

# Pacific Journal of Mathematics

## **TWO NONSEPARABLE COMPLETE METRIC SPACES DEFINED ON $[0, 1]$**

BURNETT C. MEYER AND H. D. SPRINKLE

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Let  $\mathfrak{M}$  be the set of all Lebesgue measurable subsets of the closed interval  $[0, 1]$ , and let  $A, B \in \mathfrak{M}$ . It is well-known that  $\mathfrak{M}$  becomes a pseudo-metric space if distance is defined by

$$d(A, B) = m(A - B) + m(B - A) = m[(A - B) \cup (B - A)],$$

$m$  denoting the Lebesgue measure. See [1, pp. 31-32]. It is the purpose of this paper to extend  $\mathfrak{M}$  to include the non-measurable sets and to examine some of the properties of the resulting space.

If we remove the restriction that  $A$  and  $B$  be measurable, and let them be any subsets of  $[0, 1]$ , then if

$$\rho(A, B) = m^*(A - B) + m^*(B - A), \text{ and } \delta(A, B) = m^*[A - B] \cup (B - A)]$$

(where  $m^*$  denotes the exterior Lebesgue measure), it is easily seen that pseudo-metric spaces  $\mathfrak{S}$  and  $\mathfrak{X}$  are obtained, corresponding to  $\rho$  and  $\delta$  respectively. The properties which we discuss of  $\mathfrak{S}$  and  $\mathfrak{X}$  are the same and are proved analogously, so we shall state and prove our results for the space  $\mathfrak{S}$  only, it being understood that similar theorems and proofs hold for  $\mathfrak{X}$ .

**LEMMA 1.** *A necessary and sufficient condition that  $\rho(A, B) = 0$  is the existence of sets  $Z_1$  and  $Z_2$ , both of Lebesgue measure zero, such that  $A \cup Z_1 = B \cup Z_2$ .*

*Necessity.* If  $\rho(A, B) = 0$ , then  $m(A - B) = m(B - A) = 0$ . Since  $A \cup (B - A) = A \cup B = B \cup A = B \cup (A - B)$ ,  $Z_1$  and  $Z_2$  may be taken as  $B - A$  and  $A - B$ , respectively.

*Sufficiency.* If  $A \cup Z_1 = B \cup Z_2$ , then

$$\rho(A, B) \leq \rho(A, A \cup Z_1) + \rho(A \cup Z_1, B \cup Z_2) + \rho(B \cup Z_2, B) = 0$$

The relation  $\rho(A, B) = 0$  is seen to be an equivalence relation defined on the elements of  $\mathfrak{S}$ ; hence, those elements are partitioned into equivalence classes. Let  $[A]$  denote the equivalence class which contains  $A$ . It is clear that if  $C \in [A]$  and  $D \in [B]$ , then  $\rho(A, B) = \rho(C, D)$ . If  $\mathfrak{S}^*$  is the set of all equivalence classes defined above, and if  $\rho([A], [B]) = \rho(A, B)$ , then  $\mathfrak{S}^*$  becomes a metric space with the metric  $\rho([A], [B])$ .

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LEMMA 2. *If  $B_n \in [A_n]$  for  $n = 1, 2, \dots$ , then  $[\bigcup_{n=1}^{\infty} A_n] = [\bigcup_{n=1}^{\infty} B_n]$  and  $[\bigcap_{n=1}^{\infty} A_n] = \bigcap_{n=1}^{\infty} B_n$ .*

LEMMA 3. *If  $A$  is measurable and  $B \in [A]$ , then  $B$  is measurable.*

There exist  $Z_1$  and  $Z_2$  such that  $A \cup Z_1 = B \cup Z_2$  with  $m(Z_1) = m(Z_2) = 0$ . Let  $\tilde{B}$  denote  $[0, 1] - B$ . Then  $B \cup Z_2$  is measurable and since  $B = (B \cup Z_2) - (\tilde{B} \cap Z_2)$ ,  $B$  is measurable.

It follows from Lemma 3 that the sets in each equivalence class are either all measurable or all non-measurable. Thus the space  $\mathfrak{S}^* = \mathfrak{M}^* \cup \mathfrak{N}^*$ , where  $\mathfrak{M}^*$  is the space of all equivalence classes of measurable sets, and  $\mathfrak{N}^*$  is the space of all equivalence classes of non-measurable sets. It should be noted that  $\mathfrak{M}^*$  is the metric space corresponding to the well-known pseudo-metric space  $\mathfrak{M}$  defined at the beginning of the paper.

In the following we will omit the asterisks and square brackets, and will write  $\mathfrak{S}$  for  $\mathfrak{S}^*$ , etc., and  $\rho(A, B)$  for  $\rho([A], [B])$ . When we write  $A \in \mathfrak{S}$ ,  $A$  may be considered either as an equivalence class or as a representative element of that class.

THEOREM 1. *The space  $\mathfrak{S}$  is complete.*

The proof is similar to that given in [1, p. 32].

THEOREM 2. *For every  $A \in \mathfrak{S}$  and every positive number  $\varepsilon < 1$ , there exists  $B \in \mathfrak{S}$  such that  $0 < \rho(A, B) < \varepsilon$ .*

*Proof Case I.*  $m(A) = 0$ .

If  $m(A) = 0$ , then  $A \in [\phi]$ ,  $\phi$  denoting the empty set. Let  $B \in \mathfrak{S}$  be an interval of length  $< \varepsilon$ . Then  $\rho(A, B) = \rho(\phi, B) = m(B) < \varepsilon$ .

*Case II.*  $m^*(A) > 0$ .

Let  $I \in \mathfrak{S}$  be an interval of length  $< \varepsilon$ , such that  $m^*(I \cap A) > 0$ . If  $B = A - I$ , then

$$\rho(A, B) = \rho(A, A - I) = m^*[A - (A - I)] = m^*(I \cap A) \leq m^*(I) < \varepsilon.$$

COROLLARY 1. *If in Theorem 2,  $A \in \mathfrak{M}$ , then  $B$  (as constructed)  $\in \mathfrak{M}$ .*

THEOREM 3. *If  $A \in \mathfrak{M}$  and  $\varepsilon > 0$ , then there exists  $C \in \mathfrak{N}$  such that  $0 < \rho(A, C) < \varepsilon$ .*

*Proof Case I.*  $m(A) = 0$ .

Let  $M$  be a set of real numbers such that for every measurable set  $E$ ,  $m^*(M \cap E) = m(E)$  and  $m_*(M \cap E) = 0$ ,  $m_*$  denoting the interior Lebesgue measure. (See [2], Theorem E, p. 70.) In Case I of Theorem 2, let  $C = B \cap M$ . Then

$$\rho(A, C) = \rho(\phi, C) = m^*(C) = m(B) < \varepsilon \text{ and } m_*(C) = 0 .$$

Case II.  $m(A) > 0$ .

In Case II of Theorem 2, let  $C = A - (I \cap M)$ ,  $M$  described above. Then  $\rho(A, C) = m^*(A - C) = m^*(A \cap I \cap M) \leq m(I) < \varepsilon$ , and  $m^*(A \cap I \cap M) = m(A \cap I) > 0$ ,  $m_*(A \cap I \cap M) = 0$ . Since  $(A \cap I \cap M) \in \mathfrak{R}$ ,  $C \in \mathfrak{R}$ .

THEOREM 4.  $\mathfrak{N}$  is open in  $\mathfrak{S}$ .

*Proof.* Assume Theorem 4 is false. Then there exists  $N \in \mathfrak{N}$  and sets  $M_m \in \mathfrak{M}$ ,  $m = 1, 2, \dots$ , such that  $\lim_{m \rightarrow \infty} \rho(N, M_m) = 0$ . The sequence  $M_m, m = 1, 2, \dots$ , is, therefore, a Cauchy sequence in  $\mathfrak{S}$  and so by Theorem 1 has a subsequence  $M_{m_n}, n = 1, 2, \dots$ , such that  $\lim_{m \rightarrow \infty} \rho(\lim \sup_n M_{m_n}, M_m) = 0$ . Since  $\lim \sup_n M_{m_n}$  is measurable, this means that  $N$  is measurable by Lemma 3, a contradiction.

The last few results can be summarized as follows.

THEOREM 5.  $\mathfrak{M}$  is perfect and nowhere dense in  $\mathfrak{S}$ ;  $\mathfrak{N}$  is open and dense in  $\mathfrak{S}$ .

The remainder of the work is valid for both spaces, as only the equivalence classes are dealt with (these being the same for  $\mathfrak{S}$  and  $\mathfrak{D}$ ).

After having proven completeness for  $\mathfrak{S}$  in Theorem 1, a natural question to ask is "Is the space separable?". The theorem proved here which demonstrates the existence of  $2^c (= \mathfrak{f})$ , where  $2^{c_0} = c$ , equivalence classes in  $\mathfrak{S}$  answers this question (and a similar one about a countable basis) in the negative. It is also interesting to note that the space  $\mathfrak{M}$  has exactly  $c$  equivalence classes. (In the following work  $\Omega$  is the first ordinal belonging to  $c$ .)

THEOREM 6. There exist  $\mathfrak{f}$  equivalence classes in the space  $\mathfrak{S}$ .

*Proof.* It will be sufficient to construct a well-ordered family  $\{A_\alpha \mid 0 \leq \alpha < \Omega\}$  of mutually disjoint subsets of  $[0, 1]$ , each of which has  $m^*(A_\alpha) = 1$ .

Consider  $\{B_\beta \mid 0 \leq \beta < \Omega\}$  as a well-ordering of all closed subsets  $B_\beta$  of  $[0, 1]$  which have a positive Lebesgue measure. For each  $\beta, 0 \leq \beta < \Omega$ , let  $\{x_\alpha^\beta \mid 0 \leq \alpha \leq \beta\}$  be a well-ordered subset of  $B_\beta$  such that  $x_\alpha^\beta \neq x_{\alpha'}^{\beta'}$ , if  $\beta \neq \beta'$  or  $\alpha \neq \alpha'$ . This selection is possible since, for each  $\beta$ , the set of all  $x_{\alpha'}^{\beta'}$  with  $0 \leq \alpha' \leq \beta' < \beta$  has a cardinal number  $< c$ . Set  $A_\alpha = \{x_\alpha^\beta \mid \alpha \leq \beta < \Omega\}$ , for each  $\alpha, 0 \leq \alpha < \Omega$ . By a simple argument  $A_\alpha \cap A_{\alpha'} = \phi$ , for  $\alpha \neq \alpha'$ . Now consider any  $A_\alpha$ ; if  $m^*(A_\alpha) \neq 1$ , then  $A_\alpha$  is contained in some open set  $Y$  such that  $m(Y) < 1$ . The complement of  $Y$  is closed and has  $m([0, 1] - Y) > 0$ . But  $m\{[0, \alpha] \cap ([0, 1] - Y)\}$

is a continuous function of  $x$  for  $0 \leq x \leq 1$ ; therefore, this function takes on all values between 0 and  $m([0, 1] - Y)$ , inclusive. This means that there are non-denumerably many closed sets whose measures are greater than 0 and which do not intersect  $A_\alpha$ . This is, of course, impossible by the construction of  $A_\alpha$ . Therefore,  $m^*(A_\alpha) = 1$ .

Form the set of all subsets of the set of  $A_\alpha$ 's, and take the sum of each element of this power set. Any two such sums belong to two different equivalence classes since they disagree in a set of exterior measure 1. This set of sums has cardinal  $\mathfrak{f}$ . There are, therefore, at least  $\mathfrak{f}$  equivalence classes, at most  $\mathfrak{f}$  such classes; hence, exactly  $\mathfrak{f}$ .

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