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ON THE RADICAL OF A GROUP ALGEBRA

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W. E. DESKINS

A basic result in the study of group algebras and characters states that the group algebra $\mathfrak{A}(\mathcal{G})$ of a finite group \mathcal{G} over the field \mathfrak{F} of characteristic $p \neq 0$ has a nonzero radical \mathfrak{R} if and only if p is a divisor of $o(\mathcal{G})$, the order of \mathcal{G} . This suggests that \mathfrak{R} is related in some manner to the Sylow p -groups of \mathcal{G} and that it may be possible to define \mathfrak{R} in terms of these subgroups. In [6] Jennings showed that if $o(\mathcal{G}) = p^a$, then \mathfrak{R} is of dimension $p^a - 1$ and has as a basis the set of elements $P_i - 1$. As a generalization of this define \mathfrak{N} to be the intersection of all the left ideals of $\mathfrak{A}(\mathcal{G})$ generated by the radicals of the group algebras of the Sylow p -groups of \mathcal{G} . Then \mathfrak{N} is a nilpotent ideal of $\mathfrak{A}(\mathcal{G})$ (cf. [2]), and Lombardo-Radici has shown [8] that $\mathfrak{N}' = \mathfrak{R}$ provided \mathcal{G} has a unique Sylow p -group or $o(\mathcal{G}) = pq$ where q is also a prime. Also, in [9] he demonstrated that if \mathcal{G} is the simple group of order 60 and if $p = 2$ or 3 then \mathfrak{N}' is a proper subideal of \mathfrak{R} . In this paper it will be shown that $\mathfrak{N}' = \mathfrak{R}$ if one of the following conditions is satisfied:

- (A) \mathcal{G} is homomorphic with a Sylow p -group of \mathcal{G} .
- (B) \mathcal{G} is a super-solvable group.
- (C) \mathcal{G} is a solvable group with $(o(\mathcal{G}), p^2) = p$.

In the last section of the paper an application to a related problem is made. If \mathcal{G} contains an invariant p -group then $\mathfrak{A}(\mathcal{G})$ is bound to its radical \mathfrak{R} (i.e., if a in $\mathfrak{A}(\mathcal{G})$ is an element such that $a\mathfrak{R} = \mathfrak{R}a = 0$, then a is in \mathfrak{R}). This raises the question: If $\mathfrak{A}(\mathcal{G})$ is bound to its radical \mathfrak{R} , does \mathcal{G} contain an invariant p -group? This is equivalent to the question: Does \mathcal{G} contain an invariant p -group if \mathcal{G} possesses no irreducible representation of highest kind? (An irreducible representation of highest kind is one whose dimension is divisible by the highest power of p which divides $o(\mathcal{G})$.) It is shown that if \mathcal{G} is a group such that $\mathfrak{N}' = \mathfrak{R}$ and if the Sylow p -groups of \mathcal{G} are cyclic, then the above question is answered affirmatively. Also an example is given where the answer is negative.

1. Type A. Let \mathcal{G} be a group of order of order $g = hp^a$, $(h, p) = 1$, with a normal subgroup \mathcal{H} of order h . And let \mathfrak{F} be an algebraically closed field of characteristic p . (The requirement that \mathfrak{F} be algebraically closed is only a convenience since the dimension of \mathfrak{N}' is

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unaffected by any extension of the ground field.)

THEOREM 1. *The radical \mathfrak{R} of the group algebra $\mathfrak{A}(\mathcal{G})$ of the group \mathcal{G} over the field \mathfrak{F} equals \mathfrak{R}' , the intersection of all the left ideals of $\mathfrak{A}(\mathcal{G})$ generated by the radicals of the group algebras of the Sylow p -groups of \mathcal{G} .*

Let \mathcal{P} be a Sylow p -group of \mathcal{G} : then \mathcal{G}/\mathcal{H} is isomorphic with \mathcal{P} and \mathcal{G} is an extension of \mathcal{H} by \mathcal{P} . Now $\mathfrak{A}(\mathcal{P})$, the group algebra of \mathcal{P} over \mathfrak{F} , has the radical \mathfrak{R} which is of dimension $p^a - 1$ over \mathfrak{F} and has as a basis the differences $P_i - 1$, all $P_i \in \mathcal{P}$. Form \mathfrak{M} , the left ideal of $\mathfrak{A}(\mathcal{G})$ generated by \mathfrak{R} . The ideal \mathfrak{M} is of dimension $h(p^a - 1)$ over \mathfrak{F} , and we propose to show that \mathfrak{R} , the radical of $\mathfrak{A}(\mathcal{G})$, is contained in \mathfrak{M} .

Now $\mathfrak{A}(\mathcal{H})$, the group algebra of \mathcal{H} over \mathfrak{F} , is expressible as $\mathfrak{B}_1 \oplus \dots \oplus \mathfrak{B}_n$ where \mathfrak{B}_i is a simple ideal of $\mathfrak{A}(\mathcal{H})$. Let \mathfrak{B} be one of these, and let \mathcal{P}' be the subgroup of \mathcal{P} consisting of elements P_i such that $P_i \mathfrak{B} P_i^{-1} = \mathfrak{B}$, with $o(\mathcal{P}') = r = p^c$, $0 \leq c \leq a$. The elements H of \mathcal{H} are represented by \bar{H} in \mathfrak{B} and the \bar{H} form a group \bar{H} homomorphic with \mathcal{H} . Furthermore the elements of \mathfrak{B} can be expressed linearly in terms of the elements of $\bar{\mathcal{H}}$.

If $P \in \mathcal{P}'$, then P corresponds to an automorphism of \mathfrak{B} since $P \mathfrak{B} P^{-1} = \mathfrak{B}$, and since \mathfrak{B} is central simple this automorphism is an inner automorphism of \mathfrak{B} . Thus P corresponds to a sum of elements of $\bar{\mathcal{H}}$ and so leaves the conjugate classes of $\bar{\mathcal{H}}$ invariant since these classes commute with the individual elements of $\bar{\mathcal{H}}$. Basically, therefore, we are dealing with an extension $\bar{\mathcal{G}}$ of $\bar{\mathcal{H}}$ by a p -group \mathcal{P}' in which each element of \mathcal{P}' induces an automorphism A of $\bar{\mathcal{H}}$ which leaves the conjugate classes invariant. Since the order of $\bar{\mathcal{H}}$ is prime to p it is well-known [11, p. 123] that A is an inner automorphism of $\bar{\mathcal{H}}$. Now a result due to M. Hall [4, Theorem 6.1] implies that $\bar{\mathcal{G}}$ is a direct product of \mathcal{P}' and $\bar{\mathcal{H}}$, and this leads to the conclusion that the elements of \mathcal{P}' commute elementwise with \mathfrak{B} . If $\mathfrak{Q} = \sum_{P_i \in \mathcal{P}'} P_i \mathfrak{B}$, then the radical \mathfrak{Q}' of \mathfrak{Q} equals \mathfrak{B} times the radical of $\mathfrak{A}(\mathcal{P}')$, and therefore \mathfrak{Q}' is contained in \mathfrak{M} .

If $t = p^{a-c}$ is the index of \mathcal{P}' in \mathcal{P} , then there are t distinct ideals \mathfrak{B}_i in the decomposition of $\mathfrak{A}(\mathcal{H})$ which form a set of transitivity \mathbf{T} for \mathcal{P} , with $\mathfrak{B}_1 = \mathfrak{B}$. That is, $P_i \mathfrak{B}_j P_i^{-1} \in \mathbf{T}$ if $\mathfrak{B}_j \in \mathbf{T}$ and $P_i \in \mathcal{P}$, and furthermore, if $\mathfrak{B}_i, \mathfrak{B}_j \in \mathbf{T}$, then there is a $P_k \in \mathcal{P}$ such that $\mathfrak{B}_i = P_k \mathfrak{B}_j P_k^{-1}$. Then the algebra $\mathfrak{T} = \sum P_i \mathfrak{B}_j$, all $P_i \in \mathcal{P}$ and $\mathfrak{B}_j \in \mathbf{T}$, is an ideal of $\mathfrak{A}(\mathcal{G})$, and we assert that its radical is contained in \mathfrak{M} . To

see this consider the coset expansion of \mathcal{P} relative to \mathcal{P}' , $\mathcal{P} = \sum S_i \mathcal{P}' = \sum \mathcal{P}' S_i$. Then clearly the algebra $\mathfrak{I}' = \sum_{i,j} S_i \mathcal{D}' S_j$ is a nilpotent ideal of \mathfrak{I} , while the transitivity of \mathbf{T} implies that $\mathfrak{I} - \mathfrak{I}'$ is a simple algebra. Thus \mathfrak{I}' is the radical of \mathfrak{I} and obviously is contained in \mathfrak{M} .

As the choice of \mathfrak{B} was arbitrary in the decomposition of $\mathfrak{A}(\mathcal{H})$, clearly the process above leads to the conclusion that \mathfrak{R} is contained in \mathfrak{M} . Since the choice of \mathcal{P} was arbitrary this enables us to conclude that $\mathfrak{R}' \supseteq \mathfrak{R}$. However \mathfrak{R}' is known to be nilpotent (cf [2]), hence $\mathfrak{R}' = \mathfrak{R}$.

2. Type B. A group \mathcal{G} is defined to be *super-solvable* if it possesses a sequence of subgroups $\mathcal{G}_0 = \mathcal{G} \supset \mathcal{G}_1 \supset \dots \supset \mathcal{G}_s = 1$ such that \mathcal{G}_i is normal in \mathcal{G} and $\mathcal{G}_i/\mathcal{G}_{i+1}$ is cyclic. If in addition each $\mathcal{G}_i/\mathcal{G}_{i+1}$ is contained in the center of $\mathcal{G}/\mathcal{G}_{i+1}$, then \mathcal{G} is called a *nilpotent* group. A basic result concerning nilpotent groups states that a nilpotent group is a direct product of its Sylow groups. And a principal theorem on super-solvable groups states that a super-solvable group is an extension of a nilpotent group by a nilpotent group. (For these results see Kurosch [7, pp. 216 and 228])

THEOREM 2. *The radical \mathfrak{R} of the group algebra $\mathfrak{A}(\mathcal{G})$ of a super-solvable group \mathcal{G} over the field \mathfrak{F} equals \mathfrak{R}' .*

By the theorems quoted above \mathcal{G} contains a normal nilpotent subgroup \mathcal{G}_1 such that $\mathcal{G}/\mathcal{G}_1$ is nilpotent while \mathcal{G}_1 has a normal Sylow p -group \mathcal{P}_1 . Evidently \mathcal{P}_1 is normal in \mathcal{G} since \mathcal{G}_1 is a direct product of its Sylow groups. Then the radical of $\mathfrak{A}(\mathcal{P}_1)$ generates a nilpotent ideal \mathfrak{R}_1 of $\mathfrak{A}(\mathcal{G})$ and $\mathfrak{A}(\mathcal{G}) - \mathfrak{R}_1$ is isomorphic with the group algebra $\mathfrak{A}(\mathcal{G}/\mathcal{P}_1)$ of $\mathcal{G}/\mathcal{P}_1$. Now the group $\mathcal{G}/\mathcal{P}_1$ is a group of Type A which was discussed in the preceding section. So if \mathfrak{I} is a left ideal of $\mathfrak{A}(\mathcal{G})$ generated by the radical of the group algebra of \mathcal{P} , a Sylow p -group of \mathcal{G} , then $\mathfrak{A}(\mathcal{G}) - \mathfrak{I}$ is a completely reducible left $\mathfrak{A}(\mathcal{G})$ -module since $\mathcal{P}/\mathcal{P}_1$ is a Sylow p -group of $\mathcal{G}/\mathcal{P}_1$. Hence $\mathfrak{R} = \mathfrak{R}'$.

3. Type C. Let \mathcal{G} be a solvable group whose order is divisible by p to the first power only. Then \mathcal{G} possesses a sequence of subgroups $\mathcal{G}_0 = \mathcal{G} \supset \mathcal{G}_1 \supset \dots \supset \mathcal{G}_n = 1$ such that \mathcal{G}_{i+1} is normal in \mathcal{G}_i and $\mathcal{G}_i/\mathcal{G}_{i+1}$ is a group of order q where q is a prime.

THEOREM 3. *The radical \mathfrak{R} of the group algebra $\mathfrak{A}(\mathcal{G})$ of the group \mathcal{G} over the field \mathfrak{F} equals \mathfrak{R}' .*

The proof will be by induction on n , the length of the series defined

above. If $n = 1$ the theorem is trivially true; so assume the result to be true for groups of length less than n . Now consider \mathcal{G} , which is of length $n - 1$. If $\mathcal{G}/\mathcal{G}_1$ is of order p , then the order of \mathcal{G}_1 is prime to p and the result follows by Theorem 1. So we shall restrict our attention to the case where $\mathcal{G}/\mathcal{G}_1$ is of order q , $(p, q) = 1$.

Now by a theorem due to P. Hall [5] \mathcal{G} contains a group \mathcal{H} of order t , where $pt = g$, the order of \mathcal{G} . If \mathcal{P} is a Sylow p -group of \mathcal{G} , the left ideal of $\mathfrak{A}(\mathcal{G})$ generated by the radical of $\mathfrak{A}(\mathcal{P})$. Then $\mathfrak{A}(\mathcal{G}) - \mathfrak{J} = \mathfrak{Q}$ is a left \mathcal{G} -module representable by $\mathfrak{A}(\mathcal{H})$ and is a completely reducible $\mathfrak{A}(\mathcal{G}_1)$ -module. For \mathfrak{R}_1 , the radical of $\mathfrak{A}(\mathcal{G}_1)$, is such that $\mathfrak{R}_1\mathfrak{A}(\mathcal{G})$ is contained in \mathfrak{J} and so $\mathfrak{R}_1\mathfrak{Q} = 0$. So let \mathfrak{Q}_1 be an irreducible left \mathcal{G} -submodule of \mathfrak{Q} . Then \mathfrak{Q} may be written $\mathfrak{Q} = \mathfrak{Q}_1 + \mathfrak{Q}_2$ where \mathfrak{Q}_2 is a left $\mathfrak{A}(\mathcal{G}_1)$ -module and $\mathfrak{Q}_1 \cap \mathfrak{Q}_2 = 0$. Therefore a projection T of \mathfrak{Q} onto \mathfrak{Q}_2 exists such that T annihilates the elements of \mathfrak{Q}_1 and is the identity operator on \mathfrak{Q}_2 and such that T commutes with (the representations of) the elements of $\mathfrak{A}(\mathcal{G}_1)$. Now form the projection $T' = t^{-1} \sum H_i T H_i^{-1}$, summed over the t elements of \mathcal{H} . Then T' commutes with all the elements of \mathcal{G} and hence the submodule $\mathfrak{Q}'_1 = T'\mathfrak{Q}$ of \mathfrak{Q} is a left $\mathfrak{A}(\mathcal{G})$ -module. Furthermore $\mathfrak{Q} = \mathfrak{Q}_1 + \mathfrak{Q}'_1$ where $\mathfrak{Q}_1 \cap \mathfrak{Q}'_1 = 0$. Thus \mathfrak{Q} is a completely reducible left $\mathfrak{A}(\mathcal{G})$ -module and so \mathfrak{J} contains the radical of $\mathfrak{A}(\mathcal{G})$. This proves Theorem 3.

4. A related problem. An algebra having the property that only elements of the radical can be both left and right annihilators of the radical has been termed a *bound algebra* by M. Hall [3].

THEOREM 4. *If the group \mathcal{G} contains an invariant p -subgroup \mathcal{P} , then the group algebra $\mathfrak{A}(\mathcal{G})$ of \mathcal{G} over a field of characteristic p is a bound algebra.*

If \mathcal{P} is of order $p^a = x$ and of index y , then the radical of $\mathfrak{A}(\mathcal{P})$ generates a nilpotent ideal \mathfrak{J} of $\mathfrak{A}(\mathcal{G})$ of dimension $y(x - 1)$. Now the element $P_1 + \cdots + P_x$, where P_i is in \mathcal{P} , annihilates \mathfrak{J} and is also in the center of $\mathfrak{A}(\mathcal{G})$. Hence it generates an ideal J of order y which is contained in \mathfrak{J} and $\mathfrak{J}J = J\mathfrak{J} = 0$. Since $\mathfrak{A}(\mathcal{G})$ is a Frobenius algebra, a result due to Nakayama [10] states that the set of all right annihilators of \mathfrak{J} in $\mathfrak{A}(\mathcal{G})$ forms an ideal of dimension y . Hence \mathfrak{J} contains all of the right annihilators of \mathfrak{J} . Since $\mathfrak{J} \subseteq \mathfrak{R}$, \mathfrak{J} contains the right annihilators of \mathfrak{R} , and so $\mathfrak{A}(\mathcal{G})$ is bound to \mathfrak{R} .

This raises the question: If $\mathfrak{A}(\mathcal{G})$ is bound to its radical $\mathfrak{R} \neq 0$, does \mathcal{G} contain an invariant p -subgroup? A partial answer is provided by

THEOREM 5. *If the Sylow p -groups of \mathcal{G} are cyclic and if the*

radical \mathfrak{R} of $\mathfrak{A}(\mathcal{G})$ equals \mathfrak{R}' then \mathcal{G} contains an invariant p -subgroup if $\mathfrak{A}(\mathcal{G})$ is bound to \mathfrak{R} .

Let \mathcal{P}_1 and \mathcal{P}_2 be two Sylow p -groups of \mathcal{G} and let \mathfrak{F}_1 and \mathfrak{F}_2 be the two left ideals of $\mathfrak{A}(\mathcal{G})$ generated by the radicals of $\mathfrak{A}(\mathcal{P}_1)$ and $\mathfrak{A}(\mathcal{P}_2)$ respectively. Denote by $r(\mathfrak{F}_1)$ and $r(\mathfrak{F}_2)$ the right ideals of $\mathfrak{A}(\mathcal{G})$ consisting of all elements which annihilate \mathfrak{F}_1 and \mathfrak{F}_2 , respectively, on the right. Then since $\mathfrak{R} \subseteq \bigcap \mathfrak{F}_i$ and since $r(\mathfrak{R}) \subseteq \mathfrak{R}$ it follows readily that $r(\mathfrak{F}_1)$ and $r(\mathfrak{F}_2)$ are contained in $\mathfrak{R} = \mathfrak{R}'$. In particular, the sum S of the elements of \mathcal{P}_1 is contained in \mathfrak{F}_2 . Now the only elements of \mathfrak{F}_2 which involve 1, the identity of \mathcal{G} , also involve other elements of \mathcal{P}_2 , so that the belonging of S to \mathfrak{F}_2 implies that $\mathcal{P}_1 \cap \mathcal{P}_2$ is a group containing more than one element. Then, since the \mathcal{P}_i are all cyclic, it follows readily that the p -subgroup $\mathcal{P}_1 \cap \mathcal{P}_2$ is normal in \mathcal{G} .

Now $\mathfrak{A}(\mathcal{G})$ is bound to \mathfrak{R} if and only if \mathcal{G} possesses no representation of highest kind (see [1]). If \mathcal{G} is S_5 , the symmetric group of order 120 and if $p = 2$, then the table of ordinary characters readily demonstrates that \mathcal{G} has no representation of highest kind. Yet S_5 has no invariant 2-subgroup. It may be noteworthy that this example is related to the one given by Lombardo-Radici [9] to show that \mathfrak{R} is not always equal to \mathfrak{R}' .

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