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ON THE DETERMINATION OF NUMBERS BY THEIR SUMS OF A FIXED ORDER

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ON THE DETERMINATION OF NUMBERS BY THEIR SUMS OF A FIXED ORDER

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1. Introduction. We wish to treat the following problem (suggested by a problem of L. Moser [2]):

Let $\{x\} = \{x_1, \cdots, x_n\}$ be a set of complex numbers (if one is interested in generality, one may consider them elements of an algebraically closed field of characteristic zero) and let $\{\sigma\} = \{\sigma_1, \cdots, \sigma_{\binom{n}{s}}\}$ be the set of sums of s distinct elements of $\{x\}$. To what extent is $\{x\}$ determined by $\{\sigma\}$ and what sets can be $\{\sigma\}$ sets ?

In § 2 we answer this question for $s = 2$. In § 3 we treat the question for general s.

2. The case $s = 2$.

THEOREM 1. *If $n \neq 2^k$ then the first n elementary symmetric functions of $\{\sigma\}$ can be prescribed arbitrarily and they determine $\{x\}$ uniquely.*

Proof. Instead of the elementary symmetric functions we consider the sums of powers, setting

$$\sum\nolimits_k = \sum_{i=1}^{\binom{n}{2}} \sigma_i^k\,, \qquad S_k = \sum_{i=1}^{n} x_i^k\,.$$

Then

$$\sum\nolimits_k = \sum_{i=1}^{\binom{n}{2}} \sigma_i^k = \sum_{1 \le i_1 < i_2 \le n} (x_{i_1} + x_{i_2})^k = \frac{1}{2} \sum_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{n} (x_{i_1} + x_{i_2})^k \tag{1}$$
$$= \frac{1}{2}\Big(\sum_{i_1, i_2 = 1}^{n} (x_{i_1} + x_{i_2})^k - \sum_{i=1}^{n} (2x_i)^k \Big)\,.$$

Expanding the binomials and collecting like powers we obtain

$$\sum\nolimits_k = \frac{1}{2}\Big(\sum_{l=0}^{k} \tbinom{k}{l} S_l S_{k-l} - 2^k S_k \Big)$$
$$= \frac{1}{2}(2n - 2^k) S_k + \frac{1}{2} \sum_{l=1}^{k-1} \tbinom{k}{l} S_l S_{k-l}$$

Thus, since the coefficient of S_k does not vanish, we can solve re-

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cursively for S_k in terms of $\sum_1, \cdots, \sum_k$. In particular $\sum_1, \cdots, \sum_n$ determine $S_1, \cdots, S_n$—and hence $x_1, \cdots, x_n$—uniquely.

THEOREM 2. *If $n = 2^k$ then $\sum_1, \cdots, \sum_{k+1}$ must satisfy a certain algebraic equation and $\{\sigma\}$ will not always determine $\{x\}$.*

Proof. Equation (1) for $\sum_{k+1}$ yields

$$\sum_{k+1} = \frac{1}{2} \sum_{l=1}^{k} \binom{k+1}{l} S_l S_{k+1-l} \tag{2}$$

where $S_1, \cdots, S_k$ are expressed by (1) as polynomials in $\sum_1, \cdots, \sum_k$.

To prove the second part of the theorem we proceed by induction. Assume there are two different sets $\{x_1, \cdots, x_{2^{k-1}}\}$, $\{y_1, \cdots, y_{2^{k-1}}\}$ which have the same $\{\sigma\}$. Then consider the two sets

$$\{X\} = \{x_1 + a, \cdots, x_{2^{k-1}} + a, y_1, \cdots, y_{2^{k-1}}\}$$
$$\{Y\} = \{x_1, \cdots, x_{2^{k-1}}, y_1 + a, \cdots, y_{2^{k-1}} + a\} .$$

Clearly every sum of two elements of $\{X\}$ is either σ_i or $\sigma_i + 2a$ or $x_i + y_j + a$ and the same holds for the sum of two elements of $\{Y\}$.

The sets $\{X\}$, $\{Y\}$ will clearly be different for some a. To show that they are different for any $a \neq 0$, rearrange $\{x\}$ and $\{y\}$ so that $x_i = y_i$; $i = 1, 2, \cdots, m$; $m \geq 0$, and $x_j \neq y_k$ for $j, k > m$. Then since $y_i + a = x_i + a$; $i = 1, 2, \cdots, m$, the sets $\{X\}$ and $\{Y\}$ will be different if $\{x_j \mid j > m\}$ is different from $\{x_j + a \mid j > m\}$. But this is clear for any $a \neq 0$.

Since $\{\sigma\}$ clearly does not determine $\{x\}$ for $n = 2$ the proof is complete.

In a sense we have completed the answer of the question raised in the introduction for $s = 2$, however there remain some unanswered questions in case $n = 2^k$.

1. *If $\{\sigma\}$ does not determine $\{x\}$ can there be more than two sets giving rise to same $\{\sigma\}$?*

The answer is trivially "yes" for $k = 0, 1$ and is "no" for $k = 2$. It seems probable that the answer is "no" for all $k \geq 2$, however we can see no simple way of proving this.

2. *For what values of n does there exist for all (real) $\{x\}$ a transformation $y_i = f_i(x_1, \cdots, x_n)$, different from a permutation, so that $\{x\}$ and $\{y\}$ give rise to the same $\{\sigma\}$?*

This question was suggested by T. S. Motzkin who gave the answer for $s = 2$.

LEMMA 1. *If $n > s$ and the above functions f_i exist then they are linear.*

Proof. The sets $\{y\}$, $\{x\}$ are connected by a system of equations

$$y_{i_1} + \cdots + y_{i_s} = x_{j_1} + \cdots + x_{j_s} .$$

Here the indices $i_1, \cdots, i_s$ are themselves functions of $\{x\}$. However, since they assume only a finite set of values, there exists a somewhere dense set of $\{x\}$ for which the indices are constant. We restrict our attention to that set. Let $\Delta_k^{(h)} y_i = f_i(x_1, \cdots, x_k + h, \cdots, x_n) - f_i(x_1, \cdots, x_k, \cdots, x_n)$ then we obtain

$$\Delta_k^{(h)} y_{i_1} + \cdots + \Delta_k^{(h)} y_{i_s} = 0 \text{ or } h . \tag{3}$$

If we let u_i be the difference of $\Delta_k^{(h)} y_i$ for two different sets of values of $\{x\}$ then, since the right-hand side of (3) is independent of the choice of $\{x\}$, we obtain

$$u_{i_1} + \cdots + u_{i_s} = 0 . \tag{4}$$

Summation over all sets $\{i_1, \cdots, i_s\} \subset \{1, \cdots, n\}$ yields

$$u_1 + u_2 + \cdots + u_n = 0 . \tag{5}$$

Now let t be the least positive integer so that $u_{i_1} + \cdots + u_{i_t} = 0$ for all $\{i_1, \cdots, i_t\} \subset \{1, \cdots, n\}$. Then $t \mid n$, for $n = mt + r$ with $0 < r < t$ implies

$$u_{i_1} + \cdots + u_{i_r} = u_1 + u_2 + \cdots + u_n - \sum (u_{j_1} + \cdots + u_{j_t}) = 0$$

for all $\{i_1, \cdots, i_r\} \subset \{1, \cdots, n\}$, contrary to hypothesis.

Since $n > s \geq t$ we must have $n \geq 2t$. If $t > 1$ then

$$u_j = -(u_{i_1} + \cdots + u_{i_{t-1}}) \text{ for every } j \notin \{i_1, \cdots, i_{t-1}\} .$$

But there are more than t such j, say $j_1, \cdots, j_t$. Hence

$$u_{j_1} + \cdots + u_{j_t} = -t(u_{i_1} + \cdots + u_{i_{t-1}}) = 0$$

or $u_{i_1} + \cdots + u_{i_{t-1}} = 0$ for every $\{i_1, \cdots, i_{t-1}\} \subset \{1, \cdots, n\}$ contrary to hypothesis. Thus $t = 1$ and

$$u_1 = u_2 = \cdots = u_n = 0 .$$

In other words $\Delta_k^{(h)} y_i = a_{ik}^{(h)} = \text{const}$. Thus $\Delta_k^{(h_1)} y_i + \Delta_k^{(h_2)} y_i = \Delta_k^{(h_1 + h_2)} y_i$ so that $a_{ik}^{(h)} = a_{ik} h$ and

$$y_i = \sum_k a_{ik} x_k .$$

THEOREM 3. *If $n > s$ and there exists a nontrivial transformation $y_i = f_i(x_1, \cdots, x_n)$ which preserves $\{\sigma\}$ then $n = 2s$ and the transformation is linear with matrix (up to permutations)*

$$\begin{pmatrix} -\frac{s-1}{s} & \frac{1}{s} & \cdots & \frac{1}{s} \\ \frac{1}{s} & -\frac{s-1}{s} & \cdots & \frac{1}{s} \\ \vdots & \vdots & & \vdots \\ \frac{1}{s} & \frac{1}{s} & \cdots & -\frac{s-1}{s} \end{pmatrix}$$

Proof. We know by Lemma 1 that the transformation must be linear. Let $y_i = \sum_k a_{ik} x_k$ then

$$(6) \qquad y_{i_1} + \cdots + y_{i_s} = \sum_k (a_{i_1 k} + \cdots + a_{i_s k}) x_k = x_{j_1} + \cdots + x_{j_s} .$$

Hence, for fixed k, we have

$$(7) \qquad a_{i_1 k} + \cdots + a_{i_s k} = \begin{cases} 0 \text{ for } \binom{n-1}{s} \text{ sets } \{i_1, \cdots, i_s\} \\ 1 \text{ for } \binom{n-1}{s-1} \text{ sets } \{i_1, \cdots, i_s\} . \end{cases}$$

Since $n > s$ two elements a_{ik}, a_{jk} in the same column satisfy

$$a_{ik} + a_{i_1 k} + \cdots + a_{i_{s-1} k} = 0 \text{ or } 1 ; \; a_{jk} + a_{i_1 k} + \cdots + a_{i_{s-1} k} = 0 \text{ or } 1$$

where $\{i_1, \cdots, i_{s-1}\} \subseteq \{1, \cdots, n\} - \{i, j\}$.

Hence

$$(8) \qquad a_{ik} = a_{jk} \text{ or } a_{ik} = a_{jk} \pm 1 .$$

Let the two values assumed by terms in the kth column be a_k and $1 + a_k$. From (6) we see that both values must occur. On the other hand if both a_k and $1 + a_k$ would occur more than once then $\max(a_{i_1 k} + \cdots + a_{i_s k}) - \min(a_{i_1 k} + \cdots + a_{i_s k}) \geq 2$ in contradiction to (7).

If $1 + a_k$ is assumed only once, say $a_{kk} = 1 + a_k$, then $0 = s a_k$ or

$$(9) \qquad a_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k . \end{cases}$$

According to (6) we have

$$(10) \qquad \sum_{k=1}^{n} (a_{i_1 k} + \cdots + a_{i_s k}) = s \qquad \{i_1, \cdots, i_s\} \subset \{1, \cdots, n\} .$$

We now repeat the argument that led to equation (8). Since $n > s$

we can write for any pair (i, j)

$$\sum_{k=1}^{n}(a_{i_1k} + \cdots + a_{i_{s-1}k}) + \sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n}(a_{i_1k} + \cdots + a_{i_{s-1}k}) + \sum_{k=1}^{n} a_{jk} = s$$

where $\{i_1, \cdots, i_{s-1}\} \subset \{1, \cdots, n\} - \{i, j\}$. Hence $\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{jk}$ and according to (10), $s \sum_{k=1}^{n} a_{ik} = s$ so that

$$(11) \qquad \sum_{k=1}^{n} a_{ik} = 1 \qquad i = 1, \cdots, n .$$

Combining (9) and (11) we obtain

$$(12) \qquad a_{kj} = \begin{cases} 1 & j = k \\ 0 & j \neq k . \end{cases}$$

In other words, every column contains 0 and therefore $a_k = 0$ for $k = 1, \cdots, n$. Thus the transformation is a permutation.

The only nontrivial case arises therefore if the value a_k occurs only once, say $a_{kk} = a_k$. Then $s - 1 + sa_k = 0$ and

$$(13) \qquad a_{ik} = \begin{cases} -(s-1)/s & i = k \\ 1/s & i \neq k . \end{cases}$$

Combining (11) and (13) we obtain

$$(14) \qquad \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik} = n = \frac{n(n-1)}{s} - n\frac{s-1}{s} = \frac{n}{s}(n-s)$$

and hence $n = 2s$. It is now clear from (11) that each row and column contains exactly one term $-(s-1)/s$ and that the matrix (up to permutation) is the one given in the theorem.

3. General s. The procedure which led to Theorem 1 can be generalized. First we define, for every s, a function which is a polynomial in $n, 2^k, 3^k, \cdots, s^k$. Let

$$(15) \qquad f(n, k) = \frac{1}{s}\sum_{P}(-1)^{s-t} n^{t-1} \sum_{i=1}^{r} a_i i^k$$

where the outer summation is over all permutations P on s marks, each permutation being composed of a_i i-cycles $i = 1, \cdots, r$, and $t = a_1 + \cdots + a_r$. Thus

$$(16) \quad f(n,k) = n^{s-1} - \frac{1}{2}(s-1)(2^k + s - 2)n^{s-2} + (s-1)(s-2)\Big[\frac{1}{3}(3^k + s - 3) + \frac{1}{8}(s-3)(2^{k+1} + s - 4)\Big]n^{s-3} - \cdots + (-1)^s(s-1)!\Big(\sum_{i=1}^{s-1}\frac{i^{k-1}}{s-i}\Big)n - (-1)^s(s-1)!\, s^{k-1} .$$

THEOREM 4. *For every s consider the system of Diophantine equations $f(n, k) = 0$ $k = 1, 2, \cdots, n$. If n satisfies none of these then the first n elementary symmetric functions of $\{\sigma\}$ can be prescribed arbitrarily and they determine $\{x\}$ uniquely. If $f(n, k) = 0$, then the first k elementary symmetric functions of $\{\sigma\}$ must satisfy an algebraic equation.*

Proof. In the notation of Theorem 1 we have

$$(17) \qquad \textstyle\sum_k = \sum_{1 \le i_1 < \cdots < i_s \le n} (x_{i_1} + x_{i_2} + \cdots + x_{i_s})^k = \frac{1}{s!} \sum_{D(s)} (x_{i_1} + \cdots + x_{i_s})^k$$

where by $D(t)$ is meant summation over all sets of subscripts i_j at least t of which are distinct. Hence

$$\begin{aligned} s\,!\, \textstyle\sum_k &= \sum_{D(s-1)} (x_{i_1} + \cdots + x_{i_s})^k - \binom{s}{2} \sum_{D(s-1)} (2x_{i_1} + x_{i_2} + \cdots + x_{i_{s-1}})^k \\ &= \sum_{D(s-2)} (x_{i_1} + \cdots + x_{i_s})^k - \binom{s}{2} \sum_{D(s-2)} (2x_{i_1} + x_{i_2} + \cdots + x_{i_{s-1}})^k \\ &+ 2\binom{s}{3} \sum_{D(s-2)} (3x_{i_1} + x_{i_2} \cdots + x_{i_{s-2}})^k + 3\binom{s}{4} \sum_{D(s-2)} (2x_{i_1} + 2x_{i_2} + x_{i_3} + \cdots + x_{i_{s-2}})^k . \end{aligned}$$

Continue cancelling terms until each summation is over $D(1)$. The coefficient of $\sum(m_1 x_{i_1} + \cdots + m_t x_{i_t})^k$ is just $(-1)^{s-t}$ times the number of permutations on s marks which are conjugate to one having cycles of length $m_1, \cdots, m_t$. This can be shown by a method quite similar to that used by Frobenius [1]. Hence we may write

$$(18) \qquad s\,!\, \textstyle\sum_k = \sum_P (-1)^{s-t} \sum_{D(1)} (m_1 x_{i_1} + \cdots + m_t x_{i_t})^k$$

where the outer summation is over all permutations P on s marks, and $m_1, \cdots, m_t$ are the lengths of the cycles of P. Now from the multinomial expansion we have

$$\sum_{D(1)} (m_1 x_{i_1} + \cdots + m_t x_{i_t})^k = \sum_{\substack{l_1 + \cdots + l_t = k \\ l_i \ge 0}} \frac{k\,!}{l_1\,! \cdots l_t\,!} m_1^{l_1} \cdots m_t^{l_t} S_{l_1} \cdots S_{l_t}$$

and the coefficient of S_k is $(m_1^k + \cdots + m_t^k) S_0^{t-1}$. Substituting in (18) and using (15), we obtain

$$(19) \qquad (s-1)\,!\, \textstyle\sum_k = f(n, k) S_k + \cdots$$

where the terms indicated by dots do not involve S_k. Thus, if $f(n, k) \ne 0$ for $k = 1, \cdots, n$, then (19) can be solved recursively for $S_1, \cdots, S_n$ in terms of $\sum_1, \cdots, \sum_n$.

On the other hand, if $f(n, k) = 0$ and $f(n, j) \ne 0$ for $j = 1, \cdots, k-1$ then (17) expresses $\sum_k$ as a polynomial in $S_1, \cdots, S_{k-1}$ which in turn are polynomials in $\sum_1, \cdots, \sum_{k-1}$.

COROLLARY. *If $f(n, k) = 0$ then n divides $(s-1)!\, s^{n-1}$.*

Thus $\{x\}$ will always be determined by $\{\sigma\}$ if s is less then the greatest prime factor of n.

EXAMPLE 1. $s = 3$. Here (18) becomes

$$6 \sum_k = \sum_{i_1, i_2, i_3 = 1}^{n} (x_{i_1} + x_{i_2} + x_{i_3})^k - 3 \sum_{i_1, i_2 = 1}^{n} (2x_{i_1} + x_{i_2})^k + 2 \sum_{i=1}^{n} (3x_i)^k .$$

Expanding and collecting the coefficient of S_k, we get

$$f(n, k) = n^2 - (2^k + 1)n + 2 \cdot 3^{k-1} .$$

This has obvious zeros at $n = 1$, $k = 1$; $n = 2$, $k = 1, 2$; $n = 3$, $k = 2, 3$. Also, as we know from Theorem 3, there are zeros at $n = 6$, $k = 3, 5$. For all these values of n the set $\{\sigma\}$ does not, in general, determine $\{x\}$ uniquely.

In addition we observe that $f(n, k) = 0$ has the solutions $n = 27$, $k = 5, 9$ and $n = 486$, $k = 9$. We do not know whether for these values of n the set $\{\sigma\}$ determines $\{x\}$ uniquely or not. However we do know that these are the only cases left in doubt.

THEOREM 5. *If $s = 3$ then $f(n, k) = 0$ has solutions only for $k =$ 1, 2, 3, 5, 9.*

Proof. If $f(n, k) = 0$ then

$$n = 2^a \cdot 3^b \text{ with } a = 0 \text{ or } 1 . \tag{20}$$

Substituting (20) in $f(n, k) = 0$ we obtain

$$2^a \cdot 3^b + 2^{1-a}\, 3^{k-b-1} = 2^k + 1 . \tag{21}$$

Let n be the smaller zero of $f(n, k)$ for a fixed k. Then the other zero is $n' = 2^{1-a}\, 3^{k-b-1}$ and $b \leq k - b - 1$. Hence

$$2^k \equiv -1 \pmod{3^b} \tag{22}$$

and since 2 is a primitive root of 3^b,

$$k \equiv 3^{b-1} \pmod{2 \cdot 3^{b-1}} . \tag{23}$$

But by (21) we have

$$3^{k-b-1} \leq 2^k < 3^{2k/3} \text{ or } k < 3(b+1)$$

so that

$$3^{b-1} \leq k < 3(b+1) \text{ and hence } b < 4 .$$

If $b = 3$ then $k \equiv 9 \pmod{18}$ and $k < 12$ so $k = 9$.
If $b = 2$ then $k \equiv 3 \pmod{6}$ and $k < 9$ so $k = 3$.
If $b = 1$ then $k \equiv 1 \pmod{2}$ and $k < 6$ so $k = 1, 3, 5$.
If $b = 0$ then $k < 3$.

EXAMPLE 2. $s = 4$. Here (18) becomes

$$24 \textstyle\sum_k = \sum_{i_1,i_2,i_3,i_4} (x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4})^k - 6 \sum_{i_1,i_2,i_3} (2x_{i_1} + x_{i_2} + x_{i_3})^k + 8 \sum_{i_1,i_2} (3x_{i_1} + x_{i_2})^k + 3 \sum_{i_1,i_2} (2x_{i_1} + 2x_{i_2})^k - 6 \sum_i (4x_i)^k . \tag{24}$$

Hence $f(n, k) = 0$ becomes

$$n^3 - 3(2^{k-1} + 1)n^2 + (2(3^k + 1) + 3 \cdot 2^{k-1})n - 3 \cdot 2^{2k-1} = 0 . \tag{25}$$

We first note that this has solutions $n = 1$, $k = 1$; $n = 2$, $k = 1, 2$; $n = 3$, $k = 1, 2, 3$; $n = 4$, $k = 2, 3, 4$; $n = 8$, $k = 3, 5, 7$. For these values of n, the set $\{\sigma\}$ does not generally determine $\{x\}$. When $n = 12$, $k = 6$ is a solution, and this case is left in doubt.

THEOREM 6. *If $s = 4$ then $f(n, k) = 0$ has solutions only for $n = 1, 2, 3, 4, 8, 12$.*

Proof. Let $n = 3^a \cdot 2^b$ where $a = 0$ or 1. Now if $n \geq 3(2^{k-1} + 1)$ then $2 \cdot 3^k n > 3^{k+1} \cdot 2^k > 3 \cdot 2^{2k-1}$ and the left side of (25) is positive. Hence $n < 3(2^{k-1} + 1) < 2^{k+1}$ if $k > 3$ and so $b \leq k$. (For $k \leq 3$ we have listed all solutions of (25)). If k is even then $2(3^k + 1) \equiv 4 \pmod{8}$ and if $k \geq 4$ then $8n$ divides the other terms unless $b \leq 2$. Similarly if k is odd then $2(3^k + 1) \equiv 8 \pmod{16}$ and if $k \geq 5$ then $b \leq 3$. So $b \leq 3$ in all cases. Now suppose $a = 1$. Then (25) becomes

$$2n - 3 \cdot 2^{2k-1} \equiv 0 \pmod{9}$$

or

$$2^{b+1} \equiv 2^{2k-1} \equiv 2 \pmod{3}$$

and b is even. Thus n must be $1, 2, 3, 4, 8$, or 12. It is easy to check that none of these is a root for $k > 7$.

The corollary to Theorem 4 shows that exceptional pairs (s, n) are in a certain sense quite rare. Of course it is trivial to remark that if (s, n) is exceptional, then $(n - s, n)$ is exceptional. Hence the remarks for $s = 2$ apply equally well to $s = n - 2$ and we obtain the exceptional pairs $(6, 8)$, $(14, 16)$, $(30, 32)$, $\cdots$. But there are other cases with $n > 2s$ which our method leaves in doubt.

THEOREM 7. *We can construct arbitrarily large values of s such that $f(n, k) = 0$ for some $n > 2s$.*

Proof. If $n < s$ then $\sum_k = 0$ but $S_1, \cdots, S_n$ may be prescribed arbitrarily. Hence the coefficient of S_k in the expansion of $\sum_k$ must be zero if $k \leq n$. If $n = s$ then $\sum_k = S_1^k$ but $S_2, \cdots, S_n$ may be prescribed arbitrarily. Hence $n = s$ is a zero of $f(n, k)$ for $k = 2, \cdots, n$. Thus $f(n, 1) = \prod_{i=1}^{s-1} (n - i)$; $f(n, 2) = \prod_{i=2}^{s} (n - i)$ and $f(n, 3) = (n - 2s) \prod_{i=3}^{s} (n - i)$. If we divide $f(n, 4)$ by its known factors then we obtain for $s > 2$

$$f(n, 4) = [n^2 - (6s - 1)n + 6s^2] \prod_{i=4}^{s} (n - i) \tag{26}$$

and the equation

$$n^2 - (6s - 1)n + 6s^2 = 0 \tag{27}$$

can be rewritten

$$(2n - 6s + 1)^2 - 3(2s - 1)^2 = -2 .$$

The Pell equation $u^2 - 3v^2 = -2$ has the general solution

$$u + v\sqrt{3} = \pm (1 + \sqrt{3})(2 + \sqrt{3})^r \qquad r = 0, \pm 1, \cdots .$$

Since u and v are odd, n and s are integers. It is interesting that all positive solutions are obtained in the following simple way. When $k = 4$, $(s, n) = (2, 8)$ is a solution. Hence $(6, 8)$ is a solution and putting $s = 6$ in (27) yields $(6, 27)$. Continuing in this way, we obtain $(21, 27)$, $(21, 98)$, $(77, 98)$, $(77, 363)$, $\cdots$.

In a similar manner we obtain for $s > 3$

$$f(n, 5) = [n^2 - (12s - 5)n + 12s^2](n - 2s) \prod_{i=5}^{s} (n - i) \tag{28}$$

and all integer roots of the quadratic factor may be obtained with the aid of the general solution of the Pell equation $u^2 - 6v^2 = 75$. Or we could start with $(2, 16)$ and obtain successively $(14, 147)$, $(133, 1444)$, $\cdots$. Starting with $(3, 27)$ yields $(24, 256)$, $(232, 2523)$, $\cdots$.

4. Concluding remarks. If we let $\{\tau\} = \{\tau_1, \cdots, \tau_{ns}\}$ be the set of sums of s not necessarily distinct elements of $\{x\}$, then $\{x\}$ is always determined by $\{\tau\}$. A method similar to the proof of Theorem 4 applies with the coefficient of S_k always positive. Alternatively, if the x_i are real, $x_1 \leq x_2 \leq \cdots \leq x_n$, we may determine them successively by a simple induction procedure.

Our method is applicable to the case of weighted sums $\sigma_{i_1 \cdots i_s} =$

$\sum_{j=1}^{s} a_j x_{i_j}$. The resulting Diophantine equations will however be of a rather different nature. Thus, if the a_j are all distinct then the analogue to $f(n, k) = 0$ is

$$(a_1^k + a_2^k + \cdots + a_s^k) n^{s-1} = 0 \; . \tag{29}$$

In other words the uniqueness condition is independent of n and depends on the a_i alone. For example if $a_1 + a_2 + \cdots + a_s = 0$ then $\{\sigma\}$ remains unchanged if we add the same constant to all x. It is not as easy to see what happens if (29) holds for some $k > 1$.

References

1. G. Frobenius, *Ueber die Charaktere der symmetrischen Gruppe*, S.-B. Preuss. Akad. Wiss. Berlin, (1900), 516–534.
2. L. Moser, Problem E 1248, Amer. Math. Monthly (1957), 507.

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