

Pacific Journal of Mathematics

ON THE NUMBER OF LATTICE POINTS IN $x^t + y^t = n^{t/2}$

ROY EDWIN WILD

ON THE NUMBER OF LATTICE POINTS IN $x^t + y^t = n^{t/2}$

R. E. WILD.

Introduction. Suppose that t is independent of n , $n > 1$; $t = (2M)/(2N + 1)$; $M = 1, 2, 3, \dots$; $N = 0, 1, 2, \dots$; $M \geq N + 1$, so that $t > 1$. Let $L_t(n^{t/2})$ be the number of lattice points, (x, y) , satisfying $x^t + y^t \leq n^{t/2}$. Our main objective is the proof of the relation

$$(1.1) \quad S(n) = t/2 \, n^{1-t/2} \int_0^n L_t(w^{t/2}) w^{t/2-1} dw$$

$$= c_1 n^2 - c_2 / \pi n^{(2t-1)/(2t)} \sum_{\alpha=1}^{\infty} \alpha^{-2-1/t} \cos(2\pi\sqrt{n} \alpha - \pi/(2t))$$

$$- \frac{2t}{\pi^2 \sqrt{t-1}} n^{3/4} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{\cos(2\pi H \sqrt{n} - \pi/4)}{(\alpha\beta)^{(t-2)/(2t-2)} H^{(3t-1)/(2t-2)}} + O(\sqrt{n})$$

with $t > 1$, $c_1 = \frac{2\Gamma^2(1/t)}{(t+2)\Gamma(2/t)}$, $c_2 = \frac{2^{(2t-1)/t} t^{1/t} \Gamma(1/t)}{\pi^{(t+1)/t}}$,

$H = (\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{(t-1)/t}$. The case $t = 2$ is known in connection with the classical problem of the lattice points in a circle [4, pp. 221, 235].

By choosing t as specified above the analysis is less bulky than it would be if we considered the slightly more general problem of $L_T(n^{T/2})$ corresponding to the curve $|x|^T + |y|^T = n^{T/2}$ with real $T > 0$. Expressions and estimates for $L_T(n^{T/2})$ have been obtained by Bachmann [1, pp. 447-450], Cauer [2], and van der Corput [3]. In particular van der Corput [3] found that

$$(1.2) \quad L_T(n^{T/2}) = c'_1 n - 8T^{(1-T)/T} n^{(T-1)/(2T)} \int_0^{\infty} g_1(\sqrt{n} - x) x^{(1-T)/T} dx$$

$$+ O(n^{1/3}), T > 3;$$

$$= c'_1 n - 8 \sum_{j=1}^{\infty} (-1)^{j+1} \binom{1/T}{j} \zeta(-jT) n^{(1-jT)/2}$$

$$+ O(n^{1/3}), 0 < T \leq 3, T \neq 1;$$

where

$$c'_1 = \frac{2\Gamma^2(1/T)}{T\Gamma(2/T)},$$

$g_1(x) = x - [x] - 1/2$, $[x]$ is the integral part of x , $\zeta(s)$ is the Riemann zeta function and $\binom{a}{b}$ is the binomial coefficient. From (1.2) it follows that

Received June 3, 1958. This paper is condensed from the author's Ph. D. thesis written under the guidance of Professor E. G. Straus at U. C. L. A.

$$(1.3) \quad L_T(n^{T/2}) = c'_1 n + O(n^{(T-1)/(2T)}), \quad L_T(n^{T/2}) = c'_1 n + \Omega(n^{(T-1)(2T)}), \quad T > 3.$$

These results in (1.3) and analogous results can be obtained from (1.1) also. Our methods fail to establish the analogue of (1.1) for $0 < t < 1$.

2. **First auxiliary result.** We first obtain the result

$$(2.1) \quad S(n) = n^2 \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=-\infty}^{\infty} \iint_{x^t + y^t \leq 1} (1 - x^t - y^t) \cos 2\pi\sqrt{n}(\alpha x + \beta y) dx dy, \quad t > 1.$$

In § 4 we prove that the double series is absolutely convergent.

We have [4, p. 205]

$$(2.2) \quad \int_0^W L_t(w) dw = \int_0^W \sum_{j^t + k^t \leq w} \sum_{j^t + k^t \leq w} |dw = \sum_{j^t + k^t \leq W} \sum_{j^t + k^t \leq W} \int_{j^t + k^t}^W dw \\ = \sum_{j^t + k^t \leq W} (W - j^t - k^t) = \sum_{-W^{1/t} \leq j \leq W^{1/t}} \sum_{-(W - j^t)^{1/t} \leq k \leq (W - j^t)^{1/t}} (W - j^t - k^t).$$

To this we apply the Poisson summation formula [4, p. 204] to obtain

$$(2.3) \quad \int_0^W L_t(w) dw = \sum_{\alpha=-\infty}^{\infty} \int_{-W^{1/t}}^{W^{1/t}} \cos 2\pi\alpha x \sum_{-(W - x^t)^{1/t} \leq k \leq (W - x^t)^{1/t}} (W - x^t - k^t) dx \\ = \sum_{\alpha=-\infty}^{\infty} \int_{-W^{1/t}}^{W^{1/t}} \cos 2\pi\alpha x \sum_{\beta=-\infty}^{\infty} \int_{-(W - x^t)^{1/t}}^{(W - x^t)^{1/t}} \cos 2\pi\beta y \cdot (W - x^t - y^t) dy dx.$$

Integrating by parts and applying the second mean value theorem for integrals, we have, for the inner integral,

$$\frac{t}{\pi\beta} \int_0^{(W - x^t)^{1/t}} \sin 2\pi\beta y \cdot y^{t-1} dy = \frac{t(W - x^t)^{(t-1)/t}}{\pi\beta} \int_{\xi}^{(W - x^t)^{1/t}} \sin 2\pi\beta y dy,$$

where $0 \leq \xi < (W - x^t)^{1/t}$, so that the sum over β is uniformly convergent in x . Hence we can interchange the order of operations in $\int dx \sum_{\beta}$ in (2.3) to obtain

$$(2.4) \quad \int_0^W L_t(w) dw = \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=-\infty}^{\infty} \iint_{x^t + y^t \leq W} \cos 2\pi\alpha x \cos 2\pi\beta y \cdot (W - x^t - y^t) dx dy.$$

By symmetry we can replace $\cos 2\pi\alpha x \cos 2\pi\beta y$ by $\cos 2\pi(\alpha x + \beta y)$. If also we set $w = z^{t/2}$, $x = W^{1/t}r$, $y = W^{1/t}s$, $W = n^{t/2}$, we reduce (2.4) to

$$(2.5) \quad t/2 \int_0^n L_t(z^{t/2}) z^{t/2-1} dz = n^{t/2+1} \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=-\infty}^{\infty} \iint_{r^t + s^t \leq 1} (1 - r^t - s^t) \cos 2\pi\sqrt{n}(\alpha r \beta s) dr ds$$

and then (2.1) follows upon multiplication of each side by $n^{(2-t)/t}$.

3. **Second auxiliary result.** For $t > 1$, we shall obtain from (2.1) the identity

$$(3.1) \quad S(n) = T_1 + T_2 + T_3 + T_4 + T_5$$

where

$$T_1 = c_1 n^2, \quad c_1 = \frac{2\Gamma^2(1/t)}{(t+2)\Gamma(2/t)};$$

$$T_2 = c_2 n^{5/4-1/(2t)} \sum_{\alpha=1}^{\infty} \alpha^{-3/2-1/t} J_{3/2+1/t}(2\pi\sqrt{n}\alpha),$$

where

$$c_2 = \frac{2^{(2t-1)/t} t^{1/t} \Gamma(1/t)}{\pi^{(t+1)/t}},$$

and $J_r(x)$ is the ordinary Bessel function of order r ;

$$T_3 = c_3 n^2 \sum_{\alpha=1}^{\infty} \int_0^1 f(x, t) \cos 2\pi\sqrt{n}\alpha x dx, \quad c_3 = \frac{16t}{t+1},$$

and $f(x, t) = (1 - x^t)^{(t+1)/t} - (t/2)^{(t+1)/t} (1 - x^2)^{(t+1)/t}$;

$$T_4 = -\frac{2t}{\pi\sqrt{t-1}} n \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{J_0(2\pi H\sqrt{n})}{(\alpha\beta)^{(t-2)/(2t-2)} H^{t/(t-1)}}, \quad H = (\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{(t-1)/t};$$

$$T_5 = \frac{2t}{\pi^2} n \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \int_{-\infty}^{\infty} G(u, \alpha, \beta) \cos 2\pi H\sqrt{n} v(u, \alpha, \beta) \cdot v'(u, \alpha, \beta) du,$$

where

$$v(u, \alpha, \beta) = H^{-1} A_0^{-1/t}(u), \quad A_i(u) = (-1)^i \alpha^{-t} (P\alpha - u)^{t-i} + \beta^{-t} (Q\beta + u)^{t-i},$$

$$P = \frac{\alpha^{1/(t-1)}}{\alpha^{t/(t-1)} + \beta^{t/(t-1)}}, \quad Q = \frac{\beta^{1/(t-1)}}{\alpha^{t/(t-1)} + \beta^{t/(t-1)}},$$

$$G(u, \alpha, \beta) = \frac{A_{-1}(u)A_1(u) - A_0^2(u)}{v'(u, \alpha, \beta)A_3^2(u)} - a_{-1}(\alpha, \beta) \operatorname{sgn} u [1 - v^2(u, \alpha, \beta)]^{-1/2},$$

$$a_{-1}(\alpha, \beta) = \frac{(\alpha\beta)^{t/(2t-2)}}{\sqrt{t-1}(\alpha^{t/(t-1)} + \beta^{t/(t-1)})}.$$

In the proof of (3.1) we make use of the following result on Bessel functions [5, p. 366],

$$(3.2) \quad \int_0^1 (1-x^2)^{m-1/2} \cos Kx dx = \sqrt{\pi} 2^{m-1} K^{-m} \Gamma(m+1/2) J_m(K) \quad m > -1/2.$$

First, it is convenient to break up the double sum in (2.1) as follows,

$$(3.3) \quad S(n) = \sum_{\alpha=0} \sum_{\beta=0} + \sum_{\substack{\alpha=-\infty \\ \alpha \neq 0}} \sum_{\beta=0} + \sum_{\alpha=0} \sum_{\substack{\beta=-\infty \\ \beta \neq 0}} \\ + \sum_{\alpha=1} \sum_{\beta=1} + \sum_{\alpha=-\infty}^{-1} \sum_{\beta=-\infty}^{-1} + \sum_{\alpha=-\infty}^{-1} \sum_{\beta=1}^{\infty} + \sum_{\alpha=1}^{\infty} \sum_{\beta=-\infty}^{-1} .$$

By symmetry this can be written as

$$(3.4) \quad S(n) = n^2 \iint_{x^t+y^t \le 1} (1-x^t-y^t) dx dy \\ + 4n^2 \sum_{\alpha=1}^{\infty} \iint_{x^t+y^t \le 1} (1-x^t-y^t) \cos 2\pi\sqrt{n}\alpha x dx dy \\ + 4n^2 \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \iint_{x^t+y^t \le 1} (1-x^t-y^t) \cos 2\pi\sqrt{n}(\alpha x + \beta y) dx dy \\ = S_1 + S_2 + S_3 .$$

S_1 can be evaluated in terms of gamma functions to obtain

$$(3.5) \quad S_1 = \frac{2\Gamma^2(1/t)}{(t+2)\Gamma(2/t)} n^2 = c_1 n^2 .$$

Let I_2 denote the integral in S_2 . Then

$$(3.6) \quad I_2 = 4 \int_0^1 \cos 2\pi\sqrt{n}\alpha x dx \int_0^{(1-x^t)^{1/t}} (1-x^t-y^t) dy \\ = \frac{4t}{t+1} \int_0^1 (1-x^t)^{(t+1)/t} \cos 2\pi\sqrt{n}\alpha x dx \\ = \frac{4t}{t+1} \left(\frac{t}{2}\right)^{(t+1)/t} \int_0^1 (1-x^2)^{(t+1)/t} \cos 2\pi\sqrt{n}\alpha x dx \\ + \frac{c_3}{4} \int_0^1 f(x, t) \cos 2\pi\sqrt{n}\alpha x dx$$

by the definition of $f(x, t)$ in (3.1). Applying (3.2) to (3.6) we have

$$(3.7) \quad S_2 = 4n^2 \sum_{\alpha=1}^{\infty} I_2 = T_2 + T_3 .$$

Let I_3 denote the integral in S_3 . Then by symmetry

$$(3.8) \quad I_3 = 2 \iint_{\substack{x^t+y^t \le 1 \\ \alpha x + \beta y \ge 0}} (1-x^t-y^t) \cos 2\pi\sqrt{n}(\alpha x - \beta y) dx dy .$$

The transformation

$$(3.9) \quad x = Hv(P - u/\alpha) , \quad y = Hv(Q + u/\beta)$$

transforms $x^t + y^t = 1$ into

$$(3.10) \quad v = H^{-1}A_0^{-1/t}(u)$$

where H, P, Q , and $A_i(u)$ are defined in (3.1). The transformation (3.9) is one to one for $\alpha x + \beta y \geq 0$ and the absolute value of the Jacobian is

$$(3.11) \quad J\left(\frac{x, y}{v, u}\right) = \frac{H^2 v}{\alpha \beta}.$$

The graph of (3.10) resembles that of $v = 1/(1 + u^2)$ except that the curve is not symmetric to the v axis unless $t = 2$. The curve has a relative maximum at $(0, 1)$.

Applying (3.9) to (3.8) we transform $x^t + y^t \leq 1$ and $\alpha x + \beta y \geq 0$ into $v \leq H^{-1}A_0^{-1/t}(u)$ and $v \geq 0$ respectively, so that (3.8) becomes

$$(3.12) \quad I_3 = \frac{2H^2}{\alpha\beta} \int_{-\infty}^{\infty} du \int_0^{v(u)} [1 - H^t v^t A_0(u)] v \cos 2\pi H\sqrt{n} v dv.$$

Upon integration by parts with respect to v , the integrated terms vanish and we obtain

$$(3.13) \quad \begin{aligned} I_3 &= - \frac{H}{\pi\sqrt{n}\alpha\beta} \int_{-\infty}^{\infty} du \int_0^{v(u)} [1 - (t + 1)H^t v^t A_0(u)] \sin 2\pi H\sqrt{n} v dv \\ &= - \frac{H}{\pi\sqrt{n}\alpha\beta} \int_0^1 \sin 2\pi H\sqrt{n} v dv \int_{u_-(v)}^{u_+(v)} [1 - (t + 1)H^t v^t A_0(u)] du \end{aligned}$$

where $u_+(v)$ and $u_-(v)$ refer to the first and second quadrant branches of (3.10) respectively. Since

$$(3.14) \quad \begin{aligned} A_i(u) &= (-1)^i \alpha^{-t} (P\alpha - u)^{t-i} + \beta^{-t} (Q\beta + u)^{t-i}, \\ \frac{d}{du} A_i(u) &= (t - i) A_{i+1}(u), \end{aligned}$$

we can write (3.13) as

$$(3.15) \quad \begin{aligned} I_3 &= - \frac{H}{\pi\sqrt{n}\alpha\beta} \int_0^1 [u_+(v) - H^t v^t A_{-1}(u_+(v))] \sin 2\pi H\sqrt{n} v dv \\ &\quad - \frac{H}{\pi\sqrt{n}\alpha\beta} \int_0^1 [-u_-(v) + H^t v^t A_{-1}(u_-(v))] \sin 2\pi H\sqrt{n} v dv. \end{aligned}$$

By the change of variable (3.10) this can be written as

$$(3.16) \quad I_3 = \frac{H}{\pi\sqrt{n}\alpha\beta} \int_{-\infty}^{\infty} \left[u - \frac{A_{-1}(u)}{A_0(u)} \right] \sin 2\pi H\sqrt{n} v(u) \cdot v'(u) du.$$

From (3.14) we obtain

$$(3.17) \quad u - \frac{A_{-1}(u)}{A_0(u)} = \frac{P\alpha^{1-t} - Q\beta^{1-t}}{\alpha^{-t} + \beta^{-t}} + O\left(\frac{1}{u}\right)$$

for large u , so that upon integrating by parts again we obtain

$$(3.18) \quad I_3 = \frac{t}{2\pi^2 n \alpha \beta} \int_{-\infty}^{\infty} F(u) \cos 2\pi H \sqrt{n} \bar{v}(u) \cdot v'(u) du$$

where

$$(3.19) \quad F(u) = F(u, \alpha, \beta) = \frac{A_{-1}(u)A_1(u) - A_0^2(u)}{v'(u)A_3^2(u)}.$$

The function $a_{-1} \operatorname{sgn} u [1 - v^2(u)]^{-1/2}$ is an asymptotic equivalent of $F(u)$ in the neighborhood of $(0, 1)$, even though $v(0) = 1$ and $v'(0) = 0$, if $a_{-1} = a_{-1}(\alpha, \beta)$ is determined from

$$(3.20) \quad \begin{aligned} a_{-1} &= \lim_{u \rightarrow 0+} F(u) \sqrt{1 - v^2(u)} = \lim_{u \rightarrow 0+} \frac{\sqrt{1 - v^2}}{-v'} \\ &= \lim_{u \rightarrow 0+} \frac{v v' (1 - v^2)^{-1/2}}{v''} = \frac{1}{v''(0)} \lim_{u \rightarrow 0+} \frac{v'}{\sqrt{1 - v^2}} \\ &= \frac{-1}{v''(0)a_{-1}} = \frac{1}{\sqrt{|v''(0)|}}. \end{aligned}$$

From (3.10) and (3.14) we obtain

$$(3.21) \quad v''(u) = -H^{-1} A_5^{-(1+2t)/t}(u) [-(t+1)A_1^2(u) + (t-1)A_0(u)A_2(u)]$$

from which a_{-1} , as given in (3.1), can be determined.

We now write (3.18) as

$$(3.22) \quad \begin{aligned} I_3 &= \frac{t a_{-1}}{2\pi^2 n \alpha \beta} \int_{-\infty}^{\infty} \operatorname{sgn} u [1 - v^2(u)]^{-1/2} \cos 2\pi H \sqrt{n} \bar{v}(u) \cdot v'(u) du \\ &+ \frac{t}{2\pi^2 n \alpha \beta} \int_{-\infty}^{\infty} [F(u) - a_{-1} \operatorname{sgn} u [1 - v^2(u)]^{-1/2}] \cos 2\pi H \sqrt{n} \bar{v}(u) \cdot v'(u) du \\ &= -\frac{t a_{-1}}{\pi^2 n \alpha \beta} \int_0^1 (1 - v^2)^{-1/2} \cos 2\pi H \sqrt{n} \bar{v} dv \\ &+ \frac{t}{2\pi^2 n \alpha \beta} \int_{-\infty}^{\infty} G(u) \cos 2\pi H \sqrt{n} \bar{v}(u) \cdot v'(u) du \end{aligned}$$

where $G(u) = G(u, \alpha, \beta)$ is defined in (3.1). Applying (3.2) to (3.22) we obtain

$$(3.23) \quad S_3 = 4n^2 \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} I_3 = T_4 + T_5.$$

Collecting the results of (3.4), (3.5), (3.7), and (3.23), we have (3.1).

4. Convergence investigations. We next prove that the double series in (2.1) is absolutely convergent. We write (3.18) as

$$(4.1) \quad \begin{aligned} I_3 &= \frac{t}{2\pi^2 n \alpha \beta} \left(\int_{-\infty}^0 + \int_0^\sigma + \int_\sigma^{P\alpha} + \int_{P\alpha}^\infty \right) \\ &= \frac{t}{2\pi^2 n \alpha \beta} (I_4 + I_5 + I_6 + I_7) \end{aligned}$$

where $0 < \sigma < P\alpha$.

First we consider

$$(4.2) \quad I_7 = \int_{P\alpha}^\infty F(u) \cos 2\pi H \sqrt{n} v(u) \cdot v'(u) du .$$

By (3.14) and (3.19) we have

$$(4.3) \quad F(u) = \frac{H(P\alpha - u)^{t-1} (Q\beta + u)^{t-1}}{(\alpha\beta)^t A_3^{-1/t}(u) A_1(u)} = \frac{-H[\alpha^{-t}(u - P\alpha)^t + \beta^{-t}(u + Q\beta)^t]^{(1-t)/t}}{\alpha^t(u - P\alpha)^{1-t} + \beta^t(u + Q\beta)^{1-t}}$$

From (4.3) we find that

$$(4.4) \quad \begin{aligned} \frac{dF(u)}{du} &= \frac{(1-t)H}{(\alpha\beta)^t} \left(\frac{(u - P\alpha)^t}{\alpha^t} + \frac{(u + Q\beta)^t}{\beta^t} \right)^{(1-2t)/t} \\ &\quad \times \left(\frac{-\beta^{2t}(u - P\alpha)^{2t-1} + \alpha^{2t}(u + Q\beta)^{2t-1}}{(u - P\alpha)^t(u + Q\beta)^t[\alpha^t(u - P\alpha)^{1-t} + \beta^t(u + Q\beta)^{1-t}]^2} \right) . \end{aligned}$$

From (4.3) and (4.4) we derive certain information about the graph of $F(u)$, namely,

$$(4.5) \quad \begin{aligned} F(u) &> 0, F'(u) < 0, 0 < u < P\alpha ; \\ F'(P\alpha) &= \infty, 1 < t < 2 ; F'(P\alpha) = 0, 2 < t ; \\ F(u) &< 0, P\alpha < u < \infty ; \\ F'(u) &= 0, u = u_1, P\alpha < u_1 < \infty, \beta > \alpha ; \\ F'(u) &< 0, P\alpha < u < \infty, \beta \leq \alpha . \end{aligned}$$

The point (u_1, v_1) is a relative minimum and from (4.3) and (4.4) we find that

$$(4.6) \quad \begin{aligned} u_1 &= \frac{Q\beta\alpha^{(2t)/(2t-1)} + P\alpha\beta^{(2t)/(2t-1)}}{\beta^{(2t)/(2t-1)} - \alpha^{(2t)/(2t-1)}} , \\ v_1 &= F(u_1) = -H(\alpha^{t/(2t-1)} + \beta^{t/(2t-1)})^{-(2t-1)/t} . \end{aligned}$$

Thus by (4.5) and the second mean value theorem for integrals we have, for $\beta > \alpha$, and $P\alpha \leq \xi_1 < u_1 < \xi_2 \leq \infty$,

$$(4.7) \quad I_7 = \int_{P\alpha}^{u_1} + \int_{u_1}^\infty = F(u_1) \int_{\xi_1}^{u_1} + F(u_1) \int_{u_1}^{\xi_2}$$

$$\begin{aligned}
 &= F(u_1) \int_{\xi_1}^{\xi_2} \cos 2\pi H\sqrt{n} \bar{v}(u) \cdot v'(u) du \\
 &= O\{F(u_1)H^{-1}n^{-1/2}\} = O\{(\alpha^{t/(2t-1)} + \beta^{t/(2t-1)})^{-(2t-1)/t}n^{-1/2}\} \\
 &= O\{(n\alpha\beta)^{-1/2}\}
 \end{aligned}$$

by the inequality $x^2 + y^2 \geq 2xy$, $x > 0, y > 0$. Similarly, for $\beta \leq \alpha$, and $P\alpha \leq \xi_3 < \infty$, we have

$$\begin{aligned}
 (4.8) \quad I_7 &= \int_{P\alpha}^{\infty} = F(\infty) \int_{\xi_3}^{\infty} \cos 2\pi H\sqrt{n} \bar{v}(u) \cdot v'(u) du \\
 &= O\{F(\infty)H^{-1}n^{-1/2}\} = \{(\alpha^{-t} + \beta^{-t})^{(1-t)/t}(\alpha^t + \beta^t)^{-1}n^{-1/2}\} \\
 &= O\{(\alpha\beta)^{t-1}(\alpha^t + \beta^t)^{-(2t-1)/t}n^{-1/2}\} = O\{(n\alpha\beta)^{-1/2}\}.
 \end{aligned}$$

We next consider I_5 in (4.1). By (4.3) we can write

$$(4.9) \quad I_5 = \int_0^\sigma F_1(u) \cos 2\pi H\sqrt{n} \bar{v}(u) du$$

where

$$\begin{aligned}
 (4.10) \quad -F_1(u) &= \frac{(pq)^{t-1}}{(\alpha\beta)^t A_0^2(u)}, \quad p = P\alpha - u, \quad q = Q\beta + u, \\
 &= \frac{A_0^{-2/t}(u)}{\alpha\beta} \cdot \frac{[(pq)/(\alpha\beta)]^{t-1}}{[(p/\alpha)^t + (q/\beta)^t]^{2(t-1)/t}} < \frac{A_0^{-2/t}(u)}{\alpha\beta} \cdot \frac{1}{2^{2(t-1)/t}} \\
 &< \frac{A_0^{-2/t}(0)}{\alpha\beta 2^{2(t-1)/t}} = O\left(\frac{H^2}{\alpha\beta}\right).
 \end{aligned}$$

Therefore

$$(4.11) \quad I_5 = O\left(\frac{H^2}{\alpha\beta} \int_0^\sigma du\right) = O\left(\frac{H^2\sigma}{\alpha\beta}\right).$$

Turning next to I_6 in (4.1), we note that by the first line of (4.5) we can use the second mean value theorem to write, for some ξ_4 satisfying $\sigma < \xi_4 \leq P\alpha$,

$$(4.12) \quad I_6 = F(\sigma) \int_\sigma^{\xi_4} \cos 2\pi H\sqrt{n} \bar{v}(u) \cdot v'(u) du = O\left(\frac{F(\sigma)}{H\sqrt{n}}\right).$$

To examine the question of the order of $F(u)$ in $0 < u < P\alpha$ we use (4.3) with, $p = P\alpha - u, q = Q\beta + u$, and write

$$\begin{aligned}
 (4.13) \quad F(u) &= \frac{H}{(\alpha\beta)^{1/2}} \cdot \frac{[(pq)/(\alpha\beta)]^{(t-1)/2}}{[(p/\alpha)^t + (q/\beta)^t]^{(t-1)/t}} \\
 &\quad \cdot \frac{1}{(\beta/\alpha)^{t/2}(p/q)^{(t-1)/2} + (\alpha/\beta)^{t/2}(q/p)^{(t-1)/2}} \\
 &\leq \frac{H}{(\alpha\beta)^{1/2}} \cdot \frac{1}{2^{(t-1)/2}} \cdot \frac{1}{F_2(u)}.
 \end{aligned}$$

Since $F_2(0) = 0$ and

$$(4.14) \quad F_3(u) = \frac{dF_2(u)}{du} = \frac{t-1}{2} \left[\left(\frac{\beta}{\alpha} \right)^{t/2} \left(\frac{p}{q} \right)^{(t-3)/2} \frac{1}{q^2} + \left(\frac{\alpha}{\beta} \right)^{t/2} \left(\frac{q}{p} \right)^{(t-3)/2} \frac{1}{p^2} \right],$$

we have, by the mean value theorem,

$$(4.15) \quad F(u) \leq \frac{H\lambda}{(\alpha\beta)^{1/2}} \cdot \frac{(t-1)/2}{F_3(u_3)u}, \quad \lambda = \frac{2^{(5-t)/2}}{t-1}, \quad 0 < u < P\alpha, \quad 0 < u_3 < P\alpha.$$

Setting $p_3 = P\alpha - u_3, q_3 = Q\beta + u_3$, we obtain

$$(4.16) \quad \begin{aligned} F(u) &\leq \frac{H\lambda}{(\alpha\beta)^{1/2}} \cdot \frac{p_3q_3}{[(\beta/\alpha)^{t/2}(p_3/q_3)^{(t-1)/2} + (\alpha/\beta)^{t/2}(q_3/p_3)^{(t-1)/2}]u} \\ &\leq \frac{H\lambda}{(\alpha\beta)^{1/2}} \cdot \frac{p_3q_3}{[(\beta/\alpha)^{t/2} + (\alpha/\beta)^{t/2}]u} \leq \frac{H\lambda}{(\alpha\beta)^{1/2}} \cdot \frac{4}{[(\beta/\alpha)^{t/2} + (\alpha/\beta)^{t/2}]u} \\ &= O\left\{ \frac{H(\alpha\beta)^{(t-1)/2}}{(\alpha^t + \beta^t)u} \right\}. \end{aligned}$$

Hence combining (4.11), (4.12), and (4.16), we obtain

$$(4.17) \quad I_5 + I_6 = O\left(\frac{H^2\sigma}{\alpha\beta}\right) + O\left(\frac{(\alpha\beta)^{(t-1)/2}}{(\alpha^t + \beta^t)\sigma n^{1/2}}\right) = O\left(\frac{H(\alpha\beta)^{(t-3)/4}n^{-1/4}}{(\alpha^t + \beta^t)^{1/2}}\right).$$

In the further analysis of $I_5 + I_6$ we use the inequalities,

$$(4.18) \quad 1 + x^m < (1 + x)^m, \quad 0 < x < 1, \quad m > 1,$$

$$(4.19) \quad (x + 1)^m < 2^{m-1}(x^m + 1), \quad x > 1, \quad m > 1.$$

In (4.17) suppose $1 < t \leq 2$. Since $H = (\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{(t-1)/t}$ and $t/(t-1) > t$, we have by (4.18), $H < (\alpha^t + \beta^t)^{1/t}$, and therefore, for $1 < t \leq 2$, we have

$$(4.20) \quad \frac{H(\alpha\beta)^{(t-3)/4}}{(\alpha^t + \beta^t)^{1/2}} < \frac{(\alpha^t + \beta^t)^{(2-t)/(2t)}}{(\alpha\beta)^{(3-t)/4}} < \frac{(\alpha + \beta)^{(2-t)/2}}{(\alpha\beta)^{(t-1)/4}(\alpha\beta)^{(2-t)/2}} < \frac{2^{(2-t)/2}}{(\alpha\beta)^{(t-1)/4}}.$$

Hence from (4.17) and (4.20) we have, for $1 < t \leq 2$,

$$(4.21) \quad I_5 + I_6 = O\left\{ (\alpha\beta)^{-(t-1)/4} n^{-1/4} \right\}.$$

If $t > 2$ is (4.17), then $t > t/(t-1)$ and so by (4.19) we have

$$(4.22) \quad \begin{aligned} \frac{H(\alpha\beta)^{(t-3)/4}}{(\alpha^t + \beta^t)^{1/2}} &= \frac{(\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{(t-1)/t}}{(\alpha^t + \beta^t)^{1/t}} \cdot \frac{(\alpha\beta)^{(t-3)/4}}{(\alpha^t + \beta^t)^{(t-2)/(2t)}} \\ &< 2^{(t-2)/t} \cdot \frac{(\alpha\beta)^{(t-3)/4}}{(\alpha^t + \beta^t)^{(t-2)/(2t)}} \\ &< \frac{2^{(t-2)/t}(\alpha\beta)^{(t-3)/4}}{2^{(t-2)/(2t)}(\alpha\beta)^{(t-2)/4}} = \frac{2^{(t-2)/(2t)}}{(\alpha\beta)^{1/t}}. \end{aligned}$$

Hence from (4.17) and (4.22) we have, for $t > 2$,

$$(4.23) \quad I_5 + I_6 = O\{(n\alpha\beta)^{-1/4}\} .$$

By (3.10) $v(-u, \alpha, \beta) = v(u, \alpha, \beta)$ so that an estimate for $I_5 + I_6 + I_7$ holds also for I_4 in (4.1). By this fact, and the results of (4.7), (4.8), (4.21), and (4.23), it now follows that for S_3 , defined by (3.4), (3.23), and (4.1), we have,

$$(4.24) \quad S_3 = \frac{2tn}{\pi^2} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \int_{-\infty}^{\infty} F(u) \cos 2\pi H\sqrt{n}v(u) \cdot v'(u)du \\ = O(n^{3/4}), t > 1 ,$$

the double series being absolutely convergent.

Integrating by parts and applying the second mean value theorem, we have, from (3.6), for $x_1 = [(t - 1)/t]^{1/t}$,

$$(4.25) \quad S_2 = \frac{16t}{t + 1} n^2 \sum_{\alpha=1}^{\infty} \int_0^1 (1 - x^t)^{(t+1)/t} \cos 2\pi\sqrt{n} \alpha x dx \\ = \frac{8t}{\pi} n^{3/2} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \int_0^1 (1 - x^t)^{1/t} x^{t-1} \sin 2\pi\sqrt{n} \alpha x dx \\ = \frac{8t}{\pi} n^{3/2} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \left\{ \int_0^{x_1} + \int_{x_1}^1 \right\} \\ = \frac{8t}{\pi} n^{3/2} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \left\{ (1 - x_1^t)^{1/t} x_1^{t-1} \int_{\xi_5}^{x_1} \sin 2\pi\sqrt{n} \alpha x dx \right. \\ \left. + (1 - x_1^t)^{1/t} x_1^{t-1} \int_{x_1}^{\xi_5} \sin 2\pi\sqrt{n} \alpha x dx \right\} \\ = \frac{8t}{\pi} n^{3/2} \sum_{\alpha=1}^{\infty} O\left(\frac{1}{\sqrt{n} \alpha^2}\right) = O(n), t > 1 ,$$

the series being absolutely convergent. The absolute convergence of the double series in (2.1) now follows from the results leading to (4.24) and (4.25).

5. Proof of (1.1). Finally we deduce (1.1) from (3.1). We make use of the asymptotic expansion for the general Bessel function, namely [5, p. 368],

$$(5.1) \quad J_m(K) = \sqrt{\frac{2}{\pi K}} \cos \left(K - \frac{m\pi}{2} - \frac{\pi}{4} \right) + O(K^{-3/2}) ,$$

for large K and m independent of K .

By (5.1) and the absolute convergence of the sum we have

$$(5.2) \quad T_2 = c_2 n^{5/4-1/(2t)} \sum_{\alpha=1}^{\infty} \alpha^{-3/2-1/t} \left\{ \frac{\cos [2\pi\sqrt{n}\alpha - \pi(1 + 1/(2t))]}{(\pi^2\sqrt{n}\alpha)^{1/2}} + O(n^{-3/4}\alpha^{-3/2}) \right\} \\ = - \frac{c_2}{\pi} n^{1-1/(2t)} \sum_{\alpha=1}^{\infty} \alpha^{-2-1/t} \cos (2\pi\sqrt{n}\alpha - \pi/(2t)) + O(n^{1/2-1/(2t)}) .$$

In $T_3 f'(0, t) = 0$ and $f^{(k)}(1, t) = 0, k = 0, 1, 2$. Hence if we integrate by parts twice the integrated terms vanish and we have left

$$(5.3) \quad T_3 = - \frac{4t}{\pi^2(t+1)} n \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^2} \int_0^1 f''(x, t) \cos 2\pi\sqrt{n}\alpha x dx .$$

$f''(x, t)$ is continuous in $0 \leq x \leq 1$ and independent of n and α and so it has a finite number, independent of n and α , of relative and absolute extrema whose values are also independent of n and α . Hence dividing the interval of integration into pieces in which $f''(x, t)$ is monotonic, we obtain by the second mean value theorem, for appropriate $\xi_j, \xi'_j, \xi'_{j+1}$ in the interval from 0 to 1, the ξ 's depending on n and α , the result,

$$(5.4) \quad T_3 = - \frac{4t}{\pi^2(t+1)} n^{3/4} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^2} \sum_j f''(\xi_j, t) \int_{\xi'_j}^{\xi'_{j+1}} \cos 2\pi\sqrt{n}\alpha x dx = O(\sqrt{n}) .$$

Applying (5.1) to T_4 we obtain

$$T_4 = - \frac{2t}{\pi^2\sqrt{t-1}} n^{3/4} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{\cos (2\pi H\sqrt{n} - \pi/4)}{(\alpha\beta)^{(t-2)/(2t-2)} H^{(3t-1)/(2t-2)}} \\ - \frac{2tn}{\pi\sqrt{t-1}} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{O\{(\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{-(3t-3)/(2t)} n^{-3/4}\}}{(\alpha\beta)^{(t-1)/(2t-2)} (\alpha^{t/(t-1)} + \beta^{t/(t-1)})} .$$

Since

$$(\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{-(5t-3)/(2t)} \leq 2^{-(5t-3)/(2t)} (\alpha\beta)^{-(5t-3)/(4t-4)} ,$$

the double series are absolutely convergent so that

$$(5.5) \quad T_4 = - \frac{2t}{\pi^2\sqrt{t-1}} n^{3/4} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{\cos (2\pi H\sqrt{n} - \pi/4)}{(\alpha\beta)^{(t-2)/(2t-2)} H^{(3t-1)/(2t-2)}} + O(n^{1/4}) .$$

Next we consider T_5 . We have shown that $-T_4$ and S_3 are absolutely convergent double series for $t > 1$ and hence so is their term by term sum which is identical with T_5 . We break up the interval of integration in T_5 into a finite number, independent of n, α, β , of subintervals in which $G(u, \alpha, \beta)$ is monotonic and write

$$(5.6) \quad T_5 = \frac{2tn}{\pi^2} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \sum_j \int_{\xi_j}^{\xi_{j+1}} G(u, \alpha, \beta) \cos 2\pi H\sqrt{n}v(u) \cdot v'(u) du .$$

Now $G(u, \alpha, \beta)$ is continuous in each $\xi_j \leq u \leq \xi_{j+1}$. The only doubt arises, at $u = 0$ where $v'(u) = (1 - v^2)^{1/2} = 0$, and at $u = \infty$ where $v'(u) = 0$. But, using the definitions in (3.1) and evaluating an indeterminate form, we obtain

$$(5.7) \quad G(0 + , \alpha, \beta) = \frac{-HA_{-1}(0)}{A_0^{1-1/t}(0)} - \lim_{u \rightarrow 0+} \left(\frac{1}{v'(u)} + \frac{a_{-1}}{\sqrt{1 - v^2(u)}} \right) \\ = \frac{-HA_{-1}(0)}{A_0^{1-1/t}(0)} + O\left(\frac{\alpha^{t/(t-1)} - \beta^{t/(t-1)}}{\alpha^{t/(t-1)} + \beta^{t/(t-1)}}\right)$$

which is bounded. On the other hand, by (4.3),

$$(5.8) \quad G(\infty, \alpha, \beta) = -H(\alpha\beta)^{t-1}(\alpha^t + \beta^t)^{(1-2t)/t} - a_{-1},$$

which is also bounded.

Applying the second mean value theorem to (5.6) we obtain

$$(5.9) \quad T_5 = \frac{2tn}{\pi^2} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \sum_j G(\zeta'_j, \alpha, \beta) \int_{\xi_j}^{\xi_{j+1}} \cos 2\pi H\sqrt{n}v(u) \cdot v'(u)du$$

for appropriate $\zeta'_j, \zeta_j, \zeta_{j+1}$ in the interval from ξ_j to ξ_{j+1} . Further we have

$$(5.10) \quad T_5 = \frac{2tn}{\pi^2} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \sum_j G(\zeta'_j, \alpha, \beta) \frac{O(1)}{H\sqrt{n}} \\ = \frac{2t\sqrt{n}}{\pi^2} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta H} \sum_j G(\zeta'_j, \alpha, \beta) O(1) \\ = O(\sqrt{n})$$

by the absolute convergence of the double series.

The relation (1.1) now follows from (3.1), (3.5), (5.2), (5.4), (5.5), and (5.10).

REFERENCES

1. P. Bachmann, *Zahlentheorie*, **2**, Teubner, (1894).
2. D. Cauer, *Neue Anwendungen der Pfeifferschen Methode zur Abschätzung zahlentheoretischer Funktionen*, Diss. Göttingen, **55** s. 8° (1914).
3. J. G. van der Corput, *Over roosterpunten in het platte vlak*, Diss. Leiden, Groningen: Noordhoff, **128** s. 8° (1919).
4. E. Landau, *Vorlesungen über Zahlentheorie*, **2**, Chelsea, (1947).
5. Whittaker and Watson, *Modern Analysis*, Macmillan, (1946).

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG
Stanford University
Stanford, California

R. A. BEAUMONT
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

E. G. STRAUS
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
C. E. BURGESS
M. HALL
E. HEWITT

A. HORN
V. GANAPATHY IYER
R. D. JAMES
M. S. KNEBELMAN

L. NACHBIN
I. NIVEN
T. G. OSTROM
H. L. ROYDEN

M. M. SCHIFFER
G. SZEKERES
F. WOLF
K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
THE RAMO-WOOLDRIDGE CORPORATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 8, No. 4

June, 1958

Richard Arens, <i>The maximal ideals of certain functions algebras</i>	641
Glen Earl Baxter, <i>An operator identity</i>	649
Robert James Blattner, <i>Automorphic group representations</i>	665
Steve Jerome Bryant, <i>Isomorphism order for Abelian groups</i>	679
Charles W. Curtis, <i>Modules whose annihilators are direct summands</i>	685
Wilbur Eugene Deskins, <i>On the radical of a group algebra</i>	693
Jacob Feldman, <i>Equivalence and perpendicularity of Gaussian processes</i>	699
Marion K. Fort, Jr. and G. A. Hedlund, <i>Minimal coverings of pairs by triples</i>	709
I. S. Gál, <i>On the theory of (m, n)-compact topological spaces</i>	721
David Gale and Oliver Gross, <i>A note on polynomial and separable games</i>	735
Frank Harary, <i>On the number of bi-colored graphs</i>	743
Bruno Harris, <i>Centralizers in Jordan algebras</i>	757
Martin Jurchescu, <i>Modulus of a boundary component</i>	791
Hewitt Kenyon and A. P. Morse, <i>Runs</i>	811
Burnett C. Meyer and H. D. Sprinkle, <i>Two nonseparable complete metric spaces defined on $[0, 1]$</i>	825
M. S. Robertson, <i>Cesàro partial sums of harmonic series expansions</i>	829
John L. Selfridge and Ernst Gabor Straus, <i>On the determination of numbers by their sums of a fixed order</i>	847
Annette Sinclair, <i>A general solution for a class of approximation problems</i>	857
George Szekeres and Amnon Jakimovski, <i>(C, ∞) and (H, ∞) methods of summation</i>	867
Hale Trotter, <i>Approximation of semi-groups of operators</i>	887
L. E. Ward, <i>A fixed point theorem for multi-valued functions</i>	921
Roy Edwin Wild, <i>On the number of lattice points in $x^t + y^t = n^{t/2}$</i>	929