ON THE DETERMINATION OF NUMBERS BY THEIR SUMS
OF A FIXED ORDER

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1. Introduction. We wish to treat the following problem (suggested by a problem of L. Moser [2]):

Let \( \{x\} = \{x_1, \cdots, x_n\} \) be a set of complex numbers (if one is interested in generality, one may consider them elements of an algebraically closed field of characteristic zero) and let \( \{\sigma\} = \{\sigma_1, \cdots, \sigma_{n_s}\} \) be the set of sums of \( s \) distinct elements of \( \{x\} \). To what extent is \( \{x\} \) determined by \( \{\sigma\} \) and what sets can be \( \{\sigma\} \) sets?

In § 2 we answer this question for \( s = 2 \). In § 3 we treat the question for general \( s \).

2. The case \( s = 2 \).

**Theorem 1.** If \( n \neq 2^k \) then the first \( n \) elementary symmetric functions of \( \{\sigma\} \) can be prescribed arbitrarily and they determine \( \{x\} \) uniquely.

**Proof.** Instead of the elementary symmetric functions we consider the sums of powers, setting

\[
\sum_k = \sum_{i=1}^{(n)} \sigma_i^k, \quad S_k = \sum_{i=1}^{n} x_i^k.
\]

Then

\[
\sum_k = \sum_{i=1}^{(n)} \sigma_i^k = \sum_{1 \leq i_1 < i_2 \leq n} (x_{i_1} + x_{i_2})^k = \frac{1}{2} \sum_{i_1 \neq i_2} (x_{i_1} + x_{i_2})^k
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{n} (x_{i_1} + x_{i_2})^k - \sum_{i=1}^{n} (2x_i)^k \right).
\]

Expanding the binomials and collecting like powers we obtain

\[
\sum_k = \frac{1}{2} \left( \sum_{i=0}^{k} \binom{k}{i} S_i S_{k-i} - 2^k S_k \right)
\]

\[
= \frac{1}{2} (2n - 2^k) S_k + \frac{1}{2} \sum_{i=1}^{k-1} \binom{k}{i} S_i S_{k-i}.
\]

Thus, since the coefficient of \( S_k \) does not vanish, we can solve re-

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cursively for \( S_k \) in terms of \( \sum_1, \cdots, \sum_k \). In particular \( \sum_1, \cdots, \sum_n \) determine \( S_1, \cdots, S_n \)—and hence \( x_1, \cdots, x_n \)—uniquely.

**Theorem 2.** If \( n = 2^k \) then \( \sum_1, \cdots, \sum_{k+1} \) must satisfy a certain algebraic equation and \( \{\sigma\} \) will not always determine \( \{x\} \).

**Proof.** Equation (1) for \( \sum_{k+1} \) yields

\[
\sum_{k+1} = \frac{1}{2} \sum_{l=1}^{k} \left( \frac{k+1}{l} \right) S_l S_{k+1-l}
\]

where \( S_1, \cdots, S_k \) are expressed by (1) as polynomials in \( \sum_1, \cdots, \sum_k \).

To prove the second part of the theorem we proceed by induction.

Assume there are two different sets \( \{x_1, \cdots, x_{k-1}\}, \{y_1, \cdots, y_{k-1}\} \) which have the same \( \{\sigma\} \). Then consider the two sets

\[
\{X\} = \{x_1 + a, \cdots, x_{2k-1} + a, y_1, \cdots, y_{2k-1}\}
\]

\[
\{Y\} = \{x_1, \cdots, x_{2k-1}, y_1 + a, \cdots, y_{2k-1} + a\}
\]

Clearly every sum of two elements of \( \{X\} \) is either \( \sigma_i \) or \( \sigma_i + 2a \) or \( x_i + y_j + a \) and the same holds for the sum of two elements of \( \{Y\} \).

The sets \( \{X\}, \{Y\} \) will clearly be different for some \( a \). To show that they are different for any \( a \neq 0 \), rearrange \( \{x\} \) and \( \{y\} \) so that \( x_i = y_i ; i = 1, 2, \cdots, m ; m \geq 0 \), and \( x_j \neq y_k \) for \( j, k > m \). Then since \( y_i + a = x_i + a ; i = 1, 2, \cdots, m \), the sets \( \{X\} \) and \( \{Y\} \) will be different if \( \{x_j | j > m\} \) is different from \( \{x_j + a | j > m\} \). But this is clear for any \( a \neq 0 \).

Since \( \{\sigma\} \) clearly does not determine \( \{x\} \) for \( n = 2 \) the proof is complete.

In a sense we have completed the answer of the question raised in the introduction for \( s = 2 \), however there remain some unanswered questions in case \( n = 2^k \).

1. **If** \( \{\sigma\} \) **does not determine** \( \{x\} \) **can there be more than two sets giving rise to same** \( \{\sigma\} \?)

The answer is trivially "yes" for \( k = 0, 1 \) and is "no" for \( k = 2 \). It seems probable that the answer is "no" for all \( k \geq 2 \), however we can see no simple way of proving this.

2. **For what values of** \( n \) **does there exist for all** (real) \( \{x\} \) **a transformation** \( y_i = f_i(x_1, \cdots, x_n) \), **different from a permutation**, so that \( \{x\} \) **and** \( \{y\} \) **give rise to the same** \( \{\sigma\} \?)

This question was suggested by T. S. Motzkin who gave the answer for \( s = 2 \).
LEMMA 1. If \( n > s \) and the above functions \( f_i \) exist then they are linear.

Proof. The sets \( \{y\}, \{x\} \) are connected by a system of equations

\[
y_i + \cdots + y_s = x_{j_1} + \cdots + x_{j_s}.
\]

Here the indices \( i_1, \ldots, i_s \) are themselves functions of \( \{x\} \). However, since they assume only a finite set of values, there exists a somewhere dense set of \( \{x\} \) for which the indices are constant. We restrict our attention to that set. Let \( A_k^{(h)} y_i = f(x_1, \ldots, x_k + h, \ldots, x_n) - f(x_1, \ldots, x_k, \ldots, x_n) \) then we obtain

\[
(3) \quad A_k^{(h)} y_{i_1} + \cdots + A_k^{(h)} y_{i_s} = 0 \text{ or } h.
\]

If we let \( u_i \) be the difference of \( A_k^{(h)} y_i \) for two different sets of values of \( \{x\} \) then, since the right-hand side of (3) is independent of the choice of \( \{x\} \), we obtain

\[
(4) \quad u_{i_1} + \cdots + u_{i_s} = 0.
\]

Summation over all sets \( \{i_1, \ldots, i_t\} \subset \{1, \ldots, n\} \) yields

\[
(5) \quad u_1 + u_2 + \cdots + u_n = 0.
\]

Now let \( t \) be the least positive integer so that \( u_{i_1} + \cdots + u_{i_t} = 0 \) for all \( \{i_1, \ldots, i_t\} \subset \{1, \ldots, n\} \). Then \( t \mid n \), for \( n = mt + r \) with \( 0 < r < t \) implies

\[
u_{i_1} + \cdots + u_{i_r} = u_1 + u_2 + \cdots + u_n - \sum(u_{j_1} + \cdots + u_{j_t}) = 0
\]

for all \( \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\} \), contrary to hypothesis. Since \( n > s \geq t \) we must have \( n \geq 2t \). If \( t > 1 \) then

\[
u_j = -(u_{i_1} + \cdots + u_{i_{t-1}}) \text{ for every } j \not\in \{i_1, \ldots, i_{t-1}\}.
\]

But there are more than \( t \) such \( j \), say \( j_1, \ldots, j_t \). Hence

\[
u_{j_1} + \cdots + u_{j_t} = -t(u_{i_1} + \cdots + u_{i_{t-1}}) = 0
\]

or \( u_{i_1} + \cdots + u_{i_{t-1}} = 0 \) for every \( \{i_1, \ldots, i_{t-1}\} \subset \{1, \ldots, n\} \) contrary to hypothesis. Thus \( t = 1 \) and

\[
u_i = u_1 = \cdots = u_n = 0.
\]

In other words \( A_k^{(h)} y_i = a_{ik}^{(h)} \) = const. Thus \( A_k^{(h)} y_i + A_k^{(h)} y_i = A_k^{(h+b)} y_i \) so that \( a_{ik}^{(h)} = a_{ik}^b \) and

\[
y_i = \sum_k a_{ik} x_k.
\]
THEOREM 3. If \( n > s \) and there exists a nontrivial transformation 
\( y_i = f(x_1, \ldots, x_n) \) which preserves \( \{\sigma\} \) then \( n = 2s \) and the transformation 
is linear with matrix (up to permutations)

\[
\begin{pmatrix}
-\frac{s-1}{s} & 1 & \cdots & 1 \\
\frac{1}{s} & -\frac{s-1}{s} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{s} & \frac{1}{s} & \cdots & -\frac{s-1}{s}
\end{pmatrix}
\]

Proof. We know by Lemma 1 that the transformation must be linear. Let 
\( y_i = \sum_k a_{ik}x_k \) then

\[
y_i + \cdots + y_{i_s} = \sum_k (a_{ik} + \cdots + a_{i_s})x_k = x_{j_1} + \cdots + x_{j_s}.
\]

Hence, for fixed \( k \), we have

\[
a_{i_1k} + \cdots + a_{i_sk} = \begin{cases} 
0 & \text{for } \binom{n-1}{s} \text{ sets } \{i_1, \ldots, i_s\} \\
1 & \text{for } \binom{n-1}{s-1} \text{ sets } \{i_1, \ldots, i_s\}.
\end{cases}
\]

Since \( n > s \) two elements \( a_{ik}, a_{jk} \) in the same column satisfy

\[
a_{ik} + a_{i_1k} + \cdots + a_{i_{s-1}k} = 0 \text{ or } 1; \ a_{jk} + a_{i_1k} + \cdots + a_{i_{s-1}k} = 0 \text{ or } 1
\]

where \( \{i_1, \ldots, i_{s-1}\} \subseteq \{1, \ldots, n\} \setminus \{i, j\} \).

Hence

\[
a_{ik} = a_{jk} \text{ or } a_{ik} = a_{jk} \pm 1.
\]

Let the two values assumed by terms in the \( k \)th column be \( a_k \) and 
\( 1 + a_k \). From (6) we see that both values must occur. On the other 
hand if both \( a_k \) and \( 1 + a_k \) would occur more than once then 
\[\max(a_{ik} + \cdots + a_{i_s}) - \min(a_{ik} + \cdots + a_{i_s})\geq 2\] 
in contradiction to (7).

If \( 1 + a_k \) is assumed only once, say \( a_{sk} = 1 + a_k \), then \( 0 = sa_k \) or

\[
a_{ik} = \begin{cases} 
1 & i = k \\
0 & i \neq k.
\end{cases}
\]

According to (6) we have

\[
\sum_{k=1}^n (a_{ik} + \cdots + a_{i_s}) = s \quad \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}.
\]

We now repeat the argument that led to equation (8). Since \( n > s \)
we can write for any pair \((i, j)\)
\[
\sum_{k=1}^{n} (a_{i;k} + \cdots + a_{i, s_i-k}) + \sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} (a_{i;k} + \cdots + a_{i, s_i-k}) + \sum_{k=1}^{n} a_{jk} = s
\]
where \(\{i_s, \ldots, i_{s-1}\} \subset \{1, \ldots, n\} \setminus \{i, j\}\). Hence \(\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{jk}\) and according to (10), \(s \sum_{k=1}^{n} a_{ik} = s\) so that

\[
(11) \quad \sum_{k=1}^{n} a_{ik} = 1 \quad i = 1, \ldots, n.
\]

Combining (9) and (11) we obtain

\[
(12) \quad a_{kj} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}
\]

In other words, every column contains 0 and therefore \(a_k = 0\) for \(k = 1, \ldots, n\). Thus the transformation is a permutation.

The only nontrivial case arises therefore if the value \(a_k\) occurs only once, say \(a_{kk} = a_k\). Then \(s - 1 + sa_k = 0\) and

\[
(13) \quad a_{ik} = \begin{cases} (s - 1)/s & i = k \\ 1/s & i \neq k \end{cases}
\]

Combining (11) and (13) we obtain

\[
(14) \quad \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik} = n = \frac{n(n-1)}{s} - n \frac{s-1}{s} = \frac{n}{s} (n-s)
\]

and hence \(n = 2s\). It is now clear from (11) that each row and column contains exactly one term \(-(s - 1)/s\) and that the matrix (up to permutation) is the one given in the theorem.

3. General \(s\). The procedure which led to Theorem 1 can be generalized. First we define, for every \(s\), a function which is a polynomial in \(n, 2^s, 3^s, \ldots, s^s\). Let

\[
(15) \quad f(n, k) = \frac{1}{s} \sum_{P} (-1)^{t_i} n^{t_i-1} \sum_{i=1}^{r} a_i t_i^k
\]

where the outer summation is over all permutations \(P\) on \(s\) marks, each permutation being composed of \(a_i\) \(i\)-cycles \(i = 1, \ldots, r\), and \(t = a_1 + \cdots + a_r\). Thus

\[
(16) \quad f(n, k) = n^{t-1} - \frac{1}{2} (s - 1)(2^s + s - 2)n^{t-2} + (s - 1)(s - 2) \left[ \frac{1}{3} (3^s + s - 3) 
+ \frac{1}{8} (s - 3)(2^{s+1} + s - 4) \right] n^{t-3} - \cdots + (-1)^t(s - 1)! \left( \sum_{i=1}^{t-1} \frac{i^{k-1}}{s - i} \right) n
- (-1)^t(s - 1)! s^{t-1}.
\]
Theorem 4. For every \( s \) consider the system of Diophantine equations
\[ f(n, k) = 0 \quad k = 1, 2, \ldots, n. \]
If \( n \) satisfies none of these then the first \( n \) elementary symmetric functions of \( \{\sigma\} \) can be prescribed arbitrarily and they determine \( \{x\} \) uniquely. If \( f(n, k) = 0 \), then the first \( k \) elementary symmetric functions of \( \{\sigma\} \) must satisfy an algebraic equation.

Proof. In the notation of Theorem 1 we have
\[
\sum_k = \sum_{i_1 < \cdots < i_k \leq n} (x_{i_1} + x_{i_2} + \cdots + x_{i_k})^k = \frac{1}{\delta!} \sum_i (x_{i_1} + \cdots + x_{i_\delta})^k
\]
where by \( D(t) \) is meant summation over all sets of subscripts \( i \), at least \( t \) of which are distinct. Hence
\[
s! \sum_k = \sum_{D(s-1)} (x_{i_1} + \cdots + x_{i_s})^k - \left(\begin{array}{c} s \\ 2 \end{array}\right) \sum_{D(s-2)} (2x_{i_1} + x_{i_2} + \cdots + x_{i_{s-1}})^k
\]
\[
= \sum_{D(s-2)} (x_{i_1} + \cdots + x_{i_s})^k - \left(\begin{array}{c} s \\ 2 \end{array}\right) \sum_{D(s-2)} (2x_{i_1} + x_{i_2} + \cdots + x_{i_{s-1}})^k
\]
\[
+ 2\left(\begin{array}{c} s \\ 3 \end{array}\right) \sum_{D(s-3)} (3x_{i_1} + x_{i_2} + \cdots + x_{i_{s-2}})^k + 3\left(\begin{array}{c} s \\ 4 \end{array}\right) \sum_{D(s-3)} (2x_{i_1} + 2x_{i_2} + x_{i_3} + \cdots + x_{i_{s-3}})^k.
\]
Continue cancelling terms until each summation is over \( D(1) \). The coefficient of \( \Sigma(m_1x_{i_1} + \cdots + m_x x_{i_x})^k \) is just \((-1)^{t-1}\) times the number of permutations on \( s \) marks which are conjugate to one having cycles of length \( m_1, \cdots, m_x \). This can be shown by a method quite similar to that used by Frobenius [1]. Hence we may write
\[
s! \sum_k = \sum_P (-1)^{t-1} \sum_{D(1)} (m_1x_{i_1} + \cdots + m_x x_{i_x})^k
\]
where the outer summation is over all permutations \( P \) on \( s \) marks, and \( m_1, \cdots, m_x \) are the lengths of the cycles of \( P \). Now from the multinomial expansion we have
\[
\sum_{D(1)} (m_1x_{i_1} + \cdots + m_x x_{i_x})^k = \sum_{l_1, \ldots, l_x \geq 0} \frac{k!}{l_1! \cdots l_x!} m_1^{l_1} \cdots m_x^{l_x} S_{l_1} \cdots S_{l_x}
\]
and the coefficient of \( S_k \) is \( (m_1^k + \cdots + m_x^k)S_{k-1} \). Substituting in (18) and using (15), we obtain
\[
(s - 1)! \sum_k = f(n, k)S_k + \cdots
\]
where the terms indicated by dots do not involve \( S_k \). Thus, if \( f(n, k) \neq 0 \) for \( k = 1, \cdots, n \), then (19) can be solved recursively for \( S_1, \cdots, S_n \) in terms of \( \sum_n, \cdots, \sum_n \).

On the other hand, if \( f(n, k) = 0 \) and \( f(n, j) \neq 0 \) for \( j = 1, \cdots, k - 1 \) then (17) expresses \( \Sigma_k \) as a polynomial in \( S_1, \cdots, S_k-1 \), which in turn are polynomials in \( \sum_n, \cdots, \sum_{k-1} \).
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COROLLARY. If \( f(n, k) = 0 \) then \( n \) divides \( (s - 1)! s^{n-1} \).
Thus \( \{x\} \) will always be determined by \( \{\sigma\} \) if \( s \) is less than the greatest prime factor of \( n \).

EXAMPLE 1. \( s = 3 \). Here (18) becomes

\[
6 \sum_{i} = \sum_{t_1, t_2, t_3 = 1}^{n} (x_{t_1} + x_{t_2} + x_{t_3})^k - 3 \sum_{t_1, t_2 = 1}^{n} (2x_{t_1} + x_{t_2})^k + 2 \sum_{t = 1}^{n} (3x_t)^k.
\]

Expanding and collecting the coefficient of \( S_k \), we get

\[
f(n, k) = n^2 - (2^k + 1)n + 2 \cdot 3^{k-1}.
\]

This has obvious zeros at \( n = 1, k = 1 \); \( n = 2, k = 1, 2 \); \( n = 3, k = 2, 3 \). Also, as we know from Theorem 3, there are zeros at \( n = 6, k = 3, 5 \).
For all these values of \( n \) the set \( \{\sigma\} \) does not, in general, determine \( \{x\} \) uniquely.

In addition we observe that \( f(n, k) = 0 \) has the solutions \( n = 27, k = 5, 9 \) and \( n = 486, k = 9 \). We do not know whether for these values of \( n \) the set \( \{\sigma\} \) determines \( \{x\} \) uniquely or not. However we do know that these are the only cases left in doubt.

THEOREM 5. If \( s = 3 \) then \( f(n, k) = 0 \) has solutions only for \( k = 1, 2, 3, 5, 9 \).

Proof. If \( f(n, k) = 0 \) then

\[
n = 2^a \cdot 3^b \text{ with } a = 0 \text{ or } 1.
\]

Substituting (20) in \( f(n, k) = 0 \) we obtain

\[
2^a \cdot 3^b + 2^{1-a} 3^{k-b-1} = 2^k + 1.
\]

Let \( n \) be the smaller zero of \( f(n, k) \) for a fixed \( k \). Then the other zero is \( n' = 2^{1-a} 3^{k-b-1} \) and \( b \leq k - b - 1 \). Hence

\[
2^k \equiv -1 \pmod{3^b}
\]

and since 2 is a primitive root of \( 3^b \),

\[
k \equiv 3^{b-1} \pmod{2 \cdot 3^{b-1}}.
\]

But by (21) we have

\[
3^{k-b-1} \leq 2^k < 3^{b+1} \text{ or } k < 3(b + 1)
\]

so that

\[
3^{b-1} \leq k < 3(b + 1) \text{ and hence } b < 4.
\]
If $6 = 3$ then $k \equiv 9 \pmod{18}$ and $k < 12$ so $k = 9$.
If $6 = 2$ then $k \equiv 3 \pmod{6}$ and $k < 9$ so $k = 3$.
If $6 = 1$ then $k \equiv 1 \pmod{2}$ and $k < 6$ so $k = 1, 3, 5$.
If $6 = 0$ then $k < 3$.

**Example 2.** $s = 4$. Here (18) becomes

\begin{equation}
24 \sum_i \sum_{i_1, i_2, i_3} (x_{i_1} + x_{i_2} + x_{i_3})^k
- 6 \sum_{i_1, i_2, i_3} (2x_{i_1} + x_{i_2} + x_{i_3})^k + 8 \sum_{i_1, i_2} (3x_{i_1} + x_{i_2})^k
+ 3 \sum_{i_1, i_2} (2x_{i_1} + 2x_{i_2})^k - 6 \sum_i (4x_i)^k.
\end{equation}

Hence $f(n, k) = 0$ becomes

\begin{equation}
n^3 - 3(2^{k-1} + 1)n^2 + (2(3^k + 1) + 3 \cdot 2^{k-1})n - 3 \cdot 2^{k-1} = 0.
\end{equation}

We first note that this has solutions $n = 1, k = 1$; $n = 2, k = 1, 2$; $n = 3, k = 1, 2, 3$; $n = 4, k = 2, 3, 4$; $n = 8, k = 3, 5, 7$. For these values of $n$, the set $\{\sigma\}$ does not generally determine $\{x\}$. When $n = 12, k = 6$ is a solution, and this case is left in doubt.

**Theorem 6.** If $s = 4$ then $f(n, k) = 0$ has solutions only for $n = 1, 2, 3, 4, 8, 12$.

**Proof.** Let $n = 3^a \cdot 2^b$ where $a = 0$ or 1. Now if $n \geq 3(2^{k-1} + 1)$ then $2 \cdot 3^a n > 3^{k+1} \cdot 2^k > 3 \cdot 2^{k-1}$ and the left side of (25) is positive. Hence $n < 3(2^{k-1} + 1) < 2^{k+1}$ if $k > 3$ and so $b \leq k$. (For $k \leq 3$ we have listed all solutions of (25)). If $k$ is even then $2(3^k + 1) \equiv 4 \pmod{8}$ and if $k \geq 4$ then $8n$ divides the other terms unless $b \leq 2$. Similarly if $k$ is odd then $2(3^k + 1) \equiv 8 \pmod{16}$ and if $k \geq 5$ then $b \leq 3$. So $b \leq 3$ in all cases. Now suppose $a = 1$. Then (25) becomes

\[2n - 3 \cdot 2^{2k-1} \equiv 0 \pmod{9}\]

or

\[2^{k+1} \equiv 2^{3k-1} \equiv 2 \pmod{3}\]

and $b$ is even. Thus $n$ must be $1, 2, 3, 4, 8, 12$. It is easy to check that none of these is a root for $k > 7$.

The corollary to Theorem 4 shows that exceptional pairs $(s, n)$ are in a certain sense quite rare. Of course it is trivial to remark that if $(s, n)$ is exceptional, then $(n - s, n)$ is exceptional. Hence the remarks for $s = 2$ apply equally well to $s = n - 2$ and we obtain the exceptional pairs $(6, 8), (14, 16), (30, 32), \cdots$. But there are other cases with $n > 2s$ which our method leaves in doubt.
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**Theorem 7.** We can construct arbitrarily large values of $s$ such that $f(n, k) = 0$ for some $n > 2s$.

**Proof.** If $n < s$ then $\sum_k = 0$ but $S_1, \ldots, S_n$ may be prescribed arbitrarily. Hence the coefficient of $S_k$ in the expansion of $\sum_k$ must be zero if $k \leq n$. If $n = s$ then $\sum_k = S_s$ but $S_2, \ldots, S_n$ may be prescribed arbitrarily. Hence $n = s$ is a zero of $f(n, k)$ for $k = 2, \ldots, n$. Thus $f(n, 1) = \prod_{i=1}^{s-1} (n - i) ; f(n, 2) = \prod_{i=2}^{s-1} (n - i)$ and $f(n, 3) = (n - 2s) \prod_{i=3}^{s-1} (n - i)$. If we divide $f(n, 4)$ by its known factors then we obtain for $s > 2$

$$f(n, 4) = \left[ n^2 - (6s - 1)n + 6s^2 \right] \prod_{i=4}^{s} (n - i)$$

and the equation

$$n^2 - (6s - 1)n + 6s^2 = 0$$

can be rewritten

$$(2n - 6s + 1)^2 - 3(2s - 1)^2 = -2.$$ 

The Pell equation $u^2 - 3v^2 = -2$ has the general solution

$$u + v\sqrt{3} = \pm (1 + \sqrt{3})(2 + \sqrt{3})^r \quad r = 0, \pm 1, \ldots.$$ 

Since $u$ and $v$ are odd, $n$ and $s$ are integers. It is interesting that all positive solutions are obtained in the following simple way. When $k = 4, (s, n) = (2, 8)$ is a solution. Hence $(6, 8)$ is a solution and putting $s = 6$ in (27) yields (6, 27). Continuing in this way, we obtain (21, 27), (21, 98), (77, 98), (77, 363), \ldots.

In a similar manner we obtain for $s > 3$

$$f(n, 5) = \left[ n^2 - (12s - 5)n + 12s^2 \right] \prod_{i=5}^{s} (n - i)$$

and all integer roots of the quadratic factor may be obtained with the aid of the general solution of the Pell equation $u^2 - 6v^2 = 75$. Or we could start with (2, 16) and obtain successively (14, 147), (133, 1444), \ldots. Starting with (3, 27) yields (24, 256), (232, 2523), \ldots.

4. **Concluding remarks.** If we let $\{\tau\} = \{\tau_1, \ldots, \tau_s\}$ be the set of sums of $s$ not necessarily distinct elements of $\{x\}$, then $\{x\}$ is always determined by $\{\tau\}$. A method similar to the proof of Theorem 4 applies with the coefficient of $S_k$ always positive. Alternatively, if the $x_i$ are real, $x_1 \leq x_2 \leq \cdots \leq x_n$, we may determine them successively by a simple induction procedure.

Our method is applicable to the case of weighted sums $\sigma_{t_1, \ldots, t_s} =$...
The resulting Diophantine equations will however be of a rather different nature. Thus, if the $a_i$ are all distinct then the analogue to $f(n, k) = 0$ is

\[(29) \quad (a_1^k + a_2^k + \cdots + a_s^k) n^{s-1} = 0.\]

In other words the uniqueness condition is independent of $n$ and depends on the $a_i$ alone. For example if $a_1 + a_2 + \cdots + a_s = 0$ then $\{\sigma\}$ remains unchanged if we add the same constant to all $x$. It is not as easy to see what happens if (29) holds for some $k > 1$.

**References**


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