MODULUS OF A BOUNDARY COMPONENT

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§1. PRELIMINARIES AND SUMMARY

1.1 Preliminary definitions. Let $R$ be an open Riemann surface, and let $\{G_n\}$ ($n = 1, 2, \ldots$) be an infinite sequence of subregions of $R$ such that:

(a) the relative boundary of each $G_n$ is compact,
(b) $G_n \supseteq G_{n+1}$, and
(c) $\bigcap_{0}^{\infty} \overline{G_n} = 0$.

$\{G_n\}$ is said to define a boundary component $\gamma$ of $R$ in the sense of Kerékjártó [6] and Stoilow [16]. Here two sequences of subregions $\{G_n\}$ and $\{G'_m\}$ are considered to be equivalent and to define the same $\gamma$ if each region $G_n$ includes a region $G'_m$. That this is a proper equivalence relation follows immediately.

Let $\gamma$ be a boundary component of $R$, and let $S$ be a subregion of $R$. If there exists a defining sequence $\{G_n\}$ of $\gamma$ with $G_{n_0} = S$, for some $n_0$, we call $S$ a neighborhood of $\gamma$. Throughout this paper we shall consider only neighborhoods $S$ of $\gamma$ such that the relative boundary of $S$ is a closed analytic Jordan curve $\gamma_0$.

By an exhaustion of $R$, we mean an infinite sequence $\{R_n\}$ ($n = 1, 2, \ldots$) of subregions of $R$ as follows (see [16]):

(1) each $R_n$ is compact relative to $R$ and the relative boundary $\beta_n$ of $R_n$ consists of a finite number of closed analytic Jordan curves $\beta_{ni}$,
(2) $R_n \supseteq R_{n+1}$,
(3) $\bigcup_{0}^{\infty} R_n = R$, and
(4) each connected component $S_{ni}$ of $R - \overline{R_n}$ is non-compact (relative to $R$) and its boundary consists of a single curve $\beta_{ni}$.

Each set $R - \overline{R_n}$ is said to be a boundary neighborhood of $R$. It is easy to see that, for any boundary component $\gamma$ of $R$, there exists a single connected component $S_{ni}$ which is a neighborhood of $\gamma$.

A property is said to be a boundary property (respectively a $\gamma$-property) if the following is true. If a Riemann surface $R$ has the property then every Riemann surface $R'$ which admits a conformal mapping from a boundary neighborhood of $R'$ (a neighborhood of $\gamma'$, where $\gamma'$ is a boundary
component of $R'$) onto a boundary neighborhood of $R$ (a neighborhood of $\gamma$) has the property.

Let $u$ be a harmonic function on a subregion $S$ of $R$. We shall denote by $\bar{u}$ the conjugate harmonic function of $u$ and by $D(u; S)$ the Dirichlet integral of $u$ over $S$.

1.2. Capacity of a boundary component. Let $\gamma$ be a boundary component of an open Riemann surface $R$, $P_0$ a point of $R$, and $K_z: |z| \leq 1$ a fixed parametric disc on $R$ with $z = 0$ corresponding to $P_0$. Let $\{R_n\}$ be an exhaustion of $R$ with $P_0 \in R_n$ and let $\gamma_n$ denote the curve $\beta_{ni}$ which separates $\gamma$ from $P_0$. This means that $\gamma_n$ separates a neighborhood of $\gamma$ from $P_0$.

We consider the class $\{t\}_\gamma$ of single-valued functions on $R$ which satisfy the following conditions:

(1.1) each $t$ is harmonic on $R - P_0$ and has the form

$$t = \log |z| + h(z)$$

in $K_z$, where $h$ is harmonic and $h(0) = 0$.

(1.2)

$$\int_{\gamma_n} dt = 2\pi \text{ and } \int_{\beta_{ni} \neq \gamma_n} dt = 0 ,$$

for all $n$ ,

where $\gamma_n$ and $\beta_{ni}$ are described in the positive sense with respect to $R_n$.

We further consider the corresponding class $\{t\}_{\gamma_n}$ on $R_n$, and we denote by $t_n$ the function of this class with $t_n = k_n$ on $\gamma_n$ and $t_n = k_{ni}$ on $\beta_{ni} \neq \gamma_n$, where $k_n$ and $k_{ni}$ are real numbers.

The following theorem due to Sario is proved in [14] (see also Savage [15]). Let $t \in \{t\}_\gamma$, and let

$$I(t) = \lim_{n \to \infty} \frac{1}{2\pi} \int_{\gamma_n} t \overline{t} .$$

**Theorem 1.** The sequence of functions $\{t_n\}$ is compact. Let $t_\gamma$ denote a limit function of $\{t_n\}$. Then we have the following conclusions:

(1.3) $t_\gamma \in \{t\}_\gamma$ and, for any $t$, $\min I(t) = I(t_\gamma)$.

(1.4) $I(t) = I(t_\gamma) + D(t - t_\gamma; R)$.

(1.5) $k_n \leq k_{n+1}$ and $I(t_\gamma) = \lim k_n \equiv k_\gamma$.

By (1.4), for $k_\gamma < \infty$, the minimizing function $t_\gamma$ is unique. $t_\gamma$ is called the capacity function of $R$ for $\gamma$, and the quantity $c_\gamma = e^{-k_\gamma}$ is called the capacity of $\gamma$ (with respect to $K_z$). Let $z' = az + \cdots, a \neq 0$, be a new local parameter in the neighborhood of $P_0$, and let $c'_\gamma$ denote the capacity of $\gamma$ with respect to this local parameter. It follows, from the definition of the capacity, that
Hence, the condition \( \gamma \) is called **weak** if it has a capacity \( c_{\gamma} = 0 \). The class of Riemann surfaces for which all \( \gamma \) are weak is denoted by \( C_{\gamma} \). The boundary of a Riemann surface \( R \) belonging to \( C_{\gamma} \) is called **absolutely disconnected** [14, 15].

### 1.3. Summary

Let \( R \) be an open Riemann surface, \( \gamma \) a boundary component of \( R \), \( \mathcal{S} \) a neighborhood of \( \gamma \), and \( \partial \mathcal{S} \) the relative boundary of \( \mathcal{S} \). The present paper deals with a conformal invariant of \( \mathcal{S} \) which is denoted by \( \mu(S; \partial \mathcal{S}, \gamma) \) (or, simply, for fixed \( \mathcal{S} \), by \( \mu_{\gamma} \)) and is called the **modulus of \( \mathcal{S} \) for \( \gamma \)** (or \( \gamma \) the **modulus of \( \gamma \)**).

In §2 harmonic functions \( u \) on \( \mathcal{S} \) with \( u = 0 \) on \( \partial \mathcal{S} \) and satisfying conditions (2.3) are considered, and a theorem is proved which establishes the existence of a minimizing function \( u_{\gamma} = u(z; \mathcal{S}; \partial \mathcal{S}, \gamma) \) for the Dirichlet integral \( D(u; S) \). The modulus is defined by setting \( \mu_{\gamma} = D(u_{\gamma}; S) \). The notion of a parabolic boundary component is defined by the condition \( \mu_{\gamma} = \infty \), and a theorem is proved which shows the equivalence of parabolicity and weakness.

In §3 measurable conformal metrics are considered. An important minimal property of the conformal metric \( \rho_{\gamma} = |\text{grad} u_{\gamma}| \) corresponding to a result of Wolontis [17] and Strebel [18] is proved, which connects \( \mu_{\gamma} \) with the extremal length of a certain family of curves on \( \mathcal{S} \). As an application, a characterization of a parabolic boundary component is obtained in terms of conformal metrics. Another characterization of a parabolic boundary component is given by means of the divergence of a modular series \( \sum \rho(E_n; \gamma_{n-1}, \gamma_n) \). The sufficient part of this theorem implies the modular criterion of Savage [15]. A theorem shows the equivalence of perimeter in Ahlfors and Beurling's sense and capacity in Sario's sense.

Section 4 deals with the class \( M_{\gamma} \) of Riemann surfaces for which all \( \gamma \) are parabolic in the case of a finite genus. The conformal mapping properties of \( u_{\gamma} \) and \( t_{\gamma} \) are discussed, and, for planar Riemann surfaces, the equalities \( O_{sb} = M_{\gamma} = O_{sd} \) [1, 14] are proved. Finally a theorem is proved which shows the connection between \( M_{\gamma} \) and the class of Riemann surfaces for which the continuation is topologically unique, or which do not possess essential continuations.

### §2. Harmonic Functions and Modulus

#### 2.1. Moduli of a compact subregion

Let \( S_0 \) denote a relatively compact subregion of a Riemann surface \( R \). We assume that the boundary

\[
\text{c} \gamma = |d| c' \gamma .
\]
of $S_0$ is a set $\gamma_0 \cup \alpha_0$, where $\gamma_0$ is a closed analytic Jordan curve and $\alpha_0$ consists of a finite number of closed analytic Jordan curves $\alpha_{01}, \cdots, \alpha_{0k}$ ($k \geq 1$). We assign to each $\alpha_{0i}$ ($i = 1, \cdots, k$) as positive orientation the positive sense with respect to $S_0$ and to $\gamma_0$ the sense for which $\gamma_0$ and $\alpha_0$ are homologous.

If $u$ is a harmonic function on $S_0$ then we denote the conjugate period of $u$ around $\alpha_{0i}$ by $p_i(u)$. This is defined by the integral $\int_{\alpha_{0i}} d\bar{u}$, where $\alpha'_{0i}$ is any closed Jordan curve on $S_0$ such that $\alpha_{0i}$ and $\alpha'_{0i}$ are homologous. If $u$ is harmonic on $S_0 \cup \alpha_{0i}$ then clearly $p_i(u) = \int_{\alpha_{0i}} d\bar{u}$. The period vector $(p_1(u), \cdots, p_k(u))$ will be denoted by $p(u)$.

**Lemma 1.** There is a harmonic function $u_0 = u(z; S_0, \gamma_0, k_{0q})$ on $S_0$ satisfying the following conditions:

(a) $u_0 = 0$ on $\gamma_0$ and $u_0 = \mu_{0i} = \text{const.}$ on $\alpha_{0i}$ ($i = 1, \cdots, k$),

(b) $p(u_0) = (1, 0, \cdots, 0)$.

(c) $0 < u_0(z) < \mu_{0i}$ on $S_0$ and on the boundary curves $\alpha_{02}, \cdots, \alpha_{0k}$.

**Proof.** Denote the harmonic measure of $\alpha_{0i}$ with respect to $S_0$ by $\omega_{i}$, and consider the function

$$u(z) = \sum_{i=1}^{k} \mu_i \omega_i(z),$$

where $\mu_i$ are arbitrary real numbers. Clearly, this function is harmonic on $\overline{S_0} = S_0 \cup \gamma_0 \cup \alpha_0$. Setting $a_{ij} = p_i(\omega_i)$, we obtain

$$p_i(u) = \int_{\alpha_{0i}} d\bar{u} = \sum_{j=1}^{k} a_{ij} \mu_j.$$  

We assert that this linear mapping of the $k$-dimensional cartesian space into itself is one-to-one. In fact, from Green's formula

$$D(u) = D(u; S_0) = \sum_{i=1}^{k} \int_{\alpha_{0i}} u d\bar{u} = \sum_{i=1}^{k} \mu_i p_i(u),$$

we see that the condition $p_i(u) = 0$, for all $i$, implies $D(u) = 0$, that is $u \equiv 0$ (since $u \equiv 0$ on $\gamma_0$) and consequently $\mu_i = 0$, for all $i$, which proves our assertion. Hence we deduce in particular that the above linear mapping is onto, i.e., for any $p$, there is a function $u = \sum \mu_i \omega_i(z)$ such that $p(u) = p$. Let $u_0$ denote the function (1.1) corresponding to $p_0 = (1, 0, \cdots, 0)$. This is clearly the unique bounded harmonic function on $S_0$ satisfying (a) and (b).

Now denote the maximum and the minimum of $u_0$ on the boundary of $S_0$ by $M_0$ and $m_0$ respectively. From the maximum principle, we have
It follows that \( \partial u_0 / \partial n \leq 0 \) on each boundary curve \( \gamma(M_0) \) on which \( u_0(z) = M_0 \). Here \( \partial / \partial n \) denotes the derivative in the direction of the interior normal. Since \( u_0 \) is not constant and \( \partial u_0 / \partial n \) is continuous, there exists a subarc of \( \gamma(M_0) \) on which \( \partial u_0 / \partial n < 0 \) and therefore

\[
\int_{\gamma(M_0)} d\bar{u}_0 = -\int_{\gamma(M_0)} \frac{\partial u_0}{\partial n} |dz| > 0,
\]

where \( \gamma(M_0) \) is described in the positive sense with respect to \( S_0 \). This and condition (b) implies that \( \gamma(M_0) \) coincides necessarily with \( \alpha_{a_1} \), whence \( M_0 = \mu_{a_1} \) and this maximum is attained only on \( \alpha_{a_1} \). Similarly, it can be proved that \( m_0 = 0 \) and that this minimum is attained only on \( \gamma_0 \). This completes the proof of Lemma 1.

**Lemma 2.** The function \( u_0 \) gives the minimum of \( D(u) \),

\[
\min D(u) = D(u_0),
\]

in the class of all harmonic functions \( u \) on \( S_0 \) with \( u = 0 \) on \( \gamma_0 \) and \( p(u) = (1, 0, \ldots, 0) \).

**Proof.** Clearly, the function \( u_0 \) belongs to the class of admissible functions and, by Green’s formula,

\[
D(u_0) = \sum_{i=1}^k \mu_{a_i} p_i(u_0) = \mu_{a_1} < \infty.
\]

Let \( u \) be any admissible function with \( D(u) < \infty \). Setting \( u - u_0 = h \), we have

\[
D(u) = D(u_0) + D(h) + 2D(u_0, h),
\]

where \( D(u_0, h) = D(u_0, h; S_0) \) is the mixed Dirichlet integral of \( u_0 \) and \( h \) over \( S_0 \). We shall show that \( D(u_0, h) = 0 \). If \( u \) is harmonic on \( S_0 \), then Green’s formula gives immediately

\[
D(u_0, h) = \int_{S_0} u_0 d\bar{h} = \sum_{i=1}^k \mu_{a_i} p_i(h) = 0
\]

since, for all \( i \), \( p_i(h) = p_i(u) - p_i(u_0) = 0 \). If the above assumption is not true, we consider the open set \( S_0(\varepsilon) = S_0 - \cup_{i=1}^k E_{a_i}(\varepsilon) \), where \( \varepsilon \) is a positive number, sufficiently small, and \( E_{a_i}(\varepsilon) \) is the set (of points of \( S_0 \) for which) \( \mu_{a_i} - \varepsilon \leq u_0(z) \leq \mu_{a_i} + \varepsilon \). The boundary of \( S_0(\varepsilon) \) consists only of level lines of \( u_0 \). On the other hand each level line \( c(\mu); u_0(z) = \mu \) \((0 < \mu < \mu_{a_1}, \mu \neq \mu_{a_i}, i = 1, \ldots, k) \) is a dividing cycle on \( S_0 \) (that is, \( c(\mu) \) is homologous with a sum of \( \alpha_{a_i} \)) and therefore \( \int_{c(\mu)} d\bar{h} = 0 \). Hence, Green’s formula gives again \( D(u_0, h; S_0(\varepsilon)) = 0 \) and, as \( \varepsilon \to 0 \), \( D(u_0, h) = 0 \). We conclude finally that
The uniqueness of the minimizing function $u_0$ is an immediate consequence of (2.2). For, if $D(u) = D(u_0)$, we conclude from (2.2) that $D(u - u_0) = 0$, that is $u = u_0$, since $u - u_0 = 0$ on $\gamma_0$.

The function $u_0 = u(z; S; \gamma_0, \alpha_0)$ will be called the extremal function of $S_0$ for $\gamma_0$ and $\alpha_0$. The quantity $\mu_0 = D(u_0)$ will be called the modulus of $S_0$ for $\gamma_0$ and $\alpha_0$ and denoted generally by $\mu(S_0; \gamma_0, \alpha_0)$.\[\text{2.2. Modulus of a boundary component.} \text{ Let us consider a boundary component } \gamma \text{ of an open Riemann surface } R, \text{ and let } S \text{ be a given neighborhood of } \gamma. \text{ Let } \gamma_0 \text{ be the relative boundary of } S (\text{see 1.1}). \text{ An exhaustion of } S \text{ is a sequence } \{S_n\} \text{ (} n = 1, 2, \cdots \text{) of subregions of } R \text{ such that: (1) } S_n \text{ is a relatively compact subregion of } R \text{ and the relative boundary of } S_n \text{ is a set } \gamma_0 \cup \alpha_n, \text{ where } \gamma_0 \cap \alpha_n = 0 \text{ and } \alpha_n \text{ consists of a finite number of closed analytic Jordan curves } \alpha_{n_1}, \text{ (2) } S_n \subset S_{n+1}, \text{ (3) } \bigcup_{n=1}^\infty S_n = S, \text{ and (4) each connected component of } S - S_n \text{ is non-compact and its relative boundary consists of a single } \alpha_{n_1}. \text{ We assign to each } \alpha_{n_1} \text{ as positive orientation the positive sense with respect to } S_n \text{ and to } \gamma_0 \text{ the sense for which } \gamma_0 \text{ and } \alpha_n \text{ are homologous.}

\text{Let } \gamma_n \text{ be the curve } \alpha_{n_1} \text{ which separates } \gamma \text{ from } \gamma_0, \text{ and let } \{n\}_\gamma \text{ be the class of all harmonic functions } u \text{ on } S \text{ with } u = 0 \text{ on } \gamma_0 \text{ and}
\begin{align*}
\int_{\gamma_n} d\bar{u} = 1 \text{ and } \int_{\sigma_{n_1} \setminus \gamma_n} d\bar{u} = 0,
\end{align*}

for all } n. \text{ It is easy to see, using Green's formula, that conditions (2.3) are independent of the particular exhaustion which is used.}

\text{Theorem 2.} \text{ In } \{u\}_\gamma \text{ there exists a function } u_\gamma \text{ with the property}
\begin{align*}
\min D(u; S) = D(u_\gamma; S).
\end{align*}

\text{Moreover, for any } u,
\begin{align*}
D(u; S) = D(u_\gamma; S) + D(u - u_\gamma; S).
\end{align*}

\text{Proof.} \text{ Denote by } u_n \text{ the extremal function of } S_n \text{ for } \gamma_0 \text{ and } \gamma_n, \text{ and put } \mu_n = D(\mu_n; S_n) = \text{value of } u_n \text{ on } \gamma_n; \mu_n \text{ is the modulus of } S_n \text{ for } \gamma_0 \text{ and } \gamma_n.

\text{Since the restriction of } u_{n+1} \text{ to } S_n \text{ satisfies the condition of Lemma 2 (where } S_0 \text{ is replaced by } S_n \text{ and } \alpha_0 \text{ by } \gamma_n), \text{ we have}
\begin{align*}
\mu_n = D(u_n; S_n) \leq D(u_{n+1}; S_n) \leq D(u_{n+1}; S_{n+1}) = \mu_{n+1}.
\end{align*}

\text{Similarly, we see that } \mu_n \leq \mu_\gamma, \text{ where } \mu_\gamma \text{ is the greatest lower bound of}
For a fixed $N$, let $s$ be the bounded harmonic function on $S_N$ with $s = 0$ on $\gamma_0$ and $s = d$ on $\alpha_N$, where $d$ is a constant value determined by $\int_{a_N} s \, ds = 1$. From Green's formula $\int_{a_N} u_n d\bar{s} - sd\bar{u}_n = 0$ and the boundary behavior of $u_n$ and $s$, we obtain

$$\int_{a_N} u_n d\bar{s} = d,$$

for all $n \geq N$, whence $\min_{a_N} u_n \leq d$. It follows from Harnack's principle that the sequence $\{u_n\}$ is compact. A subsequence, say again $\{u_n\}$, converges, uniformly on each $S_N$, to a function $u$. Obviously this function belongs to $\{u\}_\gamma$, so that $\mu_{\gamma} \leq D(u; S)$.  

On the other hand, the lower semicontinuity of the Dirichlet integral gives

$$D(u; S) \leq \lim D(u_n; S_n) = \lim \mu_n.$$

From the three preceding inequalities we conclude that

$$D(u; S) = \lim \mu_n = \mu_{\gamma},$$

which proves the first assertion of Theorem 2.

Let us now prove equality (2.4), for any $u$ in $\{u\}_\gamma$. This is evident if $D(u; S) = \infty$. Suppose $D(u; S) < \infty$, and put $u - u_\gamma = h$. For any real number $\varepsilon$, $u_\gamma + \varepsilon h \in \{u\}_\gamma$, and therefore

$$D(u_\gamma + \varepsilon h) = D(u_\gamma) + 2\varepsilon D(u_\gamma, h) + \varepsilon^2 D(h) \geq D(u_\gamma).$$

Since $D(u_\gamma + \varepsilon h) < \infty$, this is possible only if $D(u_\gamma, h) = 0$, so that, as $\varepsilon = 1$, we obtain (2.4).

As in Lemma 2, the uniqueness of the minimizing function $u_\gamma$ in the case $\mu_\gamma < \infty$ is an immediate consequence of (2.4).

The function $u_\gamma$ will be called the extremal function of $S$ for $\gamma_0$ and $\gamma$ and denoted generally by $u(z; S; \gamma_0, \gamma)$. The conformal invariant $\mu_\gamma = D(u_\gamma, S)$ will be called the modulus of $S$ for $\gamma_0$ and $\gamma$ or, simply, for fixed $S$, the modulus of $\gamma$. It will be denoted generally by $\mu(S; \gamma_0, \gamma)$.

2.3. Parabolic boundary components. Let $\gamma$ be a boundary component of an open Riemann surface $R$. Consider any two neighborhoods $S$ and $S'$ of $\gamma$, and denote by $\gamma_0$ and $\gamma'_0$ the relative boundaries of $S$ and $S'$, respectively.
$S'$ respectively. Set $u(z; S; \gamma_0, \gamma) = u_\gamma$, $u(z; S'; \gamma'_0, \gamma) = u'_\gamma$, $\mu(S; \gamma_0, \gamma) = \mu_\gamma$, $\mu(S'; \gamma'_0, \gamma) = \mu'_{\gamma}$.

**Lemma 3.** The moduli $\mu_\gamma$ and $\mu'_{\gamma}$ are simultaneously finite or infinite.

*Proof.* Suppose first $S \subset S'$, and let \{\$S'_n\$\} be an exhaustion of $S'$. The regions $S_n = S \cap \$S'_n$ give, for $n$ sufficiently large, an exhaustion of $S$. Set $u(z; \gamma_0, \gamma_n) = u_n, u(z; \$S'_n; \gamma'_0, \gamma_n) = u'_n, \mu(S_n; \gamma_0, \gamma_n) = \mu_n, \mu(S'_n; \gamma'_0, \gamma_n) = \mu'_n$.

From Green's formula

$$\int_{\$S'_n} (u'_n d\bar{u}_n - u_n d\bar{u}'_n) = 0,$$

it follows

$$\mu'_n - \mu_n = \int_{\gamma_0} u'_n d\bar{u}_n.$$

Hence, as $n \to \infty$, we obtain

$$\mu'_{\gamma} - \mu_\gamma = \int_{\gamma_0} u'_\gamma d\bar{u}_\gamma.$$

This proves our lemma in the particular case $S \subset S'$.

Let us now consider the general case, and construct a third neighborhood $S''$ of $\gamma$ such that $S'' \subset S \cap S'$. Let $\gamma''_0$ denote the relative boundary of $S''$, and put $\mu(S''; \gamma''_0, \gamma) = \mu''_{\gamma}$. As before, $\mu_\gamma$ and $\mu''_{\gamma}$ are simultaneously finite or infinite. The same is valid for $\mu'_\gamma$ and $\mu''_{\gamma}$ and consequently for $\mu_\gamma$ and $\mu'_\gamma$, which completes the proof of Lemma 3.

A boundary component $\gamma$ of $R$ is called *parabolic* if $\mu_\gamma = \infty$ and *hyperbolic* if $\mu_\gamma < \infty$. From Lemma 3, this condition is independent of the neighborhood $S$ which is used, i.e. the parabolicity of a $\gamma$ is a $\gamma$-property of $R$. The class of all Riemann surfaces for which all boundary components are parabolic will be denoted by $M_\gamma$. The property $R \in M_\gamma$ (or $R \notin M_\gamma$) is a boundary property of $R$.

Consider now the capacity function $t_\gamma$ of $R$ for $\gamma$ with respect to a fixed parametric disc $|z| \leq 1$. Let $\lambda$ denote a positive number which is sufficiently small such that the level line $c(\lambda): t_\gamma(z) = \log \lambda$ is a closed Jordan curve and the set $t_\gamma(z) \leq \log \lambda$ is compact. The set $S(\lambda): t_\gamma(z) > \log \lambda$ is then a neighborhood of $\gamma$. Put $u(z; S(\lambda); c(\lambda), \gamma) = u_{\gamma,\lambda}, \mu(S(\lambda); c(\lambda), \gamma) = \mu_{\gamma,\lambda}$.

**Lemma 4.** If $\lambda$ satisfies the above conditions, then

$$t_\gamma(z) - \log \lambda = 2\pi u_{\gamma,\lambda}(z),$$

(2.5)
and

\begin{equation}
\log \lambda = 2\pi \mu_{\gamma,\lambda}.
\end{equation}

**Proof.** Consider an exhaustion \( \{R_n\} \) of \( R \) as in 2.1. The regions \( S_\lambda(\lambda) = R_n \cap S(\lambda) \) give, for \( n \) sufficiently large, an exhaustion of \( S(\lambda) \). Set \( u(z; S_\lambda(\lambda); \gamma_n) = u_{n,\lambda}, \mu(S_\lambda(\lambda); \gamma) = \mu_{n,\lambda}, t_n - 2\pi u_{n,\lambda} = h_n, \) where \( t_n \) is the function on \( R_n \) defined in 1.2. From Green's formula, we have

\[
D(h_n; S_\lambda(\lambda)) = \int_{\beta_n} h_n \overline{h}_n - \int_{\gamma_n} h_n \overline{h}_n = -\int_{c_n} h_n \overline{d\gamma_n},
\]

since \( h_n = \text{const.} \) on \( \beta_n \), and \( \int_{\beta_n} \overline{d\gamma_n} = 0 \), for all \( \beta_n \). Hence, by the lower semicontinuity of the Dirichlet integral,

\[
D(h; S(\lambda)) \leq -\int_{c(\lambda)} h \overline{d\gamma} = 0,
\]

since \( h = \text{const.} = \log \lambda \) on \( c(\lambda) \) and \( \int_{c(\lambda)} \overline{d\gamma} = 0 \). We conclude that \( h \equiv \log \lambda \), which proves (2.5).

Now apply Green's formula on \( S_\lambda(\lambda) \) to \( u_{n,\lambda} \) and \( t_n \). We obtain

\[
k_n - 2\pi \mu_{n,\lambda} = \int_{c(\lambda)} t_n \overline{d\gamma_{n,\lambda}},
\]

whence, as \( n \to \infty \),

\[
k_n - 2\pi \mu_{\gamma,\lambda} = \int_{c(\lambda)} t \overline{d\gamma_{\gamma,\lambda}} = \log \lambda,
\]

which completes the proof of Lemma 4.

**Theorem 3.** A boundary component \( \gamma \) of \( R \) is parabolic if and only if it has a vanishing capacity.

**Proof.** This is evident from Lemmas 3 and 4.

**Corollary.** \( M_\gamma = C. \)

§ 3 Modulus and Conformal Metrics

**3.1. Definitions.** Consider a non-negative function \( \rho(z) \) which is defined on each parametric disc \( K_z: |z| \leq 1 \) of a subregion \( S \) of \( R \) and satisfies

\[
\rho(z) = \rho(z') \left| \frac{dz'}{dz} \right|.
\]
at corresponding points $z, z'$ of any two overlapping $K_z$ and $K_{z'}$. We say that $\rho$ is a conformal metric on $S$. We define the $\rho$-length of any cycle $c$ (finite set of closed Jordan curves) on $S$ by the lower Darboux integral (see [4])

$$l(\rho; c) = \int c \rho(z)|dz|.$$ 

A conformal metric $\rho$ is said to be measurable on $S$ if its restriction to any parametric disc is measurable in Lebesgue's sense. If $\rho$ is a measurable conformal metric on $S$, we define the $\rho$-area of $S$ by the Lebesgue integral

$$A(\rho; S) = \int_S \rho(z)d\sigma_z,$$

where $\sigma_z$ is the Lebesgue measure on $K_z$. A measurable conformal metric $\rho$ defined on $S$ is said to be $A$-bounded on $S$ if $A(\rho; S) < \infty$.

3.2. Extremal conformal metrics. Consider first the relatively compact subregion $S_0$ of 2.1. We prove the following

**Lemma 5.** The conformal metric $\rho_0 = |\text{grad}u_0|$ gives the minimum of $A(\rho; S_0)$,

$$\min A(\rho; S_0) = A(\rho_0; S_0),$$

in the class of all conformal metrics satisfying $l(\rho; c) \geq 1$, for all dividing cycles $c$ on $S_0$ which separate $\alpha_0$ from $\gamma_0$.

Moreover, for any admissible $\rho$,

$$A(\rho; S_0) \geq A(\rho_0; S_0) + A(\rho - \rho_0; S_0).$$

**Proof.** Clearly the conformal metric $\rho_0$ satisfies the condition of the lemma, and $A(\rho_0; S_0) = D(u_0; S_0) = \mu_{01} < \infty$. Let $\rho$ be any admissible conformal metric on $S_0$ with $A(\rho; S_0) < \infty$.

We evaluate the integral

$$\int_{S_0} \rho(z)\rho_0(z)d\sigma_z.$$

Take $w_0 = u_0 + i\bar{u}_0$ for the local parameter on $S_0$, so that $\rho_0(w_0) = 1$. Denote the level line $u_0(z) = \mu$ ($0 \leq \mu \leq \mu_{01}$; see Lemma 1) by $c(\mu)$. From Fubini's theorem,

$$\int_{S_0} \rho(z)\rho_0(z)d\sigma_z = \int_0^{\mu_{01}} d\mu \int_{c(\mu)} \rho(w_0)d\bar{w}_0.$$
Here the integral \( \int_{c(\mu)} \rho(w_o)\,d\bar{u}_0 \) exists almost everywhere, for \( \mu \) on the closed interval \([0, \mu_{\alpha}]\). But \( c(\mu) \) is, for any \( \mu \neq \mu_{\alpha} \), a dividing cycle on \( S_0 \) which separates \( \alpha_{\alpha} \) from \( \tau_0 \) and therefore, almost everywhere,

\[
\int_{c(\mu)} \rho(w_o)\,d\bar{u}_0 = \int_{c(\mu)} \rho(z)\,dz \geq \int_{c(\mu)} \rho(z)\,dz \geq 1
\]

From the two preceding relations it follows that

\[
\int_{\gamma_0} \rho(z)\,d\sigma \geq \mu_{\alpha}.
\]

Now put \( \rho = \rho_0 + (\rho - \rho_0) \) in \( A(\rho; S_0) \); we obtain

\[
A(\rho; S_0) = \mu_{\alpha} + A(\rho - \rho_0; S_0) + 2\int_{\gamma_0} \rho \rho_0\,d\sigma - 2\mu_{\alpha}
\]

and, from the preceding inequality, we conclude finally that

\[
A(\rho; S_0) \geq \mu_{\alpha} + A(\rho - \rho_0; S_0),
\]

which proves our lemma.

Clearly the admissible conformal metric which minimizes \( A(\rho; S_0) \) is unique. For, if \( A(\rho; S_0) = A(\rho_0; S_0) = \mu_{\alpha} < \infty \), we deduce from (3.2) that \( A(\rho - \rho_0; S_0) = 0 \), i.e. \( \rho = \rho_0 \) almost everywhere on \( S_0 \).

Now let \( \gamma \) be a boundary of \( R \), and let \( S \) be a given neighborhood of \( \gamma \). Let \( \{\rho\}_\gamma \) denote that class of all measurable conformal metrics defined on \( S \) which satisfy the condition

\[
\ell(\rho; c) \geq 1,
\]

for all dividing cycles \( c \) which separate \( \gamma \) from \( \tau_0 \). If \( u \in \{\rho\}_\gamma \), then obviously \( |\text{grad}u| \in \{\rho\}_\gamma \). This is valid, in particular, for the conformal metric \( \rho_\gamma = |\text{grad}u_\gamma| \). The \( \rho_\gamma \)-area of \( S \) is \( A(\rho_\gamma; S) = D(u_\gamma; S) = \mu_\gamma \).

**Theorem 4.** In \( \{\rho\}_\gamma \) the conformal metric \( \rho_\gamma = |\text{grad}u_\gamma| \) gives the minimum of \( A(\rho; S) \):

\[
\min A(\rho; S) = A(\rho_\gamma; S).
\]

Moreover, for any \( \rho \),

\[
A(\rho; S) \geq A(\rho_\gamma; S) + A(\rho - \rho_\gamma; S).
\]

**Proof.** If \( A(\rho; S) = \infty \), (3.5) is evident. Assume now that there exists in \( \{\rho\}_\gamma \) a conformal metric \( \rho \) which is \( A \)-bounded.

Set \( |\text{grad}u_\gamma| = \rho_\gamma \) (see 2.2). Since \( A(\rho; S) \geq A(\rho; S_\gamma) \), we conclude from Lemma 5 that
As \( n \to \infty \), Fatou's Lemma gives immediately

\[
A(\rho; S) \geq \mu_\gamma + \lim \inf A(\rho - \rho_n; S_n) \geq \mu_\gamma + A(\rho - \rho; S),
\]

which proves (3.5) and the theorem.

As in Lemma 5, the uniqueness of the minimizing conformal metric \( \rho_\gamma \) in the case \( \mu_\gamma < \infty \) is an immediate consequence of (3.5).

By Theorem 4, the quantity \( \lambda_\gamma = \mu_\gamma^{-1} \) is equal to the extremal length of the family of all dividing cycles \( c \) on \( S \) separating \( \gamma \) from \( \gamma_0 \) ([1], [5]).

### 3.3. Parabolic boundary components.

We return to the condition \( \mu_\gamma = \infty \) studied in 2.2.

**Theorem 5.** A boundary component \( \gamma \) of \( R \) is parabolic if and only if, for any neighborhood \( S \) of \( \gamma \) and for any \( A \)-bounded conformal metric \( \rho \) on \( S \), there exists a dividing cycle separating \( \gamma \) from \( \gamma_0 \) with an arbitrarily small \( \rho \)-length.

**Proof.** If \( \mu_\gamma < \infty \), the conformal metric \( \rho_\gamma \) is \( A \)-bounded and satisfies \( l(\rho; c) \geq 1 \), for all dividing cycles separating \( \gamma \) from \( \gamma_0 \). Conversely, if there is an \( A \)-bounded conformal metric \( \rho \) on \( S \) satisfying \( l(\rho; c) \geq \varepsilon > 0 \), for all dividing cycles \( c \) separating \( \gamma \) from \( \gamma_0 \), the conformal metric \( \rho^* = (1/\varepsilon)\rho \) is \( A \)-bounded and belongs to \( \{\rho\}_\gamma \). Therefore, by Theorem 4, \( \mu_\gamma < \infty \).

**Theorem 6.** Suppose \( R \) is imbedded in a larger Riemann surface \( R^* \). If a boundary component \( \gamma \) of \( R \) or a part of \( \gamma \) realized on \( R^* \) contains a continuum \( \gamma^* \), then \( \gamma \) is hyperbolic.

**Proof.** Let \( K^* : |z^*| \leq 1 \) denote a parametric disc on \( R^* \) for which \( K^* \cap \gamma^* \) contains a continuum, say again \( \gamma^* \). Since \( \gamma^* \) is a boundary continuum of \( R \), there exists a disc \( \overline{R}_0 \subset K^* \cap R \). In \( K^* \), let \( Q = ab'ab' \) be a rectangle such that its side \( a \) is completely interior to \( R_0 \) and its opposite sides \( b, b' \) have common points with \( \gamma^* \).

Set \( R - \overline{R}_0 = S \). We define a conformal metric \( \rho_0 \) on \( S \) by setting \( \rho_0(z^*) = 1 \) on \( Q \cap S \) and \( \rho_0 = 0 \) otherwise. Clearly \( \rho_0 \) is \( A \)-bounded and satisfies \( l(\rho_0; c) \geq l_0 > 0 \), where \( l_0 \) is the length of \( a \) in \( K^* \) and \( c \) is any dividing cycle separating \( \gamma \) from \( \gamma_0 \). Hence, by Theorem 5, \( \gamma \) is not parabolic.

Let \( S \) be a given neighborhood of a boundary component \( \gamma \) of \( R \), and let \( \{S_n\} \) be an exhaustion of \( S \) as in 2.2. Let \( E_n \) denote the connected component of \( S_n - S_{n-1} \) whose boundary includes \( \gamma_{n-1} \) and \( \gamma_n \). We assert that
(3.6) \[ \mu(S; \gamma_0, \gamma) \geq \sum_{n=1}^{\infty} \mu(E_n; \gamma_{n-1}, \gamma_n). \]

In fact, since the restriction of \( p \) to \( E_n \) is admissible in Lemma 5 (where \( S_0 \) is replaced by \( E_n \), \( \tau_0 \) and \( \alpha_{01} \) by \( \tau_{n-1} \) and \( \gamma_n \) respectively), we conclude that \( A(\rho_\gamma; E_n) \geq \mu(E_n; \gamma_{n-1}, \gamma_n) \). Therefore, \( \mu(S; \gamma_0, \gamma) \geq \sum_{n=1}^{\infty} A(\rho_\gamma; E_n) \geq \sum_{n=1}^{\infty} \mu(E_n; \gamma_{n-1}, \gamma_n) \), which proves (3.6).

Similarly, it may be proved that

(3.7) \[ \mu(S; \gamma_0, \gamma) \geq \mu(E_1; \gamma_0, \gamma_1) + \mu(S^*; \gamma_1, \gamma) \]

where \( S^* \) is the connected component of \( S - S_1 \) whose relative boundary is \( \gamma_1 \).

**Theorem 7.** A boundary component \( \gamma \) of \( R \) is parabolic if and only if there exists an exhaustion of \( S \) for which

(3.8) \[ \sum_{n=1}^{\infty} \mu(E_n; \gamma_{n-1}, \gamma_n) = \infty. \]

**Proof.** By (3.6), the condition (3.8) is sufficient for the parabolicity of \( \gamma \).

Conversely, assume that \( \gamma \) is parabolic, and let \( \{S_n\} \) be a given exhaustion of \( S \). Since

\[ \lim_{n \to \infty} \mu(S_n; \gamma_0, \gamma_n) = \mu(S; \gamma_0, \gamma) = \infty, \]

we can choose \( n_1 \geq 1 \) such that \( \mu(S_n; \gamma_0, \gamma_n) \geq 1 \). Let \( S^*_{n_1} \) denote the connected component of \( S - S_{n_1} \) whose relative boundary is \( \gamma_{n_1} \). \( S^*_{n_1} \) is a neighborhood of \( \gamma \). Since \( \gamma \) is parabolic, we have

\[ \lim_{n \to \infty} \mu(S^*_{n_1,n}; \gamma_{n_1}, \gamma_n) = \mu(S^*_{n_1}; \gamma_{n_1}, \gamma) = \infty, \]

where \( S^*_{n_1,n} = S^*_{n_1} \cap S_n \). Therefore, we can choose \( n_2 > n_1 \) such that \( \mu(S^*_{n_2,n}; \gamma_{n_2}, \gamma_n) \geq 1 \). Continuing this procedure, we obtain an exhaustion \( \{S_{n_k}\} \) \( (k = 1, 2, \cdots) \) of \( S \), which satisfies condition (3.8). Thus Theorem 7 is established.

**3.4. Perimeter and capacity.** Let \( |z| \leq r_0 \) be a fixed parametric disc on \( R \), and let \( S(r) \) denote the complement of the disc \( |z| \leq r \) \( (0 < r \leq r_0) \) with respect to \( R \). Set \( \mu(S(r); |z| = r, \gamma) = \mu_{r,\gamma} \). By (3.7), for \( r' < r \),

\[ \mu_{r',\gamma} \leq \frac{1}{2\pi} \log \frac{r}{r'} + \mu_{r,\gamma}. \]
or

\[ -2\pi\mu_{x,r} - \log r' \leq -2\pi\mu_{x,r} - \log r. \]

Therefore,

\[ \pi_y = \lim_{r \to 0} \frac{1}{r} e^{-2\pi\mu_{x,r}} \]

exists. According to Ahlfors and Beurling [1], we call \( \pi_y \) perimeter of \( \gamma \) with respect to the fixed parametric discs \( |z| \leq r_0 \). Let \( z' = \lambda(z) = az + \cdots, a \neq 0 \), be a new local parameter in the neighborhood of the point \( P_0 \in R \) corresponding to \( z = 0 \), and let \( \pi'_y \) denote the perimeter of \( \gamma \) with respect to the parametric disc \( |z'| \leq r'_0 \). Set \( |z| = r \) and \( |z'| = r' \).

For corresponding \( r \) and \( r' \) by \( z' = \lambda(z) \), we have

\[ |a| r (1 - \varepsilon_r) \leq r' \leq |a| r (1 + \varepsilon_r), \]

where \( \varepsilon_r \) is a positive function of \( r \) and \( \varepsilon_r \to 0 \), as \( r \to 0 \). It follows, from the conformal invariance and the monotony of modulus, that

\[ (3.9) \quad \pi_y = |a| \pi'_y. \]

We now prove the following.

**Theorem 8.** If the perimeter \( \pi_y \) and the capacity \( c_y \) are defined with respect to the same parametric disc \( |z| \leq r_0 \), then \( \pi_y = c_y \).

**Proof.** From (1.6) and (3.9), it is sufficient to prove the required equality for a particular parametric disc of the point \( P_0 \). We choose this parametric disc, say again \( |z| \geq r_0 \), such that \( t_y = \log |z| \) on \( |z| \leq r_0 \).

Then, by (2.6), we conclude immediately that

\[ \pi_y = \lim_{\lambda \to 0} \frac{1}{\lambda} e^{-2\pi\mu_{x,\lambda}} = e^{-k_y} = c_y, \]

which proves our theorem.

**Corollary.** If \( P_\gamma \) denote the class of Riemann surfaces defined by \( \pi_\gamma = 0 \), for all \( \gamma \), then \( P_\gamma = c_\gamma = M_\gamma \).

§ 4. **Riemann Surfaces of Finite Genus**

4.1. **Planar subregions.** Let \( \gamma \) be a boundary component of an open Riemann surface \( R \), and suppose that \( \gamma \) is hyperbolic and possesses a neighborhood \( S \) which is planar.

Set, as usually, \( u(z; S; \tilde{r}_0, \gamma) = u_\gamma, \mu(S; \tilde{r}_0, \gamma) = \mu_\gamma \), and consider the function \( w = F_\gamma(z) \) defined by
\[ F_y(z) = \exp 2\pi(u_y(z) + i\bar{u}_y(z)) \]

Consider an exhaustion \( \{S_n\} \) of \( S \) as in 2.2. Since \( S \) is planar, the homology group \( H^1(S) \) is generated from the boundary curves \( \alpha_n \) of \( S_n(n = 1, 2, \ldots) \), and we conclude by (2.3) that \( F_y \) is single-valued. We now prove the following [7]:

**Theorem 9.** The function \( w = F_y(z) \) maps the region \( S \) univalently onto the annulus

\[ A_{\alpha, \nu, \gamma} : 1 < |w| < e^{2\pi \nu} \]

slit along a set of circular arcs around the origin. Here the boundary circumferences \(|w| = 1 \) and \(|w|e^{2\pi \nu} \) correspond to \( \gamma_0 \) and \( \gamma \) respectively. The total area of the slits vanishes.

**Proof.** We define the function \( w = F_n(z) \) on \( S_n \) by

\[ F_n(z) = \exp 2\pi(u_n(z) + i\bar{u}(z)), \]

where \( u_n = u(z; S_n; \gamma_n, \bar{\gamma}_n) \). As before, we see that \( F_n \) is single-valued, for all \( n \).

The function \( w = F_n(z) \) gives a one-to-one conformal mapping of \( S_n \) onto the covering surface \( S_{n,w} = (S_n, w = F_n(z)) \). By the definition of \( u_n \), \( |F_n(z)| \) assumes constant values on the boundary curves of \( S_n \) and satisfies on \( S_n \):

\[ 1 < |F_n(z)| < e^{2\pi \nu_n}. \]

It follows that \( S_{n,w} \) is an unlimited covering surface of the annulus \( A_{\alpha, \nu, \gamma} \) slit along a finite number of circular arcs. On the other hand, evaluate the \( \rho_n \)-area of \( S_{n,w} \), where

\[ \rho_n(w) = \frac{1}{2\pi|w|} = \frac{1}{2\pi} \left| \frac{d}{dw} \log w \right|. \]

Since, for \( w = F_n(z) \),

\[ \rho_n(z) = |\text{grad}u_n(z)| = \frac{1}{2\pi} \left| \frac{d}{dz} \log w \right| = \rho_n(w) \left| \frac{dw}{dz} \right|, \]

we obtain

\[ A(\rho_n; S_{n,w}) = A(\rho_n; S_n) = \mu_n. \]

This is equal to the \( \rho_n \)-area of the annulus \( A_{\alpha, \nu, \gamma} \). It follows that the covering surface \( S_{n,w} \) consists necessarily of a single sheet, that is the function \( F_n \) is univalent. Since \( F_n \to F_\gamma \) uniformly on each \( S_n \), \( F_\gamma \) is also univalent.
Let us now consider the image $S_w = F_r(z)$. Denote the connected components of the boundary of $S_w$ which correspond to $\gamma_0$ and $\gamma$ by $\gamma^0_w$ and $\gamma_w$ respectively. Clearly $\gamma^0_w$ is the circumference $|w| = 1$. Further, since $\mu_n \leq \mu$, for all $n$, $S_w$ is included in the annulus $A_{\mu, \gamma}$. As before, the $\rho_w$-area of $S_w$ is

$$A(\rho_0; S_w) = A(\rho_\gamma; S) = \mu \tau,$$

since

$$\rho_\gamma(z) = \rho_\delta(w) \left| \frac{dw}{dz} \right| (w = F_r(z)).$$

This is equal to the $\rho_0$-area of the annulus $A_{\mu, \gamma}$. Accordingly, the complements of $S_w$ with respect to $A_{\mu, \gamma}$ has a (logarithmic and Euclidian) vanishing area.

Assume finally that the set $A_{\mu, \gamma} - S_w$ possesses a connected component $\gamma^* w$ which is not a point or a circular arc around the origin. Construct two circumferences $|w| = r_i$ ($i = 1, 2; r < r_1 < r_2 < e^{2\pi \gamma}$) having common points with $\gamma^* w$, and consider a point $w_0$ in the annulus $r_1 < |w| < r_2$. Let $K_t$ be the disc $|w - w_0| \leq \epsilon$. Obviously, for $\epsilon$ sufficiently small, the conformal metric $\rho_t$, defined by $\rho_t = 0$ on $K_\gamma$ and $\rho_t(w) = \rho_\delta(w)$ on $S_w - K_\gamma$, satisfies the condition (3.3), for all dividing cycles $c$ on $S_w$ separating $\gamma_w$ from $\gamma^0_w$. This contradicts Theorem 4, since $A(\rho_t; S_w) < A(\rho_0; S_w) = \mu$. Therefore, the continuum $\gamma^* w$ does not exist. In particular, $\gamma_w$ coincides with $|w| = e^{2\pi \gamma}$. Theorem 9 is completely proved.

4.2. Planar Riemann surfaces. Suppose now that $R$ itself is planar. Let $|z| \leq 1$ be a fixed parametric disc on $R$, $\gamma$ a hyperbolic boundary component of $R$, and $c_\gamma > 0$ the capacity of $\gamma$ with respect to $|z| \leq 1$. Consider the function $w = T_\gamma(z)$ defined by

$$T_\gamma(z) = c_\gamma \exp(t_\gamma(z) + it_\gamma(z)).$$

By Lemma 4 and Theorem 9, we have the following [14]:

**Theorem 10.** The function $w = T_\gamma(z)$ is univalent and single-valued on $R$ and maps $R$ onto the unit circle slit along a set of circular arcs of vanishing total area. The boundary component $\gamma$ is mapped into the unit circumference.

Let SB (SD) be the class of univalent single-valued analytic functions having a bounded modulus (a finite Dirichlet integral), and let $O_{SB}(O_{SD})$ be the class of Riemann surfaces with no functions belonging to $SB$ (SD).

**Theorem 11.** [1, 14] For planar Riemann surfaces,
Proof. Assume first that the planar surface $R$ possesses a hyperbolic boundary component $\gamma$. Then, the function $T_\gamma$ of Theorem 10 obviously belongs to the class $SB$ and $SD$.

Conversely, suppose that there exists on $R$ a function $w = T(z)$ which belongs to the class $SB$ or $SD$. In both cases, the image $R_w = T(R)$ has a finite Euclidian area. Let $K_\varepsilon: |w - w_0| \leq \varepsilon$ be a disc which is completely included in $R_w$. Denote by $\gamma_w$ the connected component of the boundary of $R_w$ which separates $w = 0$ from $w = \infty$ or contains $w = \infty$. The conformal metric $\rho(w) = 1/2\pi\varepsilon$ is clearly $A$-boundary on $R_w - K_\varepsilon$ and satisfies condition (3.3), for all dividing cycles on $R_w - K_\varepsilon$ which separate $\gamma_w$ from $|w - w_0| = \varepsilon$. We conclude that the boundary component $\gamma$ of $R$ which corresponds to $\gamma_w$ is hyperbolic.

4.3. Riemann surfaces of finite genus. A continuation of a Riemann surface $R$ is defined by (1) another Riemann surface $R'$ and (2) a one-to-one conformal mapping $T: R \rightarrow R'$, $T(R) \subset R'$. [2, 4, 8, 9, 11, 12]. If $R'$ is a compact Riemann surface, the continuation is called compact. If $R' - T(R)$ contains interior points, the continuation is called essential [9, 12].

Let $R$ be a Riemann surface of finite genus. We say that the continuation of $R$ is topologically unique if, for any two compact continuations $T_\nu: R \rightarrow R'_\nu (\nu = 1, 2)$ of $R$, there exists a topological mapping $h_{12}^* = R'_1 \rightarrow R'_2$, $h_{12}^*(R'_1) = R'_2$, with $h_{12}^* T_1(R) = h_{12}$, where $h_{12} = T_2 T_1^{-1}$. If, in addition, $h_{12}^*$ is always a conformal mapping, the continuation of $R$ is said to be conformally unique.

Let $O_{AD}$ denote the class of Riemann surfaces with no non-constant single-valued analytic functions having a finite Dirichlet integral. It is well known that the continuation of a Riemann surface $R$ of finite genus is conformally unique if and only if $R \in O_{AD}$ [1, 8, 12]. We now prove the following

THEOREM 12. For Riemann surfaces of finite genus, the following conditions are equivalent:

1. $R \in M_\gamma$
2. The continuation of $R$ is topologically unique.
3. $R$ does not possess an essential continuation.

Proof. (1) $\Rightarrow$ (2). If $R \in M_\gamma$ and $T_\nu: R \rightarrow R'_\nu (\nu = 1, 2)$ are compact continuations of $R$, then, by Theorem 6, the sets $\beta_\nu = R'_\nu - T_\nu(R)$ are totally disconnected. Set $T_2 T_1^{-1} = h_{12}$. We define a topological mapping $h_{12}^*$ of $R'_1$ onto $R'_2$ as follows. First, set $h_{12}^*(P_1) = h_{12}(P_1)$ for any $P_1 \in T_1(R)$. Now let $P_1 \in \beta_1$. Since $\beta_\nu$ is totally disconnected, there is
a fundamental sequence \( \{U_n\} \) of neighborhoods of \( P_1 \) such that the open sets \( V_n = U_n \cap T(R) \) are connected. Set \( E(P_1) = \cap_n \mathcal{H}_{P_1}(V_n) \). Clearly this is a closed and connected set. On the other hand, \( E(P_1) \subset \beta_\zeta \) and, since \( \beta_\zeta \) is totally disconnected \( E(P_1) \) contains a single point \( P_2 \). Set \( h^*_{12}(P_1) = P_2 \). It is easy to see that \( h^*_{12} \) is a topological mapping between \( R_1 \) and \( R_2 \).

(2) \( \Rightarrow \) (3). If \( R \) possesses an essential continuation \( T_1 : R \rightarrow R_1 \), we may construct in an evident manner another compact continuation \( T_2 : R \rightarrow R_2 \) of \( R \) such that \( R_1 \) and \( R_2 \) have different genera.

(3) \( \Rightarrow \) (1). Assume that \( R \notin M_\gamma \), i.e. \( R \) possesses some boundary component \( \gamma \) which is hyperbolic. Let \( S \) be a neighborhood of \( \gamma \). We have \( \mu_\gamma < \infty \). By Theorem 9, there is a one-to-one conformal mapping of \( S \) into the finite annulus \( 1 < |w| < e^{2\pi \mu_\gamma} \). Let \( K_w \) denote the set \( |w| > 1 \). Clearly the Riemann surface \( R' = (R - S) \cup K_w \) defines an essential continuation of \( R \), and therefore (3) \( \Rightarrow \) (1). Thus, Theorem 12 is established

**Corollary 1.** For Riemann surfaces of finite genus, we have \( O_{AD} \subset M_\gamma \).

Note that by a theorem of Ahlfors and Beurling [1] this inclusion is strict.

**Corollary 2.** Let \( R \in M_\gamma - O_{AD} \) and of finite genus. Then there exist two compact continuations \( T_\nu : R \rightarrow R_\nu \) (\( \nu = 1, 2 \)) of \( R \) such that the corresponding topological mapping \( h^*_{12} \) is not a conformal mapping.

In particular, we conclude from Corollary 2 that there exist Pompeiu functions which are univalent (see [3], [10], and [16]).

**References**

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