

# Pacific Journal of Mathematics

**ISOMORPHISM ORDER FOR ABELIAN GROUPS**

STEVE JEROME BRYANT

# ISOMORPHISM ORDER FOR ABELIAN GROUPS

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In the theory of isometric embedding in metric spaces the following theorem is proved: Let  $M$  be a metric space every  $n + 3$  points of which can be mapped isometrically into Euclidean  $n$ -space, then there exists an isometry from all of  $M$  into Euclidean  $n$ -space. Because of this theorem Euclidean  $n$ -space is said to have *congruence order*  $n + 3$ . [1].

L. M. Blumenthal has raised the question as to whether a notion analogous to that of congruence order could be developed for algebraic systems. In this paper a definition of *isomorphism order* is introduced for groups and a complete description of all Abelian groups having *finite* or *hyperfinite isomorphism order* is obtained.

First a well known definition to avoid any possible misunderstanding of the use of the concept of *rank*.

DEFINITION. A group  $G$  is said to have *rank*  $n$  if every finitely generated subgroup can be generated by  $n$  or fewer elements and  $n$  is the smallest natural number with this property.

For convenience we introduce the following definition.

DEFINITION. If  $k$  elements  $g_1, g_2, \dots, g_k$  of a group  $G$  generate a subgroup of  $G$  which is isomorphic to a subgroup of a group  $H$ , we will say that  $g_1, g_2, \dots, g_k$  are *embeddable* in  $H$  and that the subgroup generated by the  $g$ 's is *embeddable* in  $H$ .

Now we are ready for the definition of isomorphism order.

DEFINITION. A group  $G$  is said to have *isomorphism order*  $k$  if and only if any group  $H$  is embeddable in  $G$  whenever every  $k$  of its elements are embeddable in  $G$ .

In the above definition  $k$  may be any cardinal number, however, in this paper  $k$  will always stand for a natural number.

If  $A$  and  $B$  are two cardinal numbers such that  $A$  is less than or equal to  $B$  then it is easy to see that if a group  $G$  has isomorphism order  $A$  then  $G$  has isomorphism order  $B$ .

Every group has some isomorphism order, since if  $G$  is a group of cardinality  $M$  then  $G$  has isomorphism order  $N$  where  $N$  is any cardinal

number which is larger than  $M$ . Since the cardinals can be well ordered every group has a smallest isomorphism order. However, in what is to follow, if we say  $G$  has isomorphism order  $k$  we will not mean that  $k$  is the smallest isomorphism order of  $G$  unless we explicitly say so.

The following lemmas lead to a theorem describing all Abelian groups having finite isomorphism order.

**LEMMA 1.** *Let  $k$  be a natural number and  $p$  a fixed prime. Let  $G$  be a direct sum of  $k$  groups each of which is a cyclic group of order a power of  $p$  or a group isomorphic to  $Z(P\infty)$ . Then  $G$  has isomorphism order  $k + 1$ .*

*Proof.* Let  $H$  be a group every  $k + 1$  elements of which are embeddable in  $G$ .  $H$  is primary and has rank  $k$ . From this the conclusion easily follows. (Exercise 49, [2])

**LEMMA 2.** *An Abelian torsion group  $G$  has isomorphism order  $k$  if and only if  $G$  is a direct sum of fewer than  $k$  subgroups of the rationals mod one.*

*Proof.* Let  $G$  be an Abelian torsion group having isomorphism order  $k$ . Write  $G$  as a direct sum of primary groups that is  $G = \sum G_p$ , where  $p$  ranges over the primes and  $G_p$  consists of all elements whose order is a power of  $p$ . Now  $G_p$  does not contain the integers mod  $p$  taken  $k$  times for, if it did, arbitrarily large groups constructed by taking direct sums of the integers mod  $p$  would (by hypothesis) be embeddable in  $G$ . From this it follows that  $G_p$  has rank less than  $k$ . Hence (exercise 49, [2])  $G_p$  is a direct sum of fewer than  $k$  subgroups of  $Z(P\infty)$ , and therefore  $G$  is a direct sum of fewer than  $k$  subgroups of the rationals mod one by rearrangement of summands.

Conversely, let,  $G$  be a direct sum of fewer than  $k$  subgroups of the rationals mod one. Let  $H$  be a group every  $k$  elements of which are embeddable in  $G$ , so that  $H$  is torsion. Write  $H = \sum H_p$  and consider  $H_p$ . Every  $k$  elements of  $H_p$  are embeddable in  $G_p$ , but by Lemma 1,  $G_p$  has isomorphism order  $k$ , hence  $H_p$  is embeddable in  $G_p$  and so  $H$  is embeddable in  $G$ .

**LEMMA 3.** *A torsion free Abelian group has isomorphism order  $k$  if and only if it is a vector space over the rationals of dimension less than  $k$ .*

*Proof.* Let  $G$  be a torsion free Abelian group having isomorphism order  $k$ . Now  $G$  does not contain the direct sum of the integers taken  $k$  times, for, if it did, the group consisting of the direct sum of the

integers taken a greater number of times than the cardinality of  $G$  would have every  $k$  elements embeddable in  $G$  and hence by hypothesis would be embeddable in  $G$ , a contradiction.

Let  $m$  be the maximal number of elements of  $G$  which are independent over the integers. By what was just said  $m$  must be less than  $k$ . Any  $m$  dimensional vector space over the rationals is embeddable in  $G$ , by hypothesis. So  $G$  contains a vector space over the rationals of dimension  $m$ , call this space  $V$ . The space  $V$  is a divisible subgroup of  $G$  and hence is a direct summand so  $G = A + V$ . Let  $a$  be a nonzero element of  $A$ . Since  $m$  is the maximal number of independent elements of  $G$ ,  $na$  is in  $V$  for some nonzero integer  $n$ , but since  $na$  is in  $A$  it is zero and therefore  $a$  is zero and so  $G = V$ .

Conversely, if  $G$  is a vector space over the rationals of dimension less than  $k$  and  $H$  is a group every  $k$  elements of which are embeddable in  $G$  then  $H$  is embeddable in  $G$ . To see this, observe that  $H$  can be embedded in a vector space over the rationals consisting of all couples of the form  $(n, h,)$  when  $n$  is a nonzero integer and equivalence is defined in the natural way, and the dimension of this space is less than  $k$  for if not, there exist  $k$  elements of  $H$  not embeddable in  $G$ , which completes the proof.

**THEOREM 1.** *An Abelian group  $G$  has isomorphism order  $k$  if and only if  $G$  is the direct sum of two groups, one torsion, the other torsion free. The torsion free group is a vector space over the rationals of dimension less than  $k$ , while the torsion group can be written as a direct sum of fewer than  $k$  subgroups of the rationals mod one.*

*Proof.* Let  $G$  be an Abelian group having isomorphism order  $k$ . The theorem follows from the lemmas if  $G$  is torsion or torsion free. Now  $G$  contains a vector space  $V$  over the rationals of dimension  $n$  less than  $k$  where  $n$  is the maximal number of elements of  $G$  which are independent over the integers. This holds by an application of the argument of Lemma 3. Regard  $V$  as a group, then  $V$  is a direct summand of  $G$  since  $V$  is divisible. So  $G = A + V$  and  $A$  is torsion, for if  $x$  is in  $A$  then  $mx$  is in  $V$  for some nonzero integer  $m$ , hence  $mx = 0$ . Now apply Lemma 2 to  $A$  and obtain the necessity of the theorem.

To prove the sufficiency, let  $G$  be an Abelian group such that  $G = T + V$  where  $T = A_1 + A_2 + \dots + A_s$  and each  $A_i$  is a subgroup of the rationals mod one and  $s < k$ , and  $V$  is a vector space over the rationals of dimension less than  $k$ .

We must show that if  $H$  is an Abelian group, every  $k$  (or fewer) elements of which are embeddable in  $G$ , then  $H$  is embeddable in  $G$ .

$H$  does not contain  $k$  elements which are independent over the

integers. Hence  $H$  contains at least one subgroup  $H_0$  such that  $h \in H$  implies  $rh \in H_0$  for some natural number  $r$  and such that  $H_0$  is embeddable in  $G$ .

Let  $T^*$  be the direct sum of the rationals mod one taken  $s$  times. Let  $G^* = T^* + V$ . We will show that if  $\phi$  is an isomorphism from  $H_0$  into  $G^*$  then if  $H_0 \neq H$ ,  $\phi$  can be properly extended. Then the embeddability of  $H$  in  $G^*$  can be obtained by a transfinite argument. Finally, we will see that  $H$  is embeddable in  $G$ .

So let  $H_0$  be a subgroup of  $H$  such that  $h \in H$  implies  $rh \in H_0$  for some integer  $r$  and let  $F$  be an isomorphism from  $H_0$  into  $G^*$ . If  $H_0 = H$  we are done, if not, let  $h \notin H_0$ , and  $m$  the smallest natural number such that  $mh \in H_0$ .

*Case 1.*  $m = p$ ,  $p$  a prime. Let  $M = [z \mid pz = F(ph), z \in G^*]$ . For convenience, we will refer to  $M$  as the set of all the “ $p$ th roots” of  $F(ph)$ , and note that  $M$  is finite, and that the number of elements in  $M$  is exactly the number of “ $p$ th roots” of 0 in  $G^*$ . Now, not every element of  $M$  is in  $F(H_0)$ , for if so, a glance at the inverse images will show that the inverse image of every element of  $M$  is a “ $p$ th root” of  $ph$ . But  $F(ph)$  has at least as many “ $p$ th roots” in  $G^*$  as  $ph$  has in  $H$ . Hence  $h$  itself is in  $H_0$  a contradiction.

We conclude that some element of  $M$ , call it  $z$ , is not in  $F(H_0)$ . Furthermore, if  $0 < n < p$ , then  $nz \notin F(H_0)$  and hence  $F$  can be extended in the natural way.

*Case 2.*  $m$  not a prime, then  $m = qt$  where  $q$  is a prime. Apply the argument of Case 1 to the set of all  $q$ th roots of  $F(mh)$ .

This shows that  $H$  is embeddable in  $G^*$ . But by Lemma 2, if  $T'$  is the torsion subgroup of  $H$ ,  $T'$  is embeddable in  $T$ . Hence it is easily seen that  $H$  is actually embeddable in  $G$ , which completes the proof.

In the above theorem, nothing has been said about smallest isomorphism order. However, it is easy to see that, if  $G$  has smallest isomorphism order  $k$  then either the torsion free summand of  $G$  has rank  $k-1$  or the torsion summand cannot be written as a direct sum of fewer than  $k-1$  subgroups of the rationals mod 1.

The next step up in the hierarchy of isomorphism order is given by the following definition.

**DEFINITION.** A group  $G$  is said to have *hyperfinite isomorphism order* if, whenever every finitely generated subgroup of a group  $H$  is embeddable in  $G$ , then  $H$  is embeddable in  $G$ .

The proof of the next theorem is similar to that of Theorem 1, and

rests on the fact that a torsion group has hyperfinite isomorphism order if and only if the rank of each primary subgroup is finite, while a torsion free group has hyperfinite isomorphism order if it is a finite dimensional vector space over the rationals.

**THEOREM 2.** *An Abelian group  $G$  has hyperfinite isomorphism order if and only if it is the direct sum of two groups, one torsion, the other torsion free. The torsion free group is a finite dimensional vector space over the rationals while the torsion summand has no primary subgroup of infinite rank.*

**REMARK.** If the smallest isomorphism order  $G$  has is hyperfinite, then there is no upper bound on the ranks of the primary subgroups of  $G$ .

This concludes the analysis of Abelian groups having finite or hyperfinite isomorphism order.<sup>2</sup> In a subsequent paper, we hope to give some results concerning Abelian groups having transfinite isomorphism order.

Also, this notion can be carried over to other systems, such as rings, a direction in which some preliminary results have been obtained.

#### REFERENCES

1. L. M. Blumenthal, *Theory and applications of distance geometry*, Oxford at the Clarendon Press, 1953.
2. I. Kaplansky, *Infinite Abelian groups*, University of Michigan Press, Ann Arbor, 1954.

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