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In a recent paper I introduced the following generalization of the notion of compactness:

*A topological space $X$ is $(m, n)$-compact if from every open covering $\{O_i\}$ $(i \in I)$ of $X$ whose cardinality card $I$ is at most $n$ one can select a subcovering $\{O_{i_j}\}$ $(j \in J)$ of $X$ whose cardinality* card *$J$ is at most $m$.*

A similar definition was introduced earlier by P. Alexandroff and P. Urysohn [1]. If no inaccessible cardinals exist between $m$ and $n$ the two definitions are equivalent. The present definition has the advantage that in applications the question of the existence of inaccessible cardinals does not generally come up. The basic results on $(m, n)$-compact spaces were published by me in [8] and a detailed study of generalized compactness in the Alexandroff-Urysohn sense was made by Yu. M. Smirnov in [14] and [15]. The special case $m = \omega$ and $n = \infty$ was first studied much earlier by C. Kuratowski and W. Sierpinski in [13] and [10]. These spaces are generally known as Lindelöf spaces.

The present paper contains four types of results on $(m, n)$-compactness which were obtained since the publication of [8]. The problems and the principal results are stated in the beginnings of the individual Sections 1, 2, 3, and 4.

The following notations will be used: $\bar{A}$ and $A^i$ denote the closure and the interior of the set $A$. The symbols $O$ and $C$ stand for open and closed sets, respectively. $\phi$ denotes the empty set. $N_x$ is an arbitrary neighborhood of the point $x$ and $O_x$ denotes any open set containing $x$. Filters are denoted by $\mathscr{F}$, nets by $(x_d)$ $(d \in D)$ where $D$ stands for the directed set on which the net is formed. The set of adherence points of $\mathscr{F}$ is denoted by adh $\mathscr{F}$. Similarly the set of adherence points of a net is denoted by adh$(x_d)$. The set of limit points is denoted by lim $\mathscr{F}$ and lim $(x_d)$, respectively. A topological space $X$ is called normal if for any pair of disjoint closed sets $A$ and $B$ there exist disjoint open sets $O_A$ and $O_B$ such that $A \subseteq O_A$ and $B \subseteq O_B$.

Uniform structures for a set $X$ will be denoted by $\mathscr{U}$. The symbol $U[x]$ stands for "the vicinity $U \in \mathscr{U}$ evaluated at $x \in X$" so that $U[x] = [y : (x, y) \in U]$. The composition operator is denoted by $\circ$ and so $U \circ V$ consists of those ordered pairs $(x, z) \in X \times X$ for which there is a $y \in X$ with $(x, y) \in U$ and $(y, z) \in V$.

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**1. Characterization of $(m, n)$-compactness by filters and nets.** A topological space $X$ is compact if and only if every filter $\mathscr{F}$ in $X$ has a non-void adherence. A similar characterization of compactness can be given also in terms of nets $(x_d)$ $(d \in D)$ with values in $X$. As a matter of fact it is sufficient to prove only one of these propositions. For one can associate with every filter $\mathscr{F}$ in $X$ a net $(x_d)\,(d \in D)$ with values in $X$ such that $\operatorname{adh} \mathscr{F} = \operatorname{adh}(x_d)$ and $\lim \mathscr{F} = \lim(x_d)$ and conversely given any net with values in $X$ there is a filter $\mathscr{F}$ in $X$ having the same adherence and limit as $(x_d)\,(d \in D)$. The equivalence of filters and nets relative to adherence properties is due to R. G. Bartle [3] and the equivalence relative to both adherence and limit properties is discussed in [9].

It is natural to ask whether $(m, n)$-compactness can be characterized in term of filters and nets. We shall prove here that such characterization can be given both in terms of filters and nets. Namely for every pair of cardinals $m < n$ a class of filters called $(m, n)$-filters can be selected such that $X$ is $(m, n)$-compact if and only if each of these filters has adherence points. Similarly we can define the class of $(m, n)$-nets with values in $X$ such that $X$ is $(m, n)$-compact if and only if $\operatorname{adh}(x_d)$ is not void for every one of these nets $(x_d)\,(d \in D)$. This indicates that there is a natural correspondence between the class of $(m, n)$-filters and the class of $(m, n)$-nets and one can expect that these two classes exhibit the same adherence and limit phenomena. However it will be seen that this is not the case. Hence if we consider filters and nets in a topological space $X$ not as whole classes but in their finer classification then their behavior relative to convergence is not the same.

In the next definition we use the concept of "$m$-intersection property". A family $\{C_i\}(i \in I)$ of subsets of a set $X$ is said to have the *$m$-intersection property* if every subfamily of cardinality at most $m$ has a non-void intersection. If every finite subfamily of $\{C_i\}(i \in I)$ has a non-void intersection we say that the family has the *finite intersection property* or *1-intersection property.*

DEFINITION 1.1. A filter $\mathscr{F}$ is called an $(m, n)$-filter if it has the $m$-intersection property and if it has a base $\mathscr{B}$ of cardinality $\operatorname{card} \mathscr{B} \leqq n$.

If $\mathscr{F}$ is a filter which has a base of cardinality at most $n$ then $\mathscr{F}$ is called an $n$-filter or an $(1, n)$-filter. If the filter $\mathscr{F}$ has the $m$-intersection property we say that $\mathscr{F}$ is an $(m, \infty)$-filter. A $(1, \infty)$-filter means a filter in the usual sense.

DEFINITION 1.2. A directed set $D$ is called an $(m, n)$-directed set if every subset $S \subseteqq D$ of cardinality $\operatorname{card} S \leqq m$ has an upper bound in

$D$ and if card $D \leqq n$.

If every subset $S \subseteqq D$ of cardinality card $S \leqq m$ has an upper bound in $D$ or in other words if for every $S$ with card $S \leqq m$ there is a $d \in D$ such that $s \leqq d$ for every $s \in S$ then $D$ will be called an $m$-directed set or an $(m, \infty)$-directed set. If card $D \leqq n$ we speak about a $(1, n)$-directed set. A $(1, \infty)$-directed set means a directed set in the usual sense.

DEFINITION 1.3. An $(m, n)$-net $(x_d)(d \in D)$ with values in a set $X$ is a function $x$ defined on an $(m, n)$-directed set $D$ whose function values $x_d$ belong to the set $X$.

If the directed set $D$ is linearly ordered we call $(x_d)(d \in D)$ a linearly ordered $(m, n)$-net.

It is known that filters and nets exhibit the same convergence and adherence phenomena. The following lemmas show that the same holds for the more restricted class of $(m, \infty)$-filters and $(m, \infty)$-nets:

LEMMA 1.1. *Let $X$ be a topological space and let $(x_d)(d \in D)$ be an $(m, n)$-net in $X$. Then there exists an $(m, n)$-filter $\mathscr{F}$ in $X$ having the property that* adh $\mathscr{F}$ = adh $(x_d)$ *and* lim $\mathscr{F}$ = lim $(x_d)$.

*Proof.* For every $d \in D$ we define $B_d = [x_\delta : d \leqq \delta]$. Since $D$ is an $(m, n)$-directed set the family $\mathscr{B} = \{B_d\}(d \in D)$ has the $m$-intersection property and card $\mathscr{B} \leqq n$. Let $\mathscr{F}$ be the $(m, n)$-filter generated by the filter base $\mathscr{B}$. One shows that $\mathscr{F}$ satisfies the requirements.

LEMMA 1.2. *Let $\mathscr{F}$ be an $(m, \infty)$-filter in a topological space $X$. Then there is an $(m, \infty)$-net $(x_d)(d \in D)$ with values in $X$ and having the property that* adh $(x_d)$ = adh $\mathscr{F}$ *and* lim $(x_d)$ = lim $\mathscr{F}$.

*Proof.* Let us consider the set $D$ of all ordered pairs $d = (x, F)$ where $x \in F \in \mathscr{F}$. We say that $d_1 \leqq d_2$ if $F_1 \supseteqq F_2$. Under this ordering $D$ becomes an $(m, \infty)$-directed set. In fact if $d_i = (x_i, F_i)$ for $i \in I$ and card $I \leqq m$ then $d_i \leqq d$ for every $d = (x, F)$ where $x \in F = \cap F_i \in \mathscr{F}$. An $(m, \infty)$-net can be defined on $D$ with values in $X$ by choosing $x_d = x$ for every $d = (x, F) \in D$. Let $x \in \lim \mathscr{F}$ and let $N_x$ be arbitrary. Then there is an $F \in \mathscr{F}$ such that $F \subseteqq N_x$. Hence if $\delta = (\xi, \Phi)$ satisfies $d \leqq \delta$, or in other words if $\Phi \subseteqq F$ then $x_\delta = \xi \in \Phi \subseteqq F \subseteqq N_x$ and so $x$ is a limit point of $(x_d)(d \in D)$. Conversely let $x \in \lim (x_d)$ and let $N_x$ be given. Then there is a $d = (x, F)$ such that $x_\delta \in N_x$ for every $\delta$ satisfying $d \leqq \delta$. Using this for every $\delta = (\xi, F)(\xi \in F)$ we see that

$x_\delta = \xi \in N_x$ for every $\xi \in F$ and so $F \subseteqq N_x$. This shows that $x \in \lim \mathscr{F}$ and $\lim (x_d) = \lim \mathscr{F}$.

Now we suppose that $x \in \operatorname{adh} \mathscr{F}$ so that $N_x \cap F \neq \phi$ for every neighborhood $N_x$ and for every $F \in \mathscr{F}$. Given $N_x$ and $d = (x, F)$ we choose $\xi$ in $N_x \cap F$ and consider $\delta = (\xi, F)$. Then $d \leqq \delta$ and $x_\delta = \xi \in N_x$ and so $x \in \operatorname{adh} (x_d)$. On the other hand if $x \in \operatorname{adh} (x_d)$ then given $F \in \mathscr{F}$ and $N_x$ there is a $\delta \in D$ such that $d = (x, F) \leqq \delta = (\xi, \Phi)$ and $x_\delta \in N_x$. In other words $x_\delta = \xi \in \Phi \cap N_x \subseteqq F \cap N_x$ and so $F$ and $N_x$ intersect for every $F \in \mathscr{F}$ and for every $N_x \in \mathscr{N}(x)$. This proves that $x \in \operatorname{adh} \mathscr{F}$ and $\operatorname{adh} (x_d) = \operatorname{adh} \mathscr{F}$.

Using the same reasoning similar results can be derived for $(m, n)$-filters. For instance we can easily prove that if $X$ is an $(m, n)$-filter in a space $X$ and if card $F \leqq n$ for some $F \in \mathscr{F}$ then there is an $(m, n)$-net $(x_d)(d \in D)$ with values in $X$ and having the property that $\operatorname{adh} (x_d) = \operatorname{adh} \mathscr{F}$ and $\lim (x_d) = \lim \mathscr{F}$. If the hypothesis card $F \leqq n$ is dropped we can find only an $(m, n)$-net satisfying $\operatorname{adh} \mathscr{F} \supseteqq \operatorname{adh} (x_d)$ and $\lim \mathscr{F} \subseteqq \lim (x_d)$. None of these results will be used in the sequel.

We can easily find examples where only the strict inclusion $\operatorname{adh} \mathscr{F} \supset \operatorname{adh} (x_d)$ can be realized. For instance let $X$ be a non-countable set and let $X$ be topologized by the discrete topology. If $\mathscr{F}$ consists of the single element $X$ then $\mathscr{F}$ is an $(m, n)$-filter for any pair of cardinals $m$ and $n$. Moreover $\operatorname{adh} \mathscr{F} = X$ and so the cardinality of $\operatorname{adh} \mathscr{F}$ is greater than that of $\mathscr{F}$. On the other hand if $(x_d)(d \in D)$ is a $(1, \omega)$-net with values in $X$ then the cardinality of $\operatorname{adh} (x_d)$ is at most $\omega$. Hence $\operatorname{adh} \mathscr{F} \supset \operatorname{adh} (x_d)$ for every $(1, \omega)$-net in $X$.

This example shows that $(m, n)$-filters and $(m, n)$-nets in arbitrary topological spaces have different adherence properties. Nevertheless the following theorems show that both $(m, n)$-filters and $(m, n)$-nets can be used to characterize $(m, n)$-compactness.

THEOREM 1.1. *A topological space $X$ is $(m, n)$-compact if and only if every $(m, n)$-filter in $X$ has a non-void adherence.*

*Proof.* In [8] we proved that $X$ is $(m, n)$-compact if and only if every family $\{C_i\}$ of closed sets $C_i \subseteqq X$ having the $m$-intersection property also has the $n$-intersection property. We apply this result: Let $X$ be $(m, n)$-compact and let $\mathscr{B}$ with card $\mathscr{B} \leqq n$ be a filter base for an $(m, n)$-filter $\mathscr{F}$ in $X$. Then the family $\{\overline{B}\}$ $(B \in \mathscr{B})$ has the $m$-intersection property and so it has the $n$-intersection property. Since card $\mathscr{B} \leqq n$ this implies that $\cap \overline{B} = \operatorname{adh} \mathscr{B}$ is not void. Conversely if $X$ is not $(m, n)$-compact then there is a family $\mathscr{B}$ of closed sets with card $\mathscr{B} \leqq n$ and having the $m$-intersection property but with total intersection void. Thus $\mathscr{B}$ is a filter base for an $(m, n)$-filter $\mathscr{F}$ and $\operatorname{adh} \mathscr{F} = \Phi$.

THEOREM 1.2. *A topological space $X$ is $(m, n)$-compact if and only if every $(m, n)$-net with values in $X$ has a non-void adherence.*

*Proof.* If there is an $(m, n)$-net with values in $X$ whose adherence is void then by Lemma 1.1 there is an $(m, n)$-filter without adherence points and so by Theorem 1.1 the space $X$ is not $(m, n)$-compact. Next we prove that if every $(m, n)$-net with values in $X$ has a non-void adherence then the same is true for every $(m, n)$-filter in $X$. By Theorem 1.1 this will prove that $X$ is $(m, n)$-compact. Let $\mathscr{B} = \{B_d\}(d \in D)$ be a filter base for an $(m, n)$-filter $\mathscr{F}$ in $X$ and let card $D \leqq n$. We order $D$ by using inverse inclusion of $\mathscr{B}$: $d_1 \leqq d_2$ if $B_{d_1} \supseteqq B_{d_2}$. Under this ordering $D$ becomes an $(m, n)$-directed set. We form a net $(x_d)(d \in D)$ by choosing $x_d$ in $\overline{B}_d$. By hypothesis $(x_d)(d \in D)$ has an adherence point $x$. Given any neighborhood $N_x$ and any $d \in D$ there is a $\delta \geqq d$ such that $x_\delta \in N_x$. Hence $N_x \cap B_\delta \neq \phi$ and so by $B_\delta \subseteqq B_d$ also $N_x \cap B_d \neq \phi$. Consequently $x \in \overline{B}_d$ for every $d \in D$ and so $x \in \operatorname{adh} \mathscr{F}$.

**2. Uniformizability and $(m, n)$-compactness.** This section contains the generalization to infinite cardinals of the following results:

*A space $X$ is countably compact if and only if every infinite set $S \subseteqq X$ has an accumulation point in $X$.*

*If $X$ is a metric space such that every infinite set $S \subseteqq X$ has an accumulation point in $X$ then the open sets of $X$ have a countable base and so $X$ is a Lindelöf space.*

Countable compactness will be replaced by $(m, m')$-compactness where $m$ is an infinite cardinal and $m'$ denotes the first cardinal succeeding $m$. If $m$ denotes the symbol 1 then $1'$ is defined to be $\omega$. Instead of accumulation points we must consider $m$-accumulation points:

DEFINITION 2.1. *A point $x$ of a topological space $X$ is called an $m$-accumulation point of a set $S \subseteqq X$ if for every open set $O_x$ containing $x$ we have* card $(O_x \cap S) > m$.

If $m$ is 0, 1 or $\omega$ then the relation card $(O_x \cap S) > m$ means that $O_x \cap S$ is not void, not finite or not countable. If card $S \leqq m$ the set of its $m$-accumulation points is void. In particular if $S$ is countable then it has no $\omega$-accumulation points and if $S$ is finite then it has no 1-accumulation points. The notion of an $m$-accumulation point is related to Fréchet's "point d'accumulation maximé" (see [7]).

The metrizability condition can be rephrased as follows: There is a uniform structure $\mathscr{U}$ which is compatible with the topology of $X$ and has a countable structure base. This hypothesis will be replaced by another which requires the existence of a structure base of cardinality at most $m$.

DEFINITION 2.2. A uniformizable space $X$ is said to be of uniform cardinality $u$ if there is a base $\mathscr{U}_B$ for a uniform structure $\mathscr{U}$ compatible with the topology of $X$ whose cardinality card $\mathscr{U}_B$ is at most $u$.

Every pseudo-metric space is of uniform cardinality $\omega$. If for every uniform structure $\mathscr{U}$ compatible with the topology of $X$ and for every base $\mathscr{U}_B$ of $\mathscr{U}$ we have card $\mathscr{U}_B \geqq u$ where $u$ is a uniform cardinality of $X$ then we say that $X$ is of uniform cardinality exactly $u$. The exact uniform cardinality of a pseudo-metric-space is at most $\omega$; it can also be 1.

The first result which we mentioned in the beginning is a special case of

THEOREM 2.1. *Let $m$ be an infinite cardinal and let $m'$ denote the next cardinal. Then a topological space $X$ is $(m, m')$-compact if and only if every set $S \subseteqq X$ of cardinality* card $S > m$ *has an $m$-accumulation point in $X$.*

*Proof*. First we prove the necessity of the condition. If $X$ contains sets of cardinality greater than $m$ which have no $m$-accumulation points in $X$ then we can select a set $S$ of cardinality exactly $m'$ such that it has no $m$-accumulation points in $X$. Let $\mathscr{F}$ denote the set of all those sets $F \subseteqq S$ whose cardinality is at most $m$. For every $x \in X$ there is an open set $O_x$ such that $O_x \cap S \in \mathscr{F}$. We define $O_F$ for every $F \in \mathscr{F}$ as $O_F = \cup [O_x : O_x \cap S = F]$. The family $\{O_F\}(F \in \mathscr{F})$ is an open cover of $X$ whose cardinality card $\mathscr{F}$ is $m'$. Since every subfamily of cardinality at most $m$ would cover at most $m$ points of $S$ the family $\{O_F\}(F \in \mathscr{F})$ cannot contain such subfamilies. Hence $X$ is not $(m, m')$-compact.

The deeper part of the theorem is the sufficiency of the condition. Here we need the axiom of choice both in the form of Zorn's lemma and also in the form of the well-ordering theorem. Suppose that $X$ is not $(m, m')$-compact. Let $\{O_i\}(i \in I)$ be an open cover of cardinality card $I = m'$ which contains no subcovers of cardinality at most $m$. Let the index set $I$ be well ordered. Since $X$ is not $(m, m')$-compact there is a point $x_2 \in X$ such that $x_2 \notin O_1$. More generally for every positive integer $n > 1$ there are points $x_2, \cdots, x_n$ such that $x_j \notin O_1 \cup \cdots \cup O_{j-1}$ for every $j \leqq n$. In general we consider segments $J$ of $I$ such that a segment (or net) of points $(x_j)$ can be selected so that $x_j \notin \cup [O_j : i < j]$ for every $j \in J$. Let $\mathscr{A}$ denote the family of ordered pairs $(J, (x_j))$ where $J$ denotes a segment of $I$ and $(x_j)$ a segment of points associated with $J$. We order $\mathscr{A}$ as follows: $(J_1, (x_j)) \leqq (J_2, (y_j))$ if $J_1 \subseteqq J_2$ and $x_j = y_j$ for every $j \in J_1$. Clearly every linearly ordered subfamily of $\mathscr{A}$ has an upper bound in $\mathscr{A}$ and so by Zorn's lemma $\mathscr{A}$ has a maximal element, say $(J, (x_j))$. The maximality of $(J, (x_j))$ implies that $J$ is a limit ordinal and $\{O_j\}(j \in J)$ is a cover of $X$. We claim that the set $S = \{x_j\}(j \in J)$ has no $m$-accumulation points in $X$. For if $x \in X$ then $x \in O_j$ for some $j \in J$ and so

$$\operatorname{card}(O_j \cap S) \leqq \operatorname{card}[x_i : i < j] \leqq m .$$

Finally $\operatorname{card} S = m'$ because $\operatorname{card} S = \operatorname{card} J$ and $\{O_j\}(j \in J)$ is a cover of the not $(m, m')$-compact space $X$.

THEOREM 2.2. *If the uniformizable space $X$ is of uniform cardinality $u$ and if there is an $m$ such that every set $S \subseteqq X$ of cardinality* $\operatorname{card} X > m$ *has a non-void derived set then the open sets of $X$ have a base of cardinality at most* $\max(m, u)$.

*Proof.* Let $\mathscr{U}_B = \{U\}$ be a base of a uniform structure $\mathscr{U}$ for $X$ and let $\operatorname{card} \mathscr{U}_B \leqq u$. We may suppose that every $U \in \mathscr{U}_B$ is symmetric. We fix a vicinity $U \in \mathscr{U}_B$ and consider systems of points $\{x_i\}$ $(i \in I)$ having the property that $U[x_i] \cap U[x_j]$ is void for every $i \neq j$. Let $\mathscr{A}$ be the set of all such systems $\{x_i\}(i \in I)$. The set $\mathscr{A}$ is not void for such systems exist at least in the case when the index set $I$ consists of a single element. We order $\mathscr{A}$ by inclusion : $\{x_i\} \leqq \{y_j\}$ if $\{x_i\} \subseteqq \{y_j\}$. Every linearly ordered subset $\mathscr{L}$ of $\mathscr{A}$ has an upper bound in $\mathscr{A}$, namely $\cup [\{x_i\} : \{x_i\} \in \mathscr{L}]$ is in $\mathscr{A}$ and it majorizes every $\{x_i\} \in \mathscr{A}$. Hence Zorn's lemma can be applied to show the existence of a maximal system which we denote by $\{x_i\}(i \in I)$. If $y \in \cup [U[x_i] : i \in I]$ then by the maximality $U[y] \cap U[x_i]$ is non-void for some $i \in I$. Hence by the symmetry of $U$ we have $y \in (U \circ U)[x_i]$. Therefore the family $\{(U \circ U)[x_i]\}(i \in I)$ is a cover of the uniform space $X$.

Let $S = \{x_i\}(i \in I)$ so that $\operatorname{card} S = \operatorname{card} I$. We show that the derived set of $S$ is void and so $\operatorname{card} I \leqq m$ : Let $V$ be a symmetric vicinity in $\mathscr{U}$ such that $V \circ V \subseteqq U$ and let $x$ be an arbitrary point in $X$. If $x \in V[x_i]$ for some $i \in I$ then $V[x] \subseteqq (V \circ V)[x_i] \subseteqq U[x_i]$ and so $V[x] \cap S$ is void or contains at most the point $x_i$. If $x \notin V[x_i]$ for every $i \in I$ then by the symmetry $x_i \notin V[x]$ for every $i \in I$ and so $V[x] \cap S$ is void. It follows that $\operatorname{card} I \leqq m$.

The family $\{(U \circ U)[x_i]\}(i \in I)$ is a cover of $X$ and so the interiors of the sets $(U \circ U \circ U)[x_i](i \in I)$ form an open cover of $X$. Its cardinality is at most $m$. Hence the cardinality of the union of these families for every choice of $U \in \mathscr{U}_B$ is of cardinality at most $\max(m, u)$. Since for every vicinity $V \in \mathscr{U}$ there is a $U \in \mathscr{U}_B$ such that $U \circ U \circ U \subseteqq V$ these sets form a base for the open sets of $X$.

The results of this section can be combined to obtain the following

THEOREM 2.3. *If $X$ is a uniformizable space of uniform cardinality $u$ which is $(m, n)$-compact for some cardinals $m$ and $n$ where $m < n$ then $X$ is $(m, \infty)$-compact or $(u, \infty)$-compact according as $m \geqq u$ or $u > m$.*

*Proof.* By Theorem 2.1 every set $S$ of cardinality greater than $m$ has an $m$-accumulation point and so its derived set is not void. Theorem 2.2 implies the existence of a base of cardinality at most $\max(m, u)$ for the family of open sets of $X$. Hence the space $X$ is $(\max(m, u), \infty)$-compact.

This proof did not make use of the full force of Theorem 2.1. It is sufficient to know for instance that every set of cardinality $\operatorname{card} S > m$ has a 1-accumulation point whenever the space $X$ is $(m, n)$-compact for some $n > m$. This weaker statement can be proved without using the axiom of choice or the well ordering theorem. Nevertheless the axiom of choice is used in the proof of Theorem 2.2.

3. **Dense sets, $(m, n)$-compact spaces and complete structures.** It is knownthat if $X$ is a compact topological space then every net with values in $X$ has a non-void adherence and conversely if the adherence of every net with values in $X$ is not void then $X$ is compact. We can raise the following question : Suppose $A$ is a dense subset of $X$ and that $\operatorname{adh}(x_d)$ is not void for every net $(x_d)(d \in D)$ with values in $A$. Does it follow that $X$ is compact ? We shall prove a theorem a special case of which states that for regular spaces the answer is affirmative. The result can be formulated also in terms of filters : Every filter $\mathscr{F}$ in $A$ is a filter base in $X$. If the adherence af the filter generated by the base $\mathscr{F}$ is not void we say that the filter $\mathscr{F}$ has a non-void adherence in $X$. It was proved earlier that if $X$ is regular and if every filter in the dense set $A$ has a non-void adherence in $X$ then $X$ is compact. (See [4] p. 109 Ex. 1 a.)

The same type of question can be raised when the net $(x_d)(d \in D)$ is subject to additional restrictions : For instance we can assume that every countable net with values in $A$ has a non-void adherence in $X$ and ask whether this implies that $X$ is countably compact. It will be proved that the conclusion holds under the assumption of normality and countable compactness.

As is known a family $\mathscr{L}$ of sets $S_i \subseteqq X$ is called a locally finite system if every $x \in X$ has a neighborhood $N_x$ which meets only finitely many sets of the family $\mathscr{L}$. We shall deal only with locally finite systems which consist of open sets.

DEFINITION 3.1. A topological space $X$ is called $n$-paracompact if every open cover $\{O_i\}(i \in I)$ satisfying $\operatorname{card} I \leqq n$ admits a refinement $\{Q_j\}(j \in J)$ which is a locally finite system.

Clearly every topological space is 1-paracompact and we agree that $\infty$-paracompactness means paracompactness in the usual sense. Using this definition we can state the following

THEOREM 3.1. *Let $X$ be a normal $n$-paracompact space which contains a dense set $A$ such that every $(m, m')$-net with values in $A$ (or every $(m, m')$-filter in $A$) has a non-void adherence in $X$. Then $X$ is $(m, n)$-compact.*

Since every regular paracompact space is normal in the special case when $n = +\infty$ normality can be replaced by the formally weaker requirement of regularity. However this is not a real improvement of the result. If $m = 1$ then by our agreement $m' = \omega$ and so if $X$ is countably compact then every $(m, m')$-net with values in $X$ has a non-void adherence. Hence as a corollary we have the following result due to R. Arens and J. Dugundji [2]:

COROLLARY. *If $X$ is regular, paracompact and countably compact then $X$ is compact.*

Since every pseudo-metric space is paracompact (see [17]) the corollary is a generalization of the following known result: If the pseudometric space $X$ is countably compact then it is compact. A weaker form of the corollary was obtained by Miss A. Dickinson who proved in [5] that every uniformizable space with a unique structure is countably compact and a paracompact space with a unique structure is compact.

In the proof of the theorem we shall use the following known lemmas:

LEMMA 3.1. *If $\{S_i\}(i \in I)$ is a locally finite system of sets then $\overline{\cup S_i} = \cup \overline{S_i}$.*

A short proof can be found for instance in [16].

LEMMA 3.2. *Let $\{O_i\}(i \in I)$ be a locally finite open cover of the normal space $X$. Then there is an open cover $\{Q_i\}(i \in I)$ of $X$ such that $\overline{Q_i} \subseteqq O_i$ for every $i \in I$.*

Proofs of this lemma can be found in [12], [9], [6] or [11],

LEMMA 3.3. *If* $\{\omega_i\}(i \in I)$ *is a set of ordinals such that* $\omega_i < m'$ *for every* $i \in I$ *and if* $\operatorname{card} I \leqq m$ *then* $\operatorname{lub}\{\omega_i\} < m'$.

*Proof.* Since $m'$ is the first ordinal of cardinality $m'$ we have $\omega_i < m'$, that is, $\operatorname{card} \omega_i \leqq m$ for every $i \in I$. Hence by $\operatorname{card} I \leqq m$ the cardinality of $\operatorname{lub}\{\omega_i\}$ is $m$.

*Proof of Theorem* 3.1. We assume that $X$ is normal and $n$-paracompact but it is not $(m, n)$-compact. We shall construct a linear $(m, m')$-net $(x_d)(d \in D)$ with values in $A$ and such that it has no adherence points in $X$. Then the sets $[x_\delta : d \leqq \delta](d \in D)$ form a filter base for an $(m, m')$ filter in $A$ which has no adherence points in $X$.

Let $\{O_i\}(i \in I)$ be an open cover of cardinality at most $n$ which contains no subcover of cardinality at most $m$. Since $X$ is $n$-paracompact $\{O_i\}(i \in I)$ admits a refinement $\{Q_j\}(j \in J)$ which is a locally finite system of open sets. The space being normal by Lemma 3.2 we may assume that $\overline{Q}_j \subseteqq O_i$ for every $j \in J$ and for a suitable $i = i(j) \in I$. Since $\{Q_j\}(j \in J)$ is locally finite Lemma 3.1. can be applied to any subfamily of this cover.

Let the index set $J$ be well ordered; for the sake of simplicity we assume that the elements of $J$ are ordinals. Denote by $S_k$ the open set

$$S_k = Q_k - \cup [\overline{Q}_j : j < k] .$$

Let $D$ be the set of those indices $k \in J$ for which $S_k$ is not void. We prove that $\kappa = \operatorname{card} D \geqq m'$.

For let $\mathscr{C}$ be the class of those initial segments $K \subseteqq J$ for which $\cup[\overline{Q}_j : j \in K] \subseteqq \cup[\overline{Q}_j : j \in D]$. Then $\mathscr{C}$ is not void because $(1, \cdots, \kappa) \in \mathscr{C}$. It can be ordered by inclusion: $K_1 \leqq K_2$ if $K_1 \subseteqq K_2$. There is a maximal element in $\mathscr{C}$ namely $K_m = \cup[K : K \in \mathscr{C}]$ itself is an element of $\mathscr{C}$. We prove that $K_m = J$. For let $K \in \mathscr{C}$ be a proper subset of $J$ which contains $\kappa$ and let $k'$ be the first index not in $K$. We set $K' = K \cup \{k'\}$ and obtain by $\kappa < k'$

$$\begin{aligned} \cup[\overline{Q}_j : j \in K'] &= \cup[\overline{Q}_j : j \in K] \cup \overline{Q}_{k'} = \cup[\overline{Q}_j : j \in K] \\ &\subseteqq \cup[\overline{Q}_j : j \in D] . \end{aligned}$$

Hence $K' \in \mathscr{C}$ and $K$ is not maximal. Consequently $K_m = J$ and this implies that

$$\cup[Q_j : j \in J] \subseteqq \cup[\overline{Q}_j : j \in D] .$$

However on the one hand $\{Q_j\}(j \in J)$ is a cover of $X$ and so $X =$

$\cup[\overline{Q}_j : j \in D]$. On the other hand $\{\overline{Q}_j\}(j \in J)$ is a refinement of $\{O_i\}$ $(i \in I)$ and by hypothesis $\{O_i\}(i \in I)$ does not contain a subcover of cardinality at most $m$. Hence we have $\operatorname{card} D \geqq m'$.

The well-ordered set $D$ is order isomorphic to an initial segment of the ordinals which segment contains every ordinal preceding $m'$. If we discard from $D$ every element corresponding to $m'$ and to the ordinals succeeding $m'$ we obtain a subset of $D$ of cardinality at least $m'$. We denote this subset again by $D$. By Lemma 3.3 the new $D$ is an $(m, m')$-directed set.

The open sets $S_d$ are not void for every $d \in D$ and $A$ is dense in $X$. Hence we can choose a point $a_d \in A$ in each of the sets $S_d(d \in D)$. The linear net $(a_d)(d \in D)$ is an $(m, m')$-net with values in $A$ and it has no adherence points in $X$: In fact $\{Q_j\}(j \in J)$ being locally finite for every point $x \in X$ we can find an open set $O_x$ such that $O_x \cap Q_d$ is not void only for finitely many indices $d \in D$. If $d$ is larger than any of these finitely many indices then $O_x \cap Q_\delta = \phi$ for every $\delta \geqq d$ and so $a_\delta \notin O_x$ for every $\delta \geqq d$. This, however, shows that $x$ is not an adherence point of the net $(a_d)(d \in D)$. This completes the proof of Theorem 3.1.

Now we turn to uniformizable spaces:

THEOREM 3.2. *Let $X$ be a uniformizable space of uniform cardinality $u$. Suppose that $X$ contains a dense subset $A$ such that every $(m, n)$-filter in $A$ has a non-void adherence in $X$. Then $X$ is $(m, \infty)$-compact.*

It is sufficient to prove that $X$ is $(m, n)$-compact. The $(m, \infty)$-compactness follows from Theorem 2.3. The proof of the $(m, u)$-compactness can be modified such that we obtain the following known result (see [7] p. 150, Proposition 7):

*Let $A$ be a dense subset of a uniform space $X$ with uniform structure $\mathscr{U}$. If every Cauchy filter in $A$ is convergent to some point of $X$ then the structure $\mathscr{U}$ is complete.*

*Proof.* Let $\mathscr{F}$ be a filter (or an $(m, u)$-filter) in $X$. Consider the family $\mathscr{B} = \{U[F] \cap A\}$ $(U \in \mathscr{U}$ and $F \in \mathscr{F})$. Since $A$ is dense in $X$ every set $U[F] \cap A$ is non-void and

$$(U[F_1] \cap A) \cap (U[F_2] \cap A) \supseteqq U[F_1 \cap F_2] \cap A .$$

Hence $\mathscr{B}$ is a filter base in $X$. (Moreover if $\mathscr{F}$ is an $(m, u)$-filter and $\mathscr{U}$ is of uniform cardinality $u$ then $\mathscr{B}$ is a base for an $(m, u)$-filter in $A$.) If $\mathscr{F}$ is a Cauchy filter then $\mathscr{B}$ is a base for a Cauchy filter because if $F \times F \subseteqq V$ where $V$ is symmetric then $V[F] \times V[F] \subseteqq$

$V \circ V \circ V$: In fact if $x \in V[F]$ and $y \in V[F]$ then $(x, a) \in V$ and $(b, y) \in V$ for some $a, b \in F$. Thus by $(a, b) \in F \times F \subseteqq V$ we have $(x, y) = (x, a) \circ (a, b) \circ (b, y) \in V \circ V \circ V$. By hypothesis $\mathscr{B}$ as a Cauchy filter base in $X$ (as a base for an $(m, u)$-filter in $X$) is convergent to some point $x \in X$. We show that $x \in \operatorname{adh} \mathscr{F}$ which is equivalent of saying that $x \in \lim \mathscr{F}$. Given any an open neighborhood $O_x$ of $x$ there is a $U \in \mathscr{U}$ such that $U[x] \subseteqq O_x$. We determine the symmetric $V \in \mathscr{U}$ such that $V \circ V \subseteqq U$. Since $x \in \operatorname{adh} \mathscr{B}$ and $V[F] \cap A$ is an element of $\mathscr{B}$ we have $V[x] \cap V[F] \neq \phi$ for every $F \in \mathscr{F}$. Hence there is an $a \in A$ and an $f \in F$ such that $(x, a) \in V$ and $(a, f) \in V$. Therefore $(x, f) \in V \circ V \subseteqq U$ and $f \in U[x] \cap F$. This shows that $U[x]$ and $F$ intersect for every $U \in \mathscr{U}$ and for every $F \in \mathscr{F}$. Thus $x \in \operatorname{adh} \mathscr{F}$.

**4. Additional results and notes.** In my first paper on $(m, n)$-compactness I introduced the notion of a hereditary or completely $(m, n)$-compact topological space: $X$ is completely $(m, n)$-compact if every subspace $Y$ of $X$ is $(m, n)$-compact. It can be easily proved that if every open set $Y$ is an $(m, n)$-compact subspace then $X$ is completely $(m, n)$-compact. In the same paper I gave a number of equivalent characterizations of complete $(m, n)$-compactness. At that time time I did not notice that one of these criteria (Theorem 4, condition (ii) in [8]) involves $n$ only in a formal way.[1] I should have added as a corollary the following.

THEOREM 4.1. *If $X$ is completely $(m, n)$-compact for some cardinals $m < n$ then $X$ is completely $(m, \infty)$-compact.*

*Proof.* Suppose that $X$ is not completely $(m, \infty)$-compact. Then there is a family of open sets $O_i$ $(i \in I)$ in $X$ such that $\cup O_{i_j}$ is a proper subset of $\cup O_i$ whenever $\operatorname{card} J \leqq m$. Let the index set $I$ be well ordered. Let $O_{i_1}$ be the first non-void $O_i$ and let $O_{i_2}$ be the first $O_i$ such that $O_{i_2} \nsubseteqq O_{i_1}$. In general we consider initial segments $J$ of the ordinals $1, 2, \cdots, j, \cdots$ and sets $O_{i_j} (j \in J)$ such that for every $j \in J$ the set $O_{i_j}$ is the first $O_i$ set which is not a subset of $\cup [O_{i_k} : k < j]$. By hypothesis $\{O_i\} (i \in I)$ does not admit a subfamily $\{O_{i_j}\} (j \in J)$ satisfying $\cup O_i = \cup O_{i_j}$ with $\operatorname{card} J \leqq m$. Hence using Zorn's lemma we can find initial segments $J$ and corresponding sets $O_{i_j}$ such that $\operatorname{card} J \geqq m'$ where $m'$ is the first cardinal greater than $m$. We restrict ourselves to ordinals preceding $m'$ so that $J = [j : j < m']$ and $O_{i_j} \nsubseteqq \cup [O_{i_k} : k < j]$ for every $j \in J$. The family $\{O_{i_j}\} (j \in J)$ is of cardinality $\operatorname{card} J = m'$ and if $\operatorname{card} K < m'$ where $K \subset J$ then by Lemma 3.3

[1] This was first noticed by Mr. R. D. Joseph.

$$\cup[O_{i_k} : k \in K] \subset \cup[O_{i_j} : j \in J] .$$

This shows that $X$ is not completely $(m, m')$-compact and so the theorem is proved.

Let $\mathscr{B} = \{B\}$ be a base for the open sets of a space $X$ and let $Y \subseteqq X$. Then $\mathscr{B}_Y = \{B \cap Y\}(B \in \mathscr{B}\}$ is a base for the subspace $Y$. Hence if $\mathscr{B}$ is a base for the topology of the space $X$ then every subspace of $X$ is (card $\mathscr{B}$, $\infty$)-compact. Applying this remark to the situation described in Theorem 2.3 we obtain

THEOREM 4.2. *Let $X$ be a uniformizable space of uniform cardinality $u$. If $X$ is $(m, n)$-compact for some $m < n$ then $X$ is completely $(m, \infty)$-compact or completely $(u, \infty)$-compact according as $m \geqq u$ or $m \leqq u$.*

If $n = \infty$ this result can be obtained directly by using the definition of $(m, \infty)$-compactness and of hereditary $(m, \infty)$-compactness.

The product of a $(1, \infty)$-compact space with an $(m, n)$-compact space is $(m, n)$-compact. This was proved a few years ago by Yu. M. Smirnov [14]. Not knowing the existence of this paper, I proved in [8] (Theorem 8) the result in the special case when $n = \infty$, but a slight modification in my reasoning gives a new proof of Smirnov's theorem: Start again by replacing the open cover $\{O_i\}(i \in I)$ where card $I \leqq n$ by a family of sets $O_x^y \times O_y^x$. However instead of forming the intersection $O_{y^1}^{x_1} \cap \cdots \cap O_{y^m}^{x_m}$ form the intersection of those sets $O_{i_1}^{(y)}, \cdots, O_{i_m}^{(y)}$ of the given family which have the property that

$$O_{x_1}^y \times O_{y^1}^{x_1} \subseteqq O_{i_1}^{(y)}, \cdots, O_{x_m}^y \times O_{y^m}^{x_m} \subseteqq O_{i_m}^{(y)} .$$

Since card $I \leqq n$ there are at most $n$ distinct ones among the finite intersections $Q_y = O_{i_1}^{(y)} \cap \cdots \cap O_{i_m}^{(y)}$. The rest of the reasoning then is the same as in [8].

We end by stating two unsolved problems: Professor Erdös mentioned to me that he was thinking without success of the following problem: Let $m$ be an infinite cardinal. We say that $X$ is $[m]$-compact if from every open covering of $X$ one can select a subcovering having fewer than $m$ elements. Is there an infinite cardinal $m$ such that the product of any two $[m]$-compact spaces is again $[m]$-compact?

It is known that given any filter $\mathscr{F}$ in a set $X$ there exists an ultrafilter $\mathscr{M}$ such that $\mathscr{F} \subseteqq \mathscr{M}$. Let $\mathscr{F}$ be an $(m, \infty)$-filter. The corresponding ultrafilter $\mathscr{M}$ need not be an $(m, \infty)$-filter and in general there is no $(m, \infty)$-ultrafilter $\mathscr{M}$ satisfying the requirement $\mathscr{F} \subseteqq \mathscr{M}$. We can ask the following question: Is there any infinite cardinal $m$ such that for every $(m, \infty)$-filter $\mathscr{F}$ the ultrafilter $\mathscr{M}$ can be chosen such that $\mathscr{M}$ is an $(m, \infty)$-filter and $\mathscr{F} \subseteq \mathscr{M}$?

I do not know to what extent $n$-paracompactness is necessary in the hypothesis of Theorem 3.1. The only example that I know of shows that there exists a non-compact space $X$ which contains a dense set $A$ such that every filter in $A$ has a non-void adherence in $X$: We choose $X$ to be the interval $[-1, 1]$ and call $O$ open if it can be obtained from an open set in the usual sense by omitting points of the form $x = \pm 1, \pm \frac{1}{2}, \cdots$. We can choose $A = X - \{\pm 1, \pm \frac{1}{2}, \cdots\}$. The space $X$ is neither regular nor compact. It can be proved that $X$ is not countably paracompact.

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