MINIMAL COVERINGS OF PAIRS BY TRIPLES

Marion K. Fort, Jr. and G. A. Hedlund
1. Introduction. Let $F$ be a finite set with $n$ members, $n \geq 3$. An $F$-covering of pairs by triples, which we abbreviate $F$-copt, is a set $S$ of triples of distinct members of $F$ which has the property that each pair of distinct members of $F$ is contained in at least one member of $S$. If $n$ is a positive integer, $n \geq 3$, then an $n$-copt is an $F$-copt for the set $F = \{1, 2, \ldots, n\}$. We assume throughout that $n \geq 3$.

For any finite set $A$, let $C(A)$ denote the number of members of $A$. An $F$-copt $S$ is minimal if $C(S) \leq C(S')$ for every $F$-copt $S'$. If $n = 1 \pmod{6}$ or $n = 3 \pmod{6}$, then a minimal $n$-copt $S$ turns out to be exact in the sense that each pair is contained in exactly one member of $S$. Such exact coverings are called Steiner triple systems. The existence of Steiner triple systems for all $n$ (of form $6h + 1$ or $6h + 3$) was proved by M. Reiss [2] in 1859.

Let $S$ be a minimal $n$-copt and let $C(S) = \mu(n)$. The main result of this paper is obtained in §2, where we determine $\mu(n)$ explicitly for $n \geq 3$. In §3 we discuss certain properties of minimal $n$-copts, and give several methods for constructing minimal $n$-copts.

2. Determination of $\mu(n)$. Let $S$ be a minimal $n$-copt. For each integer $i$, $1 \leq i \leq n$, we define $\alpha(i)$ to be the number of members of $S$ that contain $i$. Then

$$\sum_{i=1}^{n} \alpha(i) = 3 \cdot C(S).$$

Since $i$ must appear in members of $S$ with $n - 1$ other numbers we have $\alpha(i) \geq \lfloor n/2 \rfloor$. ($\lfloor x \rfloor$ is the largest integer which is not greater than $x$.) Thus,

$$\mu(n) = C(S) \geq \frac{n \lfloor n/2 \rfloor}{3}.$$

Since $(n/3) \lfloor n/2 \rfloor$ may not be an integer, we define $\varphi(n)$ to be the least integer which is not less than $(n/3) \lfloor n/2 \rfloor$. It is easy to compute

$$\varphi(n) = \begin{cases} 
\frac{n^2}{6} & \text{if } n = 6k, \\
\frac{n(n-1)}{6} & \text{if } n = 6k + 1 \text{ or } n = 6k + 3, \\
\frac{(n^2 + 2)}{6} & \text{if } n = 6k + 2 \text{ or } n = 6k + 4, \\
\frac{(n^2 - n + 4)}{6} & \text{if } n = 6k + 5.
\end{cases}$$

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We may clearly improve (1) to

(3) \[ \mu(n) = C(S) \geq \varphi(n). \]

Our main theorem proves that in (3) equality holds for every \( n \).

Let \( A, B \) and \( C \) be pairwise disjoint sets, each having the same number \( n \) of members. A tricover for the system \((A, B, C)\) is a set \( K \) of triples \((x, y, z)\), \( x \in A, y \in B, z \in C \) such that each pair \( uv, u \) and \( v \) in different ones of \( A, B, C \), is contained in exactly one member of \( K \).

**Lemma 1.** If \( n \) is a positive integer and \( A, B, C \) are pairwise disjoint sets each of which has \( n \) members, then a tricover \( K \) for \((A, B, C)\) exists. Moreover, if \( a \in A, b \in B \) and \( c \in C \), then \( K \) may be chosen so that \((a, b, c) \in K\).

**Proof.** Let the members of \( A, B, C \) be respectively

\[ a_1, a_2, \ldots, a_n; \quad b_1, b_2, \ldots, b_n; \quad c_1, c_2, \ldots, c_n, \]

where \( a_i = a, b_i = b, c_i = c \). We define \( K \) to be the set of all triples \((a_i, b_j, c_k)\) for which \( k = i + j - 1 \) \((\text{mod} \ n)\), \( 1 \leq i, j, k \leq n \). The set \( K \) obviously has the desired properties.

**Remark.** Any tricover for \((A, B, C)\) must have \( n^3 \) members.

**Lemma 2.** Let \( A, B, C \) be pairwise disjoint sets, each having \( n \) members. Let \( p \) be an integer such that \( 0 < p \leq n/2 \). Let \( A^* \subseteq A, B^* \subseteq B, C^* \subseteq C \) be sets, each of which has \( p \) members and let \( K^* \) be a tricover for \((A^*, B^*, C^*)\). Then there exists a tricover \( K \) for \((A, B, C)\) such that \( K^* \subseteq K \).

**Proof.** Let

\[ A = \{a_1, a_2, \ldots, a_n\}, \]
\[ B = \{b_1, b_2, \ldots, b_n\}, \]
\[ C = \{c_1, c_2, \ldots, c_n\}. \]

We can assume that

\[ A^* = \{a_1, a_2, \ldots, a_p\}, \]
\[ B^* = \{b_1, b_2, \ldots, b_p\}, \]
\[ C^* = \{c_1, c_2, \ldots, c_p\}. \]

For \( 1 \leq i, j \leq p \), let \( m_{ij}^* \) be the unique integer \( k \) such that \((a_i, b_j, c_k) \in K^* \). Clearly \( 1 \leq m_{ij}^* \leq p \) and the square array \((m_{ij}^*)\) is a Latin square of order \( p \). It follows from a theorem of Marshall Hall [1] that there exists a Latin square \((m_{ij})\), \( 1 \leq i, j \leq n \), such that \( m_{ij} = m_{ij}^* \).
Let \[ K = \{(a_i, b_j, c_m) | 1 \leq i, j \leq n\} \].

The set \(K\) is the desired tricover.

In order to produce an inductive proof of our main theorem, it is convenient to restrict ourselves to a special type of minimal \(n\)-copt for the case \(n \equiv 5 \pmod{6}\). Also, for \(n \equiv 3 \pmod{6}\), there is a special type of minimal \(n\)-copt whose existence we wish to establish, and it is possible to include this result in our main theorem. For these reasons we introduce the notion of "admissible \(F\)-copt."

An \(F\)-copt \(S\) is admissible if \(C(S) = \varphi(n), n = C(F)\), and:

1. \(n \equiv 0, 1, 2, \text{ or } 4 \pmod{6}\);
2. \(n \equiv 3 \pmod{6}\) and \(S\) contains a set of pairwise disjoint triples whose union is \(F\); or
3. \(n \equiv 5 \pmod{6}\) and \(S\) contains four elements of the form \((a, b, x), (a, b, y), (a, b, z), (x, y, z)\).

**Theorem 1.** If \(n\) is a positive integer, \(n \geq 3\), then there exists an admissible \(n\)-copt.

**Proof.** Our proof is by induction on \(n\). However, it is necessary to prove independently that there are admissible \(n\)-copts for \(n = 3, 5, 7, 9, 11, 13, \text{ and } 15\). We accomplish this by exhibiting such admissible \(n\)-copts.

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Our proof now divides into six cases. In Case \( r, 0 \leq r \leq 5 \), we assume that \( n \equiv r \pmod{6} \), that \( n > 3 \) and that there exist admissible \( m \)-copts for \( 3 \leq m < n \). We then show that these assumptions imply that there exists an admissible \( n \)-copt.

**Case 0.** Let \( S_1 \) be an admissible \((n - 1)\)-copt having \((1, 2, 3), (1, 2, 4), \) and \((1, 2, 5)\) as three of its members. If we delete \((1, 2, 3)\) from \( S_1 \) and add 

\[
(1, 3, n), (2, 3, n), (4, 5, n), (6, 7, n), \ldots, (n - 2, n - 1, n),
\]

we obtain a set \( S \) of triples which is an \( n \)-copt. Since \( S_1 \) has

\[
[(n - 1)^2 - (n - 1) + 4] / 6 = (n^2 - 3n + 6) / 6
\]

members, \( S \) has

\[
(n^2 - 3n + 6) / 6 - 1 + n / 2 = n^2 / 6 = \varphi(n)
\]

members.

**Case 1.** We have exhibited admissible \( n \)-copts for \( n = 7 \) and \( n = 13 \). Therefore we may assume \( n = 6h + 1, h > 2 \).

We consider two subcases.

1. **Subcase i.** Either \( h \equiv 0 \) or \( h \equiv 1 \pmod{3} \). Then there exists \( k \) such that \( 2h + 1 = 6k + 1 \) or \( 2h + 1 = 6k + 3 \).

Let

\[
A_1 = \{1, \ldots, 2h, n\}
\]

\[
A_2 = \{2h + 1, \ldots, 4h, n\}
\]

\[
A_3 = \{4h + 1, \ldots, 6h, n\}
\]

and let \( S_j \) be an admissible \( A_j \)-copt for \( j = 1, 2, 3 \). Let \( T \) be a tricover for \( \{1, \ldots, 2h\}, \{2h + 1, \ldots, 4h\}, \{4h + 1, \ldots, 6h\} \). We now define \( S = S_1 \cup S_2 \cup S_3 \cup T \). It is easy to verify that \( S \) is an \( n \)-copt, and that

\[
3 \cdot \frac{(2h + 1)2h}{6} + (2h)^2 = \frac{n(n - 1)}{6} = \varphi(n)
\]

members.
Subcase ii. $h \equiv 2 \pmod{3}$. In this case there exists $k$ such that $2h + 1 = 6k + 5$. We define $A_1, A_2, A_3$ as above. Now, for $j = 0, 1, 2$, we let $S_{j+1}$ be an admissible $A_{j+1}$-copt such that $S_{j+1}$ contains a subset $R_{j+1}$ whose members are:

$$(2jh + 1, 2jh + 2, 2jh + 3)$$
$$(2jh + 1, 2jh + 2, 2jh + 4)$$
$$(2jh + 1, 2jh + 2, n)$$
$$(2jh + 3, 2jh + 4, n).$$

Let $T$ be a tricover for $(\{1, \cdots, 4\}, \{2h + 1, \cdots, 2h + 4\}, \{4h + 1, \cdots, 4h + 4\})$, and let $T^*$ be a tricover for $(\{1, \cdots, 2h\}, \{2h + 1, \cdots, 4h\}, \{4h + 1, \cdots, 6h\})$ that is an extension of $T$. Since $h \geq 5$, the existence of such a tricover follows from Lemma 2. We next take an admissible copt $U$ for

$$\{1, \cdots, 4, 2h + 1, \cdots, 2h + 4, 4h + 1, \cdots, 4h + 4, n\}.$$

Finally, we define

$$S = (S_1 - R_1) \cup (S_2 - R_2) \cup (S_3 - R_3) \cup (T^* - T) \cup U.$$

It is easy to check that $S$ is an $n$-copt. The number of member of $S$ is

$$3 \cdot \left\lfloor (2h + 1)^2 - (2h + 1) + 4 \right\rfloor + \left\lfloor (2h)^2 - 16 \right\rfloor + 26$$

$$= 6h^2 + h = \frac{n(n - 1)}{6}.$$

Thus, $S$ is admissible.

Case 2. Let $S_1$ be an admissible $(n-1)$-copt. We define $S$ to be the set of triples obtained by adding to $S_1$ the triples

$$(1, 2, n), (3, 4, n), \cdots, (n - 3, n - 2, n), (n - 2, n - 1, n).$$

Then, $S$ is an $n$-copt and $S$ has

$$\frac{(n - 1)(n - 2)}{6} + \frac{n}{2} = \frac{n^2 + 2}{6}$$

members. Thus $S$ is admissible.

Case 3. There exists $h$ such that $n = 6h + 3$. Since we have listed admissible $n$-copts for $n = 3, 9, 15$, we may assume $h > 2$. We consider two subcases.
Subcase i. $h \equiv 0$ or $h \equiv 1 \pmod{3}$. In this case there exists $k$ such that $2h + 1 = 6k + 1$ or $2h + 1 = 6k + 3$. Let $S_1$ be an admissible $(2h+1)$-copt. For each triple $(a, b, c) \in S_1$, we choose a tricover for $\{3a - 2, 3a - 1, 3a\}, \{3b - 2, 3b - 1, 3b\}, \{3c - 2, 3c - 1, 3c\}$. The union of all such tricovers, together with the triples $(1, 2, 3), (4, 5, 6), \ldots, (n-2, n-1, n)$ is an $n$-copt $S$. The number of members of $S$ is

$$9 \cdot \frac{(2h + 1) \cdot 2h}{6} + (2h + 1) = \frac{(2h + 1)(3h + 1)}{6} = \frac{n(n-1)}{6}.$$ 

If follows that $S$ is admissible.

Subcase ii. $h \equiv 2 \pmod{3}$. In this case there exists $k$ such that $2h + 1 = 6k + 5$. We choose an admissible $(2h+1)$-copt $S_1$ that contains the triples $(1, 2, 3), (1, 2, 4), (1, 2, 5), (3, 4, 5)$. If $(a, b, c)$ is any other member of $S_1$, we choose a tricover for $\{3a - 2, 3a - 1, 3a\}, \{3b - 2, 3b - 1, 3b\}, \{3c - 2, 3c - 1, 3c\}$. Let $S_2$ be the 15-copt exhibited at the beginning of our proof. We now define $S$ to be the set whose members are the members of $S_2$, the members of the chosen tricovers, and the triples $(16, 17, 18), \ldots, (n-2, n-1, n)$. $S$ is an $n$-copt, and the number of members of $S$ is

$$35 + 9 \left[ \frac{(2h + 1)^2 - (2h + 1) + 4 - 4}{6} \right] + \frac{n - 15}{3} = \frac{n(n-1)}{6}.$$ 

Since $S$ has $(1, 2, 3), (4, 5, 6), \ldots, (n-2, n-1, n)$ as members, $S$ is admissible.

Case 4. For this case, the construction is exactly the same as in Case 2.

Case 5. We first observe that numbers of the form $6h + 5$, $h$ a non-negative integer, form the same set as numbers of the form $3s - 4$, $s$ an odd integer and $s > 1$. We have listed an admissible 5-copt, and an admissible 11-copt. Thus, we may assume $n = 6h + 5 = 3s - 4, s > 5$. We consider two subcases.

Subcase i. There exists $k$ such that $s = 6k + 1$ or $s = 6k + 3$. In this case, we let

$$A_1 = \{1, \ldots, s - 2\},$$

$$A_2 = \{s - 1, \ldots, 2s - 4\},$$

$$A_3 = \{2s - 3, \ldots, 3s - 6\}.$$ 

There is a tricover $K$ of $(A_i, A_s, A_3)$ such that $(1, s - 1, 2s - 3) \in K$. For $i = 1, 2, 3$ we define
and let $S_i$ be an admissible $R_i$-copt such that $(1, 3s - 5, 3s - 4) \in S_i$, $(s - 1, 3s - 5, 3s - 4) \in S_2$ and $(2s - 3, 3s - 5, 3s - 4) \in S_3$. We define

$$S = K \cup S_1 \cup S_2 \cup S_3.$$  

It is easy to see that $S$ is an $n$-copt, and that $S$ has

$$(s - 2)^2 + \frac{3s(s - 1)}{6} = \frac{3s^2 - 9s + 8}{2} = \frac{n^2 - n + 4}{2}$$

members. Since $(1, 3s - 5, 3s - 4), (s - 1, 3s - 5, 3s - 4), (2s - 3, 3s - 5, 3s - 4), (1, s - 1, 2s - 3)$ are members of $S$, $S$ is admissible.

Subcase ii. There exists $k$ such that $s = 6k + 5$. We define

$$A_1 = \{1, \ldots, s - 2\},$$

$$A_2 = \{s - 1, \ldots, 2s - 4\},$$

$$A_3 = \{2s - 3, \ldots, 3s - 6\}$$

and let $R_i = A_i \cup \{3s - 5, 3s - 4\}$ for $i = 1, 2, 3$. By the inductive hypothesis, there exists an admissible $R_i$-copt $S_i$ such that $S_i$ contains the set $B_i$, where

$$B_i = \{(1, 2, 3), (1, 3s - 5, 3s - 4), (2, 3s - 5, 3s - 4), (3, 3s - 5, 3s - 4)\},$$

$$B_2 = \{(s - 1, s, s + 1), (s - 1, 3s - 5, 3s - 4), (s, 3s - 5, 3s - 4),$$

$$(s + 1, 3s - 5, 3s - 4)\},$$

$$B_3 = \{(2s - 3, 2s - 2, 2s - 1), (2s - 3, 3s - 5, 3s - 4), (2s - 2, 3s - 5, 3s - 4),$$

$$(2s - 1, 3s - 5, 3s - 4)\}.$$

Let $G = \{1, 2, 3, s - 1, s, s + 1, 2s - 3, 2s - 2, 2s - 1, 3s - 5, 3s - 4\}$. $G$ has 11 members, and hence there exists an admissible $G$-copt $M$.

We choose a tricover $T_1$ for $\{(1, 2, 3), \{s - 1, s, s + 1\}, \{2s - 3, 2s - 2, 2s - 1\}\}$ and extend $T_1$ to a tricover $T$ for $(A_1, A_2, A_3)$.

We now define

$$S = (S_1 - B_1) \cup (S_2 - B_2) \cup (S_3 - B_3) \cup M \cup (T - T_1).$$

It is a routine matter to verify that $S$ is an $n$-copt. The number of members of $S$ is

$$3\left[\frac{s^2 - s + 4}{6} - 4\right] + 19 + \left[(s - 2)^2 - 9\right] = \frac{3s^2 - 9s + 8}{2} = \frac{n^2 - n + 4}{6}.$$ 

Since $S \supset M$ and $M$ is admissible, it follows that $S$ is admissible.

3. Properties of minimal $n$-copts. Let $S$ be a minimal $n$-copt. If $n \equiv r \pmod{6}$, for $r = 0, 2, 4, 5$, then the covering is not exact and some
pairs must be contained in more than one member of $S$. However, it is possible to state precisely the way in which this sort of "multiple covering" takes place. Our results are contained in the next three theorems.

**Theorem 2.** Let $n = 6k$, and let $S$ be an $n$-copt for which $C(S) = \phi(n)$. There exists a partition of $\{1, 2, \ldots, n\}$ into $3k$ pairs $P_1, P_2, \ldots, P_{3k}$, each of which is contained in exactly two members of $S$. Every other pair $(u, v), 1 \leq u < v \leq n$, is contained in exactly one member of $S$.

**Proof.** For $1 \leq j \leq n$, let $f(j)$ be the number of members of $S$ that contain $j$. It is clear that $f(j)$ is at least $n/2$, so that $f(j) = n/2 + g(j)$, $g(j) \geq 0$. We obtain

$$\sum_{j=1}^{n} f(j) = 3\phi(n).$$

Thus

$$\sum_{j=1}^{n} \left(\frac{n}{2} + g(j)\right) = 3 \cdot \frac{n^2}{6},$$

and

$$\frac{n^2}{2} + \sum_{j=1}^{n} g(j) = \frac{n^2}{2}.$$ 

We see that $g(j) = 0$ for $j = 1, \ldots, n$ and $f(j) = n/2$. Since for each $k \neq j$ there is at least one member of $S$ which contains $(j, k)$, there must exist $j^* \neq j$ such that $(j, j^*)$ is contained in exactly two members of $S$, and $(j, k)$ is contained in exactly one member of $S$ for $j \neq k \neq j^*$. Moreover, $j^{**} = j$, and hence the pairs $(j, j^*)$ are the $n/2$ pairs $P_1, P_2, \ldots, P_{3k}$.

**Theorem 3.** Let $n = 6k + 2$ or $n = 6k + 4$, and let $S$ be an $n$-copt for which $C(S) = \phi(n)$. There exist $n/2 + 1$ pairs $P_1, \ldots, P_{n/2+1}$ which are contained in exactly two members of $S$. Every other pair is contained in exactly one member of $S$. There exists an integer $m$ which is contained in exactly three of the pairs $P_1, \ldots, P_{n/2+1}$. Every other integer is contained in exactly one of the pairs $P_1, \ldots, P_{n/2+1}$.

**Proof.** Let $f(j)$ be the number of members of $S$ that contain the integer $j$. Since $f(j) \geq n/2$, we can write

$$f(j) = \frac{n}{2} + g(j), \quad g(j) \geq 0.$$

Then
Thus $\sum_{j=1}^{n} g(j) = 1$. There exists an integer $m$ such that $g(m) = 1$ and $g(j) = 0$ for $j \neq m$.

Now suppose $j \neq m$. There must exist $j^*$ such that $(j, j^*)$ is contained in exactly two members of $S$, and $(j, h)$ is contained in exactly one member of $S$ for $j \neq h \neq j^*$.

Since there are $n/2 + 1$ members of $S$ that contain $m$, and each pair $(m, j)$ is contained in at least one and not more than two members of $S$, there exist $a, b, c$, such that $(m, a), (m, b), (m, c)$ are each contained in exactly two members of $S$, and $(m, j)$ is contained in exactly one member of $S$ if $j \neq a, j \neq b$, and $j \neq c$.

If $j$ is a member of $T = \{1, \cdots, n\} - \{m, a, b, c\}$, then $j^{**} = j$. Hence $T$ is partitioned into pairs $P_1, P_2, \cdots, P_{(n-1)/2}$, each of which is contained in exactly two members of $S$. These pairs, together with $(m, a), (m, b), (m, c)$ form the set $\mathcal{P}$.

THEOREM 4. If $n = 6k + 5$ and $S$ is a minimal $n$-copt for which $\varphi(n) = (n^2 - n + 4)/6$, then one pair is contained in three members of $S$ and every other pair is contained in exactly one member of $S$.

Proof. For $1 \leq j \leq n$, we define $f(j)$ to be the number of members of $S$ that contain $j$. Clearly $f(j) \geq (n - 1)/2$. We define $g(j) = f(j) - (n - 1)/2$. Since $\sum_{j=1}^{n} f(j) = 3\varphi(n) = (n^2 - n + 4)/2$, we obtain

$$\sum_{j=1}^{n} g(j) = 2.$$  

There exists $j_i$ such that $g(j_i) > 0$. Since there are more than $(n - 1)/2$ triples of $S$ that contain $j_i$, there exists $j_2$ such that the pair $(j_i, j_2)$ is contained in at least two triples $(j_1, j_2, j_3), (j_i, j_2, j_4)$. The integer $j_2$ must be in triples with $n - 4$ integers other than $j_1, j_3, j_4$, and it requires at least $(n - 3)/2$ triples to satisfy this condition. Thus $f(j_2) \geq (n + 1)/2$ and $g(j_2) > 0$. We now see that $g(j_i) = g(j_2) = 1$ and $g(j) = 0$ if $j_1 \neq j_2$.

It now follows that if $(u, v)$ is a pair for which $g(u) = 0$ or $g(v) = 0$, then $(u, v)$ is contained in exactly one member of $S$. Since $3\varphi(n) = n(n - 1)/2 + 2$, the pair $(j_i, j_2)$ must be contained in three members of $S$.

Our Theorem 1 is of a constructive nature, and indicates how minimal $n$-copts can be constructed out of minimal $m$-copts for $m < n$. There are other methods, however, of constructing minimal $n$-copts out of minimal $m$-copts for $m < n$. We give a lemma and theorem due to Reiss [2] which are useful in this connection. Our final theorem is analogous to the Reiss Theorem.
Reiss Lemma. Let \( n \) be a positive integer. Let
\[
P = \{(u, v) \mid 1 \leq u < v \leq 2n\}.
\]
Then there exists a partition of \( P \) into sets \( S_1, S_2, \ldots, S_{2n-1} \), each containing \( n \) elements, such that for each \( i, i = 1, 2, \ldots, 2n - 1 \), the coordinates of the \( n \) pairs in \( S_i \) constitute the integers \( 1, 2, \ldots, 2n \).

Proof. Let \( j \) be an integer such that \( 1 \leq j \leq 2n - 1 \). We define
\[
T_j = \{(a, b) \mid 1 \leq a < b \leq j + 1 \text{ and } a + b = j + 2\}
\]
and
\[
R_j = \{(a, b) \mid j + 1 < a < b < 2n \text{ and } a + b = j + 2n + 1\}.
\]
Let \( S_{2n-1} = T_{2n-1} \). For \( j \) even, \( 1 \leq j \leq 2n - 2 \), let
\[
S_j = T_j \cup R_j \cup \left\{\left(\frac{j + 2}{2}, 2n\right)\right\}.
\]
For \( j \) odd, \( 1 \leq j \leq 2n - 3 \), let
\[
S_j = T_j \cup R_j \cup \left\{\left(\frac{j + 1 + 2n}{2}, 2n\right)\right\}.
\]
It may be verified that the sets \( S_j \) have the desired properties.

Reiss Theorem. Let \( m \) be odd and let \( S \) be an \( m \)-copt for which \( C(S) = \varphi(m) \). Then there exists a \((2m + 1)\)-copt \( T \) such that \( T \supset S \) and \( C(T) = \varphi(2m + 1) \).

Proof. Let \( P = \{(u, v) \mid m < u < v \leq 2m + 1\} \). We use the Reiss lemma to partition \( P \) into sets \( S_1, \ldots, S_m \), each containing \((m + 1)/2\) elements, such that for each \( i, i = 1, 2, \ldots, m, \) the coordinates of the \((m + 1)/2\) pairs in \( S_i \) constitute the integers \( m + 1, m + 2, \ldots, 2m + 1 \). We now define
\[
T = S \cup \{(i, j, k) \mid 1 \leq i \leq m \text{ and } (j, k) \in S_i\}.
\]
It is easily verified that \( T \) is a \((2m + 1)\)-copt. If \( m \equiv 1 \) or \( m \equiv 3 \) (mod 6), then
\[
C(S) = \frac{m(m - 1)}{6} + \frac{m(m + 1)}{2} = \frac{4m^2 + 2m}{6} = \frac{(2m + 1)(2m)}{6} = \varphi(2m + 1).
\]
If \( m \equiv 5 \) (mod 6), then
\[
C(S) = \frac{m^2 - m + 1}{6} + \frac{m(m + 1)}{2} = \frac{4m^2 + 2m + 4}{6} = \frac{(2m + 1)^2 - (2m + 1) + 4}{6} = \varphi(2m + 1).
\]
Theorem 5. Let \( n \) be an even integer and let \( S \) be an \( n \)-copt for which \( C(S) = \varphi(n) \). Then there exists a \( 2n \)-copt \( T \) such that \( C(T) = \varphi(2n) \) and \( S \subset T \).

Proof. According to the Reiss Lemma there exists a partition of the set

\[ P = \{(u, v) \mid n + 1 \leq u < v \leq 2n\} \]

into \( n - 1 \) sets \( A_1, A_2, \ldots, A_{n-1} \), such that for each \( i, i = 1, 2, \ldots, n - 1 \), the coordinates of the \( n/2 \) pairs in \( A_i \) constitute the integers \( \{n + 1, \ldots, 2n\} \). Let \( A_n = A_{n-1} \), and let

\[ T = S \cup \{(i, j, k) \mid i = 1, 2, \ldots, n \; ; \; (j, k) \in A_i\} \]

It is easy to prove that \( T \) satisfies the desired conditions.

References

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