ON THE NUMBER OF LATTICE POINTS IN $x^t + y^t = n^{t/2}$

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ON THE NUMBER OF LATTICE POINTS IN \( x^t + y^t = n^{1/2} \)

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Introduction. Suppose that \( t \) is independent of \( n, n > 1 \); \( t = (2M)/(2N+1); M = 1, 2, 3, \cdots; N = 0, 1, 2, \cdots; M \geq N + 1 \), so that \( t > 1 \). Let \( L_t(n^{1/2}) \) be the number of lattice points, \((x, y)\), satisfying \( x^t + y^t \leq n^{1/2} \). Our main objective is the proof of the relation

\[
S(n) = \frac{c_t n^2 - c_2 n t^{(2t-1)/2}}{n^{t-1} \int_0^n L_t(n^{1/2}) n^{t/2-1} \, dw}
\]

\[
= c_t n^2 - c_2 \pi n^{(2t-1)/2} \sum_{a=1}^{\infty} \alpha^{-2a/t} \cos(2\pi \sqrt{n} \alpha - \pi/(2t))
\]

\[
- \frac{2t}{\pi^3 \sqrt{t-1} n^{t-2}} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\cos(2\pi H \sqrt{n} - \pi/4)}{(a, b)^{t-2}(3t-2)H(3t-2)/(3t-2) + O(\sqrt{n})}
\]

with \( t > 1 \), \( c_t = \frac{2\Gamma^{(t-1)/4} \Gamma(1/t)}{(t+2)\Gamma(2/t)} \), \( c_2 = \frac{2(3-1/t)\Gamma(1/t)}{\pi^{(t+1)/t}} \).

\( H = (\alpha^{t(t-1)} + \beta^{t(t-1)})^{t-1/t} \). The case \( t = 2 \) is known in connection with the classical problem of the lattice points in a circle \([4, pp. 221, 235]\).

By choosing \( t \) as specified above the analysis is less bulky than it would be if we considered the slightly more general problem of \( L_t(n^{1/2}) \) corresponding to the curve \(|x|^r + |y|^r = n^{1/2}\) with real \( T > 0 \). Expressions and estimates for \( L_t(n^{1/2}) \) have been obtained by Bachmann \([1, pp. 447-450]\), Cauer \([2]\), and van der Corput \([3]\). In particular van der Corput \([3]\) found that

\[
L_t(n^{1/2}) = c_t n - 8T^{(1-r)/2} n^{(r-1)/(2r)} \int_0^n g_t(\sqrt{n} - x) x^{(1-r)/r} \, dx
\]

\[
+ O(n^{1/3}), T > 3 \;
\]

\[
= c_t n - 8 \sum_{j=1}^{\infty} (-1)^{j+1} \left(\frac{1}{j}\right) \zeta(-jT)n^{(1-jr)/2}
\]

\[
+ O(n^{1/3}), 0 < T \leq 3, T \neq 1 \;
\]

where

\[
c_t = \frac{2\Gamma^{(1-T)}(1/T)}{TT^{(2/T)}}
\]

\( g_t(x) = x - \lfloor x \rfloor - 1/2 \), \( \lfloor x \rfloor \) is the integral part of \( x \), \( \zeta(s) \) is the Riemann zeta function and \( \left(\frac{a}{b}\right) \) is the binomial coefficient. From (1.2) it follows that

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These results in (1.3) and analogous results can be obtained from (1.1) also. Our methods fail to establish the analogue of (1.1) for 0 < t < 1.

2. First auxiliary result. We first obtain the result

\[ S(n) = n^t \sum_{\alpha=\infty}^{\infty} \sum_{\beta=\infty}^{\infty} \int_{x^t}^{x^t} (1 - x^t - y^t) \cos 2\pi \sqrt{n(\alpha x + \beta y)} dx dy, \]

\( t > 1. \)

In § 4 we prove that the double series is absolutely convergent.

We have [4, p. 205]

\[ \int_{0}^{w} L(t) dw = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \int_{j^t + k^t \leq w}^{j^t + k^t \leq w} dw = \sum_{j^t + k^t \leq w}^{w} \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} (W - j^t - k^t). \]

To this we apply the Poisson summation formula [4, p. 204] to obtain

\[ \int_{0}^{w} L(t) dw = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \int_{x^t}^{x^t} \cos 2\pi ax \cdot \sum_{0 \leq \xi \leq 1} \left( W - x^t - y^t \right) dy. \]

Integrating by parts and applying the second mean value theorem for integrals, we have, for the inner integral,

\[ \frac{\pi^t}{\pi^t} \int_{0}^{(W - x^t)^{1/t}} \sin 2\pi \beta y \cdot y^{t-1} dy = \frac{\pi}{\pi^t} \int_{0}^{(W - x^t)^{1/t}} \sin 2\pi \beta y dy, \]

where \( 0 \leq \xi < (W - x^t)^{1/t} \), so that the sum over \( \beta \) is uniformly convergent in \( x \). Hence we can interchange the order of operations in \( \int dx \sum_{b} \) in

(2.3) to obtain

\[ \int_{0}^{w} L(t) dw = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \int_{x^t}^{x^t} \cos 2\pi ax \cos 2\pi \beta y \cdot (W - x^t - y^t) dx dy. \]

By symmetry we can replace \( \cos 2\pi ax \cos 2\pi \beta y \) by \( \cos 2\pi (ax + \beta y) \). If also we set \( w = z^{1/t}, x = W^{1/t}r, y = W^{1/t}s, W = n^{1/t}, \) we reduce (2.4) to

\[ \frac{t}{2} \int_{0}^{n} L(t) z^{(t-1)/t} dz = n^{(t-1)/t} \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \int_{r^t + s^t \leq 1} (1 - r^t - s^t) \cos 2\pi \sqrt{n(\alpha r^t + \beta s^t)} dr ds, \]

and then (2.1) follows upon multiplication of each side by \( n^{(t-1)/t} \).
3. Second auxiliary result. For \( t > 1 \), we shall obtain from (2.1) the identity

\[
S(n) = T_1 + T_2 + T_3 + T_4 + T_5
\]

where

\[
T_1 = c_1 n^2, \quad c_1 = \frac{2I^n(1/t)}{(t + 2)I'(2/t)};
\]

\[
T_2 = c_2 n^{5/2} \sum_{s=1}^{\infty} \alpha^{-3/2-1/t}J_{3/2+1/t}(2\pi\sqrt{n}\alpha),
\]

where

\[
c_2 = \frac{2^{(2t-1)/t}t^{1/t}I'(1/t)}{\pi^{(t+1)/t}},
\]

and \( J_r(x) \) is the ordinary Bessel function of order \( r \);

\[
T_3 = c_3 n^2 \int_0^1 f(x, t) \cos 2\pi\sqrt{n}\alpha xdx, \quad c_3 = \frac{16t}{t + 1},
\]

and \( f(x, t) = (1 - x^t)^{(t+1)/t} - (t/2)^{(t+1)/t}(1 - x^{2t})^{(t+1)/t} \);

\[
T_4 = -\frac{2t}{\pi\sqrt{t - 1}} \sum_{s=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{J_s(2\pi H\sqrt{n})}{(\alpha^s\beta^s)^{(s-3)(s-2)}} H^{s/(t-1)}, \quad H = (\alpha^s(t-1) + \beta^s(t-1))^{(s-1)/t};
\]

\[
T_5 = \frac{2t}{\pi^2} \sum_{s=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \int_{-\infty}^{\infty} G(u, \alpha, \beta) \cos 2\pi H\sqrt{n} v(u, \alpha, \beta) \cdot v'(u, \alpha, \beta) du,
\]

where

\[
v(u, \alpha, \beta) = H^{-1} A_s^{1/2}(u), \quad A_s(u) = (-1)^s \alpha^{-s} (P\alpha - u)^{t-s} + \beta^{-s} (Q\beta + u)^{t-s},
\]

\[
P = \frac{\alpha^s(t-1)}{\alpha^s(t-1)} + \beta^s(t-1), \quad Q = \frac{\beta^s(t-1)}{\alpha^s(t-1)} + \beta^s(t-1),
\]

\[
G(u, \alpha, \beta) = \frac{A_s(u)A_s(u) - A_s^{1/2}(u)}{v'(u, \alpha, \beta)A_s^{1/2}(u)} - a_{-1}(\alpha, \beta) \operatorname{sgn} u[1 - v'(u, \alpha, \beta)]^{-1/2},
\]

\[
a_{-1}(\alpha, \beta) = \frac{(\alpha\beta)^{s(t-2)}}{\sqrt{t - 1}} \frac{(\alpha^s(t-1) + \beta^s(t-1))}{(s-1)!}.
\]

In the proof of (3.1) we make use of the following result on Bessel functions [5, p. 366],

\[
\int_0^1 (1 - x^2)^{m-1/2} \cos Kxdx = \sqrt{\pi} 2^{m-1} K^{-m} I(m + 1/2) J_m(K) \quad m > -1/2.
\]

First, it is convenient to break up the double sum in (2.1) as follows,


\[ S(n) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{\alpha=\infty}^{a} \sum_{\beta=\infty}^{b} + \sum_{a=0}^{\infty} \sum_{b=\infty}^{a} \sum_{\alpha=\infty}^{b} \sum_{\beta=\infty}^{a} + \sum_{a=\infty}^{\infty} \sum_{b=\infty}^{\infty} \sum_{\alpha=\infty}^{a} \sum_{\beta=\infty}^{b} + \sum_{a=\infty}^{\infty} \sum_{b=\infty}^{\infty} \sum_{\alpha=\infty}^{b} \sum_{\beta=\infty}^{a} . \]

By symmetry this can be written as

\[ S(n) = n^2 \int_{x^2 + y^2 \leq 1} (1 - x^t - y^t) dx dy \]

\[ + 4n^2 \sum_{a=1}^{\infty} \int_{x^2 + y^2 \leq 1} (1 - x^t - y^t) \cos 2\pi \sqrt{n} \alpha x dx dy \]

\[ + 4n^2 \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \int_{x^2 + y^2 \leq 1} (1 - x^t - y^t) \cos 2\pi \sqrt{n} (\alpha x + \beta y) dx dy \]

\[ = S_1 + S_2 + S_3 . \]

\( S_1 \) can be evaluated in terms of gamma functions to obtain

\[ S_1 = \frac{2\Gamma((1/2)t)}{(t + 2)\Gamma(2/2t)} n^2 = c n^2 . \]

Let \( I_2 \) denote the integral in \( S_2 \). Then

\[ I_2 = 4 \int_0^t \cos 2\pi \sqrt{n} \alpha x dx \int_0^{(1 - z^t)/t} (1 - x^t - y^t) dy \]

\[ = \frac{4t}{t + 1} \int_0^1 (1 - x^t)^{(t+1)/t} \cos 2\pi \sqrt{n} \alpha x dx \]

\[ = \frac{4t}{t + 1} \left( \frac{t}{2} \right)^{(t+1)/t} \int_0^1 (1 - x^t)^{(t+1)/t} \cos 2\pi \sqrt{n} \alpha x dx \]

\[ + \frac{c_3}{4} \int_0^1 f(x, t) \cos 2\pi \sqrt{n} \alpha x dx \]

by the definition of \( f(x, t) \) in (3.1). Applying (3.2) to (3.6) we have

\[ S_2 = 4n^3 \sum_{a=1}^{\infty} I_2 = T_2 + T_3 . \]

Let \( I_3 \) denote the integral in \( S_3 \). Then by symmetry

\[ I_3 = 2 \int_{x^2 + y^2 \leq 1} (1 - x^t - y^t) \cos 2\pi \sqrt{n} (\alpha x - \beta y) dx dy . \]

The transformation

\[ x = Hv(P - u/\alpha) , \ y = Hv(Q + u/\beta) \]
transforms \( x^t + y^t = 1 \) into
\[
(3.10) \quad v = H^{-1}A_t^{-1/2}(u)
\]
where \( H, P, Q, \) and \( A_t(u) \) are defined in (3.1). The transformation (3.9) is one to one for \( \alpha x + \beta y \geq 0 \) and the absolute value of the Jacobian is
\[
(3.11) \quad J(x, y, v, u) = \frac{H^2v}{\alpha \beta}.
\]
The graph of (3.10) resembles that of \( v = 1/(1 + u^2) \) except that the curve is not symmetric to the \( v \) axis unless \( t = 2 \). The curve has a relative maximum at \((0, 1)\).

Applying (3.9) to (3.8) we transform \( x^t + y^t \leq 1 \) and \( \alpha x + \beta y \geq 0 \) into \( v \leq H^{-1}A_t^{-1/2}(u) \) and \( v \geq 0 \) respectively, so that (3.8) becomes
\[
(3.12) \quad I_3 = \frac{2H^2}{\alpha \beta} \int_{-\infty}^{\infty} du \int_0^{u(u)} [1 - H^t v^t A_t(u)] v \cos 2\pi H \sqrt{\frac{v}{n}} dv.
\]
Upon integration by parts with respect to \( v \), the integrated terms vanish and we obtain
\[
(3.13) \quad I_3 = -\frac{H}{\pi \sqrt{n} \alpha \beta} \int_{-\infty}^{\infty} du \int_0^{u(u)} [1 - (t + 1)H^t v^t A_t(u)] \sin 2\pi H \sqrt{n} v dv
\]
where \( u_+(v) \) and \( u_-(v) \) refer to the first and second quadrant branches of (3.10) respectively. Since
\[
(3.14) \quad A_t(u) = (-1)^t \alpha^{-t}(P\alpha - u)^{t-1} + \beta^{-t}(Q\beta + u)^{t-1},
\]
we can write (3.13) as
\[
(3.15) \quad I_3 = -\frac{H}{\pi \sqrt{n} \alpha \beta} \int_0^{u_+(v)} [u_+(v) - H^t v^t A_{t-1}(u_+(v))] \sin 2\pi H \sqrt{n} v dv
\]
\[
-\frac{H}{\pi \sqrt{n} \alpha \beta} \int_0^{u-(v)} [-u_-(v) + H^t v^t A_{t-1}(u_-(v))] \sin 2\pi H \sqrt{n} v dv.
\]
By the change of variable (3.10) this can be written as
\[
(3.16) \quad I_3 = \frac{H}{\pi \sqrt{n} \alpha \beta} \int_{-\infty}^{\infty} \left[ u - A_{t-1}(u) \right] \sin 2\pi H \sqrt{n} v(u) \cdot v'(u) du.
\]
From (3.14) we obtain
for large $u$, so that upon integrating by parts again we obtain

$$I_3 = \frac{t}{2\pi n\alpha\beta} \int_{-\infty}^{\infty} F(u) \cos 2\pi H\sqrt{u} v(u) \cdot v'(u) du$$

where

$$F(u) = F(u, \alpha, \beta) = \frac{A_{-1}(u)A_3(u) - A_3^2(u)}{v'(u)A_3^2(u)}.$$

The function $a_{-1} \text{sgn} u[1 - v^2(u)]^{-1/2}$ is an asymptotic equivalent of $F(u)$ in the neighborhood of $(0, 1)$, even though $v(0) = 1$ and $v'(0) = 0,$ if $a_{-1} = a_{-1}(\alpha, \beta)$ is determined from

$$a_{-1} = \lim_{u+0} \sqrt{1 - v^2(u)} = \lim_{u+0} \frac{1}{\sqrt{1 - v^2}}$$

$$= \lim_{u+0} \frac{v'(1 - v^2)^{-1/2}}{v''} = \frac{1}{v''(0)} \lim_{u+0} \frac{1}{\sqrt{1 - v^2}}$$

$$= \frac{1}{v''(0)}.$$

From (3.10) and (3.14) we obtain

$$v''(u) = -H^{-1}A_5(1+2\alpha)(u)[-(t+1)A_3(u) + (t-1)A_3(u)A_3(u)]$$

from which $a_{-1}$, as given in (3.1), can be determined.

We now write (3.18) as

$$I_3 = \frac{ta_{-1}}{2\pi n\alpha\beta} \int_{-\infty}^{\infty} \text{sgn} u[1 - v^2(u)]^{-1/2} \cos 2\pi H\sqrt{u} v(u) \cdot v'(u) du$$

$$+ \frac{t}{2\pi n\alpha\beta} \int_{-\infty}^{\infty} [F(u) - a_{-1} \text{sgn} u[1 - v^2(u)]^{-1/2}] \cos 2\pi H\sqrt{u} v(u) \cdot v'(u) du$$

$$= -\frac{ta_{-1}}{\pi n\alpha\beta} \int_{0}^{\infty} (1 - v^2)^{-1/2} \cos 2\pi H\sqrt{v} v dv$$

$$+ \frac{t}{2\pi n\alpha\beta} \int_{-\infty}^{\infty} G(u) \cos 2\pi H\sqrt{u} v(u) \cdot v'(u) du$$

where $G(u) = G(u, \alpha, \beta)$ is defined in (3.1). Applying (3.2) to (3.22) we obtain

$$S_3 = 4\pi^2 \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} I_3 = T_4 + T_5.$$

Collecting the results of (3.4), (3.5), (3.7), and (3.23), we have (3.1).
4. Convergence investigations. We next prove that the double series in (2.1) is absolutely convergent. We write (3.18) as

\[ I_3 = \frac{t}{2\pi^2 n a^2}(\int_{-\infty}^{0} + \int_{0}^{\sigma} + \int_{\sigma}^{\infty} + \int_{\infty}^{p_a}) \]

where \(0 < \sigma < Pa\).

First we consider

\[ I_\tau = \int_{p_a}^{\infty} F(u) \cos 2\pi H_n v(u) \cdot v'(u) du . \]

By (3.14) and (3.19) we have

\[ F(u) = \frac{H(P^\alpha - u)}{(\alpha \beta)^{\alpha - 1} \beta^{\alpha - 1}} - \frac{H[u - P^\alpha]^\alpha + \beta^{-\alpha}(u + Q^\beta)]^{(1 - \tau)/t}}{\alpha^\tau(u - P^\alpha)^{-\alpha} + \beta^\alpha(u + Q^\beta)^{-\alpha}} . \]

From (4.3) we find that

\[ \frac{dF(u)}{du} = \frac{(1 - \tau)H[(u - P^\alpha)^\alpha + (u + Q^\beta)^\beta]}{(\alpha \beta)^{\alpha - 1} \beta^{\alpha - 1}} - \frac{2\tau(u - P^\alpha)^{\alpha - 1} + \beta^{\beta}(u + Q^\beta)^{\beta - 1}}{(u - P^\alpha)^{\alpha + (u + Q^\beta)^{\beta + 1}} . \]

From (4.3) and (4.4) we derive certain information about the graph of \(F(u)\), namely,

\[ F(u) > 0, F(u) < 0, 0 < u < P^\alpha ; \]
\[ F'(P^\alpha) = \infty, 1 < \tau < 2 ; F'(P^\alpha) = 0, 2 < \tau ; \]
\[ F(u) < 0, P^\alpha < u < \infty ; \]
\[ F(u) = 0, u = u_i, P^\alpha < u_i < \infty, \beta > \alpha ; \]
\[ F(u) < 0, P^\alpha < u < \infty, \beta \leq \alpha . \]

The point \((u_i, v_i)\) is a relative minimum and from (4.3) and (4.4) we find that

\[ u_i = \frac{Q^\beta \alpha^{\beta}(u_i)^{\beta - 1}}{\beta^{\beta}(u_i)^{\alpha - 1} - \alpha^\alpha} + \frac{P^\alpha \beta^{\alpha}(u_i)^{\alpha - 1}}{\beta^\beta(u_i)^{\beta - 1}} , \]
\[ v_i = F(u_i) = \frac{H(u_i)^{\alpha - 1} + \beta^{\beta}(u_i)^{\beta - 1}}{\alpha^{\alpha}(u_i)^{\beta - 1}} . \]

Thus by (4.5) and the second mean value theorem for integrals we have, for \(\beta > \alpha\), and \(P^\alpha \leq \xi_i < u_i < \xi_2 \leq \infty , \)

\[ I_\tau = \int_{p_a}^{u_1} + \int_{u_1}^{\infty} = F(u_i) \int_{u_1}^{u_1} + F(u_i) \int_{u_1}^{\xi_2} \]
by the inequality $x^2 + y^2 \geq 2xy, x > 0, y > 0$. Similarly, for $\beta \leq \alpha$, and $P\alpha \leq \xi_i < \infty$, we have

\begin{equation}
I_7 = \int_{\xi_i}^{\infty} = F(\infty) \int_{\xi_i}^{\sigma} \cos 2\pi H_v \sqrt{n} v(u) \cdot v'(u) du
\end{equation}

\begin{align*}
&= O\{F(\infty) H^{-1} n^{-1/2}\} = \{(\alpha^{-i} + \beta^{-i})(1-t)^{t-1}((\alpha^i + \beta^i)n^{-1/2}\}
&= O\{(n\alpha\beta)^{-1/2}\}.
\end{align*}

We next consider $I_5$ in (4.1). By (4.3) we can write

\begin{equation}
I_5 = \int_{0}^{\sigma} F_1(u) \cos 2\pi H_v \sqrt{n} v(u) du
\end{equation}

where

\begin{equation}
F_1(u) = \frac{(pq)^{i-1}}{(\alpha \beta)^{1/2} A_\delta(u)}, \quad p = P\alpha - u, \quad q = Q\beta + u,
\end{equation}

\begin{align*}
&= A_5^{1/2}(u) \cdot \frac{[(pq)/(\alpha \beta)]^{i-1}}{[(p/\alpha)^t + (q/\beta)^t]^{t-1/2}} < \frac{1}{2^{i-1/2}} \\
&\leq \frac{A_5^{1/2}(0)}{\alpha \beta 2^{t-1/2}} = O\left(\frac{H^2}{\alpha \beta}\right).
\end{align*}

Therefore

\begin{equation}
I_5 = \left(\frac{H^2}{\alpha \beta}\right) \int_{0}^{\sigma} du = O\left(\frac{H^2\sigma}{\alpha \beta}\right).
\end{equation}

Turning next to $I_6$ in (4.1), we note that by the first line of (4.5) we can use the second mean value theorem to write, for some $\xi_i$ satisfying $\sigma < \xi_i \leq P\alpha$,

\begin{equation}
I_6 = F(\sigma) \int_{\sigma}^{\xi_i} \cos 2\pi H_v \sqrt{n} v(u) \cdot v'(u) du = O\left(\frac{F(\sigma)}{H_n}\right).
\end{equation}

To examine the question of the order of $F(u)$ in $0 < u < P\alpha$ we use (4.3) with, $p = P\alpha - u, q = Q\beta + u$, and write

\begin{equation}
F(u) = \frac{H}{(\alpha \beta)^{1/2}} \cdot \frac{[(pq)/(\alpha \beta)]^{(t-1)/2}}{[(p/\alpha)^t + (q/\beta)^t]^{t-1/2}} \cdot \\
\cdot \frac{1}{\{\beta^t(p/q)^{(t-1)/2} + (\alpha/\beta)^t(q/p)^{(t-1)/2}\}^{t-1/2}} \leq \frac{H}{(a\beta)^{1/2}} \cdot \frac{1}{2^{(t-1)/2}} \cdot \frac{1}{F_4(u)}.
\end{equation}
Since \( F_2(0) = 0 \) and

\[
(4.14) \quad F_3(u) = \frac{dF_3(u)}{du} = t - 1 \left[ \left( \frac{\beta}{\alpha} \right)^{t/2} \left( \frac{p}{q} \right)^{(t-1)/2} - \frac{1}{q^2} + \left( \frac{\alpha}{\beta} \right)^{t/2} \left( \frac{q}{p} \right)^{(t-1)/2} \frac{1}{p^2} \right],
\]

we have, by the mean value theorem,

\[
(4.15) \quad F(u) \leq \frac{H^\lambda}{(\alpha \beta)^{t/2}} \cdot \frac{(t-1)/2}{t-1} \cdot 0 < u < P \alpha, \quad 0 < u_3 < P \alpha.
\]

Setting \( p_3 = P \alpha - u_3, q_3 = Q \beta + u_3 \), we obtain

\[
(4.16) \quad F(u) \leq \frac{H^\lambda}{(\alpha \beta)^{t/2}} \cdot \left[ (\beta/\alpha)^{t/2} (p_3/q_3)^{(t-1)/2} + (\alpha/\beta)^{t/2} (q_3/p_3)^{(t-1)/2} \right] u
\]

\[
\leq \frac{H^\lambda}{(\alpha \beta)^{t/2}} \cdot \left[ (\beta/\alpha)^{t/2} + (\alpha/\beta)^{t/2} \right] u
\]

\[
\leq \frac{H^\lambda}{(\alpha \beta)^{t/2}} \cdot \frac{4}{(\alpha^t + \beta^t) u}.
\]

Hence combining (4.11), (4.12), and (4.16), we obtain

\[
(4.17) \quad I_5 + I_6 = O\left( \frac{H^{t \sigma}}{\alpha / \beta} \right) + O\left( \frac{(\alpha \beta)^{(t-1)/2}}{(\alpha^t + \beta^t) \sigma n^{1/2}} \right).
\]

In the further analysis of \( I_5 + I_6 \) we use the inequalities,

\[
(4.18) \quad 1 + x^m < (1 + x)^m, \quad 0 < x < 1, \quad m > 1,
\]

\[
(4.19) \quad (x + 1)^m < 2^{m-1}(x^m + 1), \quad x > 1, \quad m > 1.
\]

In (4.17) suppose \( 1 < t \leq 2 \). Since \( H = (\alpha^t + \beta^t) (t-1)/t \) and \( t/(t-1) > t \), we have by (4.18), \( H < (\alpha^t + \beta^t) t \), and therefore, for \( 1 < t \leq 2 \), we have

\[
(4.20) \quad \frac{H(\alpha \beta)^{(t-3)/4}}{(\alpha^t + \beta^t)^{(t-1)/2}} < \frac{(\alpha^t + \beta^t)^{(t-1)/2}}{(\alpha \beta)^{(t-1)/4}} \left( \frac{(\alpha^t + \beta^t)^{(t-3)/4}}{(\alpha \beta)^{(t-1)/4}} \right) < \frac{2^{(t-1)/2}}{(\alpha \beta)^{(t-1)/4}}.
\]

Hence from (4.17) and (4.20) we have, for \( 1 < t \leq 2 \),

\[
(4.21) \quad I_5 + I_6 = O\left( (\alpha \beta)^{(t-1)/4} n^{-1/4} \right).
\]

If \( t > 2 \) is (4.17), then \( t > t/(t-1) \) and so by (4.19) we have

\[
(4.22) \quad \frac{H(\alpha \beta)^{(t-3)/4}}{(\alpha^t + \beta^t)^{(t-1)/2}} = \frac{(\alpha^t + \beta^t)^{(t-1)/4}}{(\alpha^t + \beta^t)^{(t-1)/4}} \frac{(\alpha \beta)^{(t-3)/4}}{(\alpha^t + \beta^t)^{(t-3)/4}} \left( \frac{(\alpha \beta)^{(t-3)/4}}{(\alpha \beta)^{(t-1)/4}} \right) < \frac{2^{(t-3)/2}}{(\alpha \beta)^{(t-3)/4}} \left( \frac{2^{(t-3)/4}}{(\alpha \beta)^{(t-3)/4}} \right).
\]
Hence from (4.17) and (4.22) we have, for \( t > 2 \),

(4.23) \[ I_5 + I_6 = O\{(n\alpha\beta)^{-1/4}\} \, . \]

By (3.10) \( v(-u, \alpha, \beta) = v(u, \alpha, \beta) \) so that an estimate for \( I_5 + I_6 + I_7 \) holds also for \( I_4 \) in (4.1). By this fact, and the results of (4.7), (4.8), (4.21), and (4.23), it now follows that for \( S_n \), defined by (3.4), (3.23), and (4.1), we have,

(4.24) \[ S_n = 2t \pi \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha \beta} \int_{-\infty}^{\infty} F(u) \cos 2\pi n v(u) \cdot v'(u) du \]

\[ = O(n^{3/4}), \, t > 1 , \]

the double series being absolutely convergent.

Integrating by parts and applying the second mean value theorem, we have, from (3.6), for \( x_1 = [(t - 1)/t]^{1/4} , \)

(4.25) \[ S_n = \frac{16t}{\pi} \pi \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha \beta} \int_{0}^{\infty} (1 - x^t)^{(t+1)/t} \cos 2\pi n \alpha x dx \]

\[ = 8t \pi \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha \beta} \left\{ \left[ \int_{0}^{1} + \int_{x_1}^{1} \right] \right\} \]

\[ = 8t \pi \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha \beta} \int_{0}^{1} (1 - x^t)^{(t+1)/t} x^{-1} \sin 2\pi \sqrt{n} \alpha x dx \]

\[ + (1 - x_1^{(t+1)/t} x_1^{-1} \int_{x_1}^{1} \sin 2\pi \sqrt{n} \alpha x dx) \]

\[ = 8t \pi \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} O \left( \frac{1}{\sqrt{n} \alpha^2} \right) = O(n), \, t > 1 , \]

the series being absolutely convergent. The absolute convergence of the double series in (2.1) now follows from the results leading to (4.24) and (4.25).

5. Proof of (1.1). Finally we deduce (1.1) from (3.1). We make use of the asymptotic expansion for the general Bessel function, namely [5, p. 368],

(5.1) \[ J_m(K) = \sqrt{\frac{2}{\pi K}} \cos \left( K - \frac{m\pi}{2} - \frac{\pi}{4} \right) + O(K^{-3/2}) , \]

for large \( K \) and \( m \) independent of \( K \).

By (5.1) and the absolute convergence of the sum we have
(5.2) \[ T_2 = c_2 p^{5/4 - 1/(2t)} \sum_{\alpha = 1}^{\infty} \alpha^{-3/2 - 1/t} \left\{ \cos \frac{2\pi \sqrt{n} \alpha - \pi (1 + 1/(2t))}{(\pi^2 \sqrt{n} \alpha)^{1/2}} + O(n^{-3/4}) \right\} \]

\[ = - \frac{c_2}{\pi} p^{t-1/(2t)} \sum_{\alpha = 1}^{\infty} \alpha^{-3/2 - 1/t} \cos \left( \frac{2\pi \sqrt{n} \alpha - \pi (2t)}{(n^{1/2 - 1/(2t))}} \right) + O(n^{-3/4}) \].

In $T_3$, $f'(0, t) = 0$ and $f^{(k)}(1, t) = 0$, $k = 0, 1, 2$. Hence if we integrate by parts twice the integrated terms vanish and we have left

(5.3) \[ T_3 = - \frac{4t}{\pi (t + 1)} p^{t-1/(2t)} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} f''(\xi, t) \cos \frac{2\pi \sqrt{n} \alpha}{\alpha} dx = O(\sqrt{n}) \cdot \]

$f''(x, t)$ is continuous in $0 \leq x \leq 1$ and independent of $n$ and $\alpha$ and so it has a finite number, independent of $n$ and $\alpha$, of relative and absolute extrema whose values are also independent of $n$ and $\alpha$. Hence dividing the interval of integration into pieces in which $f''(x, t)$ is monotonic, we obtain by the second mean value theorem, for appropriate $\xi_j, \xi^*_j, \xi^*_{j+1}$ in the interval from 0 to 1, the $\xi$'s depending on $n$ and $\alpha$, the result,

(5.4) \[ T_3 = - \frac{4t}{\pi (t + 1)} p^{t-1/(2t)} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} f''(\xi_j, t) \cos \frac{2\pi \sqrt{n} \alpha}{\alpha} dx = O(\sqrt{n}) \cdot \]

Applying (5.1) to $T_4$ we obtain

\[ T_4 = - \frac{2t}{\pi^2 \sqrt{t} - 1} n^{3/(4)} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} \cos \left( \frac{2\pi H \sqrt{n} - \pi/4}{(\pi^2 \sqrt{n})^{1/2}} \right) \frac{1}{(\alpha \beta)^{(t-1)/2(t-2)} H^{(t-1)/(2t-2)}} \]

\[ - \frac{2tn}{\pi \sqrt{t} - 1} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} \frac{O(\alpha \beta)}{(\alpha \beta)^{(t-1)/2(t-2)} (\alpha \beta)^{(t-1)/2(t-2)} + \beta^t(t-1)} \cdot \]

Since

\[ (\alpha^t(t-1) + \beta^t(t-1) - (\alpha^t(t-1))/(2t-2) \leq 2 - (\alpha^t(t-1))/(2t-2) \cdot (\alpha \beta)^{-2(t-1)/(4-1)} \]

the double series are absolutely convergent so that

(5.5) \[ T_4 = - \frac{2t}{\pi^2 \sqrt{t} - 1} n^{3/(4)} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} \cos \left( \frac{2\pi H \sqrt{n} - \pi/4}{(\pi^2 \sqrt{n})^{1/2}} \right) + O(n^{1/4}) \cdot \]

Next we consider $T_5$. We have shown that $-T_4$ and $S_4$ are absolutely convergent double series for $t > 1$ and hence so is their term by term sum which is identical with $T_5$. We break up the interval of integration in $T_5$ into a finite number, independent of $n, \alpha, \beta, \nu$ of subintervals in which $G(u, \alpha, \beta)$ is monotonic and write

(5.6) \[ T_5 = \frac{2tn}{\pi \sqrt{t} - 1} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} \frac{1}{\alpha \beta} \sum_{j=1}^{f_{j+1}} G(u, \alpha, \beta) \cos \frac{2\pi H \sqrt{n} \nu(u)}{\nu'(u)} du \cdot \]
Now $G(u, \alpha, \beta)$ is continuous in each $\xi_j \leq u \leq \xi_{j+1}$. The only doubt arises, at $u = 0$ where $v'(u) = (1 - v^2)^{1/2} = 0$, and at $u = \infty$ where $v'(u) = 0$. But, using the definitions in (3.1) and evaluating an indeterminate form, we obtain

\begin{align*}
(5.7) \quad G(0^+, \alpha, \beta) & = -\frac{HA_{-1}(0)}{A_{-1}(0)} - \lim_{u \to 0^+} \left( \frac{1}{\sqrt{1 - v^2(u)}} + \frac{a_{-1}}{\sqrt{1 - v^2(u)}} \right) \\
& = -\frac{HA_{-1}(0)}{A_{-1}(0)} + O\left(\frac{\alpha^{i(\xi-1)} + \beta^{i(\xi-1)}}{\alpha^{i(\xi-1)} - \beta^{i(\xi-1)}}\right)
\end{align*}

which is bounded. On the other hand, by (4.3),

\begin{align*}
(5.8) \quad G(\infty, \alpha, \beta) & = -H(\alpha \beta)^{(\xi - 1)(\alpha^i + \beta^i)(1 - 2^i)/\xi} - a_{-1},
\end{align*}

which is also bounded.

Applying the second mean value theorem to (5.6) we obtain

\begin{align*}
(5.9) \quad T_5 = \frac{2tn}{\pi^2} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} \frac{1}{\alpha \beta} \sum_{j} G(\zeta_j', \alpha, \beta) \int_{\xi_j}^{\xi_{j+1}} \cos 2\pi H\sqrt{n}v(u) \cdot v'(u)du
\end{align*}

for appropriate $\zeta_j', \zeta_j, \zeta_{j+1}$ in the interval from $\xi_j$ to $\xi_{j+1}$. Further we have

\begin{align*}
(5.10) \quad T_5 & = \frac{2tn}{\pi^2} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} \frac{1}{\alpha \beta} \sum_{j} G(\zeta_j', \alpha, \beta) \frac{O(1)}{H\sqrt{n}} \\
& = \frac{2t\sqrt{n}}{\pi^2} \sum_{\alpha = 1}^{\infty} \sum_{\beta = 1}^{\infty} \frac{1}{\alpha \beta H} \sum_{j} G(\zeta_j', \alpha, \beta)O(1) \\
& = O(\sqrt{n})
\end{align*}

by the absolute convergence of the double series.

The relation (1.1) now follows from (3.1), (3.5), (5.2), (5.4), (5.5), and (5.10).

**References**

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