorphism order* is introduced for groups and a complete description of all Abelian groups having *finite* or *hyperfinite isomorphism order* is obtained.

First a well known definition to avoid any possible misunderstanding of the use of the concept of *rank*.

**Definition.** A group $G$ is said to have rank $n$ if every finitely generated subgroup can be generated by $n$ or fewer elements and $n$ is the smallest natural number with this property.

For convenience we introduce the following definition.

**Definition.** If $k$ elements $g_1, g_2, \ldots, g_k$ of a group $G$ generate a subgroup of $G$ which is isomorphic to a subgroup of a group $H$, we will say that $g_1, g_2, \ldots, g_k$ are *embeddable* in $H$ and that the subgroup generated by the $g$'s is *embeddable* in $H$.

Now we are ready for the definition of isomorphism order.

**Definition.** A group $G$ is said to have isomorphism order $k$ if and only if any group $H$ is embeddable in $G$ whenever every $k$ of its elements are embeddable in $G$.

In the above definition $k$ may be any cardinal number, however, in this paper $k$ will always stand for a natural number.

If $A$ and $B$ are two cardinal numbers such that $A$ is less than or equal to $B$ then it is easy to see that if a group $G$ has isomorphism order $A$ then $G$ has isomorphism order $B$.

Every group has some isomorphism order, since if $G$ is a group of cardinality $M$ then $G$ has isomorphism order $N$ where $N$ is any cardinal...
number which is larger than $M$. Since the cardinals can be well ordered every group has a smallest isomorphism order. However, in what is to follow, if we say $G$ has isomorphism order $k$ we will not mean that $k$ is the smallest isomorphism order of $G$ unless we explicitly say so.

The following lemmas lead to a theorem describing all Abelian groups having finite isomorphism order.

**Lemma 1.** Let $k$ be a natural number and $p$ a fixed prime. Let $G$ be a direct sum of $k$ groups each of which is a cyclic group of order a power of $p$ or a group isomorphic to $\mathbb{Z}(P\infty)$. Then $G$ has isomorphism order $k + 1$.

*Proof.* Let $H$ be a group every $k + 1$ elements of which are embeddable in $G$. $H$ is primary and has rank $k$. From this the conclusion easily follows. (Exercise 49, [2])

**Lemma 2.** An Abelian torsion group $G$ has isomorphism order $k$ if and only if $G$ is a direct sum of fewer than $k$ subgroups of the rationals mod one.

*Proof.* Let $G$ be an Abelian torsion group having isomorphism order $k$. Write $G$ as a direct sum of primary groups that is $G = \sum G_p$, where $p$ ranges over the primes and $G_p$ consists of all elements whose order is a power of $p$. Now $G_p$ does not contain the integers mod $p$ taken $k$ times for, if it did, arbitrarily large groups constructed by taking direct sums of the integers mod $p$ would (by hypothesis) be embeddable in $G$. From this it follows that $G_p$ has rank less than $k$. Hence (exercise 49, [2]) $G_p$ is a direct sum of fewer than $k$ subgroups of $\mathbb{Z}(P\infty)$, and therefore $G$ is a direct sum of fewer than $k$ subgroups of the rationals mod one by rearrangement of summands.

Conversely, let, $G$ be a direct sum of fewer than $k$ subgroups of the rationals mod one. Let $H$ be a group every $k$ elements of which are embeddable in $G$, so that $H$ is torsion. Write $H = \sum H_p$, and consider $H_p$. Every $k$ elements of $H_p$ are embeddable in $G_p$, but by Lemma 1, $G_p$ has isomorphism order $k$, hence $H_p$ is embeddable in $G_p$ and so $H$ is embeddable in $G$.

**Lemma 3.** A torsion free Abelian group has isomorphism order $k$ if and only if it is a vector space over the rationals of dimension less than $k$.

*Proof.* Let $G$ be a torsion free Abelian group having isomorphism order $k$. Now $G$ does not contain the direct sum of the integers taken $k$ times, for, if it did, the group consisting of the direct sum of the
Let $m$ be the maximal number of elements of $G$ which are independent over the integers. By what was just said $m$ must be less than $k$. Any $m$ dimensional vector space over the rationals is embeddable in $G$, by hypothesis. So $G$ contains a vector space over the rationals of dimension $m$, call this space $V$. The space $V$ is a divisible subgroup of $G$ and hence is a direct summand so $G = A + V$. Let $a$ be a nonzero element of $A$. Since $m$ is the maximal number of independent elements of $G$, $na$ is in $V$ for some nonzero integer $n$, but since $na$ is in $A$ it is zero and therefore $a$ is zero and so $G = V$.

Conversely, if $G$ is a vector space over the rationals of dimension less than $k$ and $H$ is a group every $k$ elements of which are embeddable in $G$ then $H$ is embeddable in $G$. To see this, observe that $H$ can be embedded in a vector space over the rationals consisting of all couples of the form $(n, h)$ when $n$ is a nonzero integer and equivalence is defined in the natural way, and the dimension of this space is less than $k$ for if not, there exist $k$ elements of $H$ not embeddable in $G$, which completes the proof.

**Theorem 1.** An Abelian group $G$ has isomorphism order $k$ if and only if $G$ is the direct sum of two groups, one torsion, the other torsion free. The torsion free group is a vector space over the rationals of dimension less than $k$, while the torsion group can be written as a direct sum of fewer than $k$ subgroups of the rationals mod one.

**Proof.** Let $G$ be an Abelian group having isomorphism order $k$. The theorem follows from the lemmas if $G$ is torsion or torsion free. Now $G$ contains a vector space $V$ over the rationals of dimension $n$ less than $k$ where $n$ is the maximal number of elements of $G$ which are independent over the integers. This holds by an application of the argument of Lemma 3. Regard $V$ as a group, then $V$ is a direct summand of $G$ since $V$ is divisible. So $G = A + V$ and $A$ is torsion, for if $x$ is in $A$ then $mx$ is in $V$ for some nonzero integer $m$, hence $mx = 0$. Now apply Lemma 2 to $A$ and obtain the necessity of the theorem.

To prove the sufficiency, let $G$ be an Abelian group such that $G = T + V$ where $T = A_1 + A_2 + \cdots + A_s$ and each $A_i$ is a subgroup of the rationals mod one and $s < k$, and $V$ is a vector space over the rationals of dimension less than $k$.

We must show that if $H$ is an Abelian group, every $k$ (or fewer) elements of which are embeddable in $G$, then $H$ is embeddable in $G$. $H$ does not contain $k$ elements which are independent over the
integers. Hence $H$ contains at least one subgroup $H_d$ such that $h \in H$ implies $rh \in H_o$ for some natural number $r$ and such that $H_o$ is embeddable in $G$.

Let $T^*$ be the direct sum of the rationals mod one taken $s$ times. Let $G^* = T^* + V$. We will show that if $\phi$ is an isomorphism from $H$ into $G^*$ then if $H_o \neq H$, $\phi$ can be properly extended. Then the embeddability of $H$ in $G^*$ can be obtained by a transfinite argument. Finally, we will see that $H$ is embeddable in $G$.

So let $H_o$ be a subgroup of $H$ such that $h \in H$ implies $rh \in H_o$ for some integer $r$ and let $F$ be an isomorphism from $H_o$ into $G^*$. If $H_o = H$ we are done, if not, let $h \notin H_o$, and $m$ the smallest natural number such that $mh \in H_o$.

Case 1. $m = p$, $p$ a prime. Let $M = [z \mid pz = F(ph), z \in G^*]$. For convenience, we will refer to $M$ as the set of all the "pth roots" of $F(ph)$, and note that $M$ is finite, and that the number of elements in $M$ is exactly the number of "pth roots" of 0 in $G^*$. Now, not every element of $M$ is in $F(H_o)$, for if so, a glance at the inverse images will show that the inverse image of every element of $M$ is a "pth root" of $ph$. But $F(ph)$ has at least as many "pth roots" in $G^*$ as $ph$ has in $H$. Hence $h$ itself is in $H_o$, a contradiction.

We conclude that some element of $M$, call it $z$, is not in $F(H_o)$. Furthermore, if $0 < n < p$, then $nz \notin F(H_o)$ and hence $F$ can be extended in the natural way.

Case 2. $m$ not a prime, then $m = qt$ where $q$ is a prime. Apply the argument of Case 1 to the set of all $qth$ roots of $F(mh)$.

This shows that $H$ is embeddable in $G^*$. But by Lemma 2, if $T'$ is the torsion subgroup of $H$, $T'$ is embeddable in $T$. Hence it is easily seen that $H$ is actually embeddable in $G$, which completes the proof.

In the above theorem, nothing has been said about smallest isomorphism order. However, it is easy to see that, if $G$ has smallest isomorphism order $k$ then either the torsion free summand of $G$ has rank $k-1$ or the torsion summand cannot be written as a direct sum of fewer than $k-1$ subgroups of the rationals mod 1.

The next step up in the hierarchy of isomorphism order is given by the following definition.

**Definition.** A group $G$ is said to have **hyperfinite isomorphism order** if, whenever every finitely generated subgroup of a group $H$ is embeddable in $G$, then $H$ is embeddable in $G$.

The proof of the next theorem is similar to that of Theorem 1, and
rests on the fact that a torsion group has hyperfinite isomorphism order if and only if the rank of each primary subgroup is finite, while a torsion free group has hyperfinite isomorphism order if it is a finite dimensional vector space over the rationals.

**Theorem 2.** An Abelian group $G$ has hyperfinite isomorphism order if and only if it is the direct sum of two groups, one torsion, the other torsion free. The torsion free group is a finite dimensional vector space over the rationals while the torsion summand has no primary subgroup of infinite rank.

**Remark.** If the smallest isomorphism order $G$ has is hyperfinite, then there is no upper bound on the ranks of the primary subgroups of $G$.

This concludes the analysis of Abelian groups having finite or hyperfinite isomorphism order. In a subsequent paper, we hope to give some results concerning Abelian groups having transfinite isomorphism order.

Also, this notion can be carried over to other systems, such as rings, a direction in which some preliminary results have been obtained.

**References**


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