Pacific Journal of Mathematics

LINEAR OPERATORS AND THEIR CONJUGATES

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Vol. 9, No. 1 May 1959

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Introduction. In a paper of Taylor and Halberg ([3]), a complete systematic account of the theorems about the range and inverse of a bounded linear operator T and its conjugate T' was presented. For example, questions concerning T and the corresponding questions concerning T' such as the following were answered:

Does $Tx = y$ have a solution x for each given y? If not, for which y's does a solution exist?

Does the operator T have an inverse T^{-1}, and if so, is T^{-1} bounded?

These matters were considered for a bounded linear operator T defined on all of a normed linear space X with values in a second normed linear space Y.

The purpose of this paper is to investigate the same questions for T and T', where now T is defined on a linear manifold $\mathscr{D}$ dense in X, and moreover, T need not be bounded. It is shown that most of the theorems are still valid under these weakened hypotheses. Examples are constructed to show which theorems no longer hold.

Next, by imposing the condition that T be a closed linear operator on $\mathscr{D}$, we show that we obtain the same results as for the case that T be bounded on all of X.

1. The conjugate transformation. Throughout this paper we shall use X and Y to denote normed linear spaces over the real or complex scalar field. The space of all continuous linear functionals on X will be written as X'.

The following theorem is well known.

THEOREM 1.1. *Let Y be complete. If T is a bounded linear transformation on $\mathscr{D} \subset X$ to Y with norm $\|T\|$, then T has a unique extention $\hat{T}$ on $\overline{\mathscr{D}}$ and $\|\hat{T}\| = \|T\|$.*

DEFINITION 1. Let T be a linear operator (not necessarily bounded) with domain $\mathscr{D}$ dense in X and range $\mathscr{R} \subset Y$. The *conjugate transformation* T' is defined as follows: Its domain $\mathscr{D}(T')$ consists of the sets of all $y \in Y'$ for which $y'T$ is continuous on $\mathscr{D}$; for such a y' we define $T'y' = x'$ where x' is the bounded linear extension of $y'T$ to X.

THEOREM 1.1 assures the existense of such an x' which is unique. Thus T' is well defined. It is easy to see that $\mathscr{D}(T')$ is a linear manifold and that T' is a closed linear operator. We refer to T' as the

Received June 17, 1958. The author gratefully acknowledges the advice and counsel of Professor Angus E. Taylor in the preparation of this paper.

conjugate of T.

Unless otherwise indicated. T and T' will be as in Definition 1.

DEFINITION 2. A set F contained in the space of all linear functionals on X is called *total* if $x'x = 0$ for all $x' \in F$ implies $x = 0$.

The following theorem is due to Phillips [2, Theorem 2.11.9, p. 43].

THEOREM 1.2. *If T is closed, then $\mathscr{D}(T')$ is total.*

REMARKS. The converse of this theorem need not hold. For let $\mathscr{D}$ be such that $\overline{\mathscr{D}} = X$ but $\mathscr{D} \neq X = Y$, and let T be tne identity operator on $\mathscr{D}$. However, we easily prove the following.

THEOREM 1.3. *If $\mathscr{D}(T) = X$ and $\mathscr{D}(T')$ is total, then T is closed.*

Proof. Let $\lim x_n = x$ and $\lim Tx_n = y$. All we need show is that $y = Tx$. If this were not the case, there would exist a $y' \in \mathscr{D}(T')$ such that $y'(y - Tx) \neq 0$. Since $y'T$ is continuous on X, we have that

$$y'y = \lim y'Tx_n = y'Tx$$

which is a contradiction.

2. The state of a linear operator and its conjugate. To discuss the range of linear operator T, we consider the following three possibilities, where $\mathscr{R}(T)$ will denote the range of T.

I. $\mathscr{R}(T) = Y$,

II. $\mathscr{R}(T) \neq Y$, but $\mathscr{R}(T)$ is dense in Y,

III. $\mathscr{R}(T)$ is not dense in Y, that is $\overline{\mathscr{R}(T)} \neq Y$.

If T has an inverse, then the inverse mapping T^{-1} is a linear operator from the normed linear space $\mathscr{R}(T)$ into the normed linear space X. As regards the inverse of T, we consider the following three possibilities:

1. T has a bounded inverse,
2. T has an unbounded inverse,
3. T has no inverse.

By the various pairings of I, II, or III with 1, 2, or 3, nine conditions can thus be described relating to $\mathscr{R}(T)$ and T^{-1}. For instance, it may be that $\mathscr{R}(T) = Y$, and that T has a bounded inverse. This we will describe by saying that T is in state I_1, (written $T \in \mathrm{I}_1$).

Since T' is a linear operator from $\mathscr{D}(T')$ into X', we can use the above classifications for $\mathscr{R}(T')$ and the inverse of T'. To the ordered pair of operators (T, T') we now make correspond an ordered pair of conditions which we call the "state" of (T, T'). Thus if $T \in \mathrm{I}_3$ and $T' \in \mathrm{III}_1$, we say that (T, T') is in state $(\mathrm{I}_3, \mathrm{III}_1)$ (written $(T, T') \in (\mathrm{I},_3 \mathrm{III}_1)$).

At times we shall use a notation such as $(T, T') \in (\mathrm{I}_2, 3)$ to mean that $T \in \mathrm{I}_2$ and T' has no inverse.

The question arises as to whether (T, T') can be in each of the 81 states. It will be shown that only 16 states can occur if no additional assumptions are made about X, Y or T. However, if we require that X be reflexive, Y complete and T closed, the number of actually possible cases drops to 7.

We shall now exhibit several theorems which will enable us to determine which states can or cannot occur for the pair (T, T').

THEOREM 2.1. *If T' has a continuous inverse, then $\mathscr{R}(T')$ is closed.* (T' cannot be in II_1).

Proof. Suppose there exists a sequence $\{y_n'\}$ from $\mathscr{D}(T')$ with $T'y_n' \to x'$. The sequence $\{y_n'\}$ is a Cauchy sequence since $||y_n' - y_m'|| \leqq M||T'y_n' - T'y_m'||$ where M is the norm of $(T')^{-1}$ as an operator on $\mathscr{R}(T')$. But Y' is complete, therefore there exists a $y' \in Y'$ such that $\lim y_n' = y'$. Hence $y' \in \mathscr{D}(T')$ and $T'y' = x'$ since T' is closed.

Theorems 2.2 through 2.5 are due to Phillips [2 pp. 44–45].

THEOREM 2.2. *A necessary and sufficient condition that $\overline{\mathscr{R}(T)} = Y$ is that T' have an inverse.*

THEOREM 2.3. *If $\mathscr{R}(T')$ is w^* dense in X', then T has an inverse.*

THEOREM 2.4. *If $\overline{\mathscr{R}(T)} = Y$ and T^{-1} exists, then $(T^{-1})' = (T')^{-1}$; furthermore, T has a bounded inverse if and only if T' has a bounded inverse defined on all of X'.*

THEOREM 2.5. *$\mathscr{R}(T') = X'$ if and only if T has a bounded inverse.*

The following theorem will show that three more states for (T, T') cannot exist if we require that Y be complete.

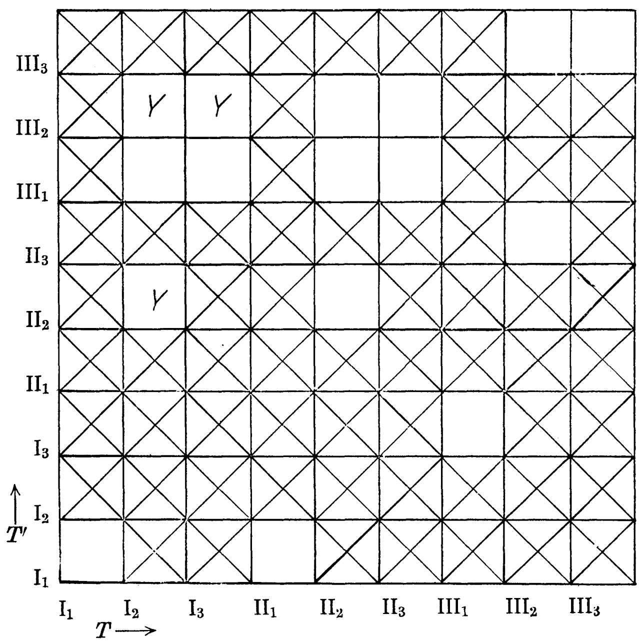
THEOREM 2.6. *If Y is complete and $\mathscr{R}(T) = Y$, then T' has a continuous inverse.* (States (I, 2) and (I, 3) cannot exist if Y is complete).

Proof. If T' did not have a continuous inverse, there would exist a sequence y_n' in Y' such that $||y_n'|| \to \infty$ and $||T'y_n'|| \to 0$. Since $\mathscr{R}(T) = Y$, it follows that $||y_n'y|| \to 0$ for each $y \in Y$. Hence we can conclude that the sequence $\{||y_n'||\}$ is bounded, by a theorem due to Banach [1 p. 80, Theorem 5]. We have thus reached a contradiction.

3. **The state diagram of pairs (T, T').** In order to present systematically which states can or cannot occur for pairs (T, T'), it will be

convenient to construct a "state diagram" conceived by Taylor [3 p. 100]. This diagram is a large square divided into 81 congruent smaller squares arranged in rows and columns. We label each column at the bottom denoting a given state for T, and each row by a "state" symbol placed at the left, denoting a certain state for T'. The square which is the intersection of a certain column and row will thus denote the state of the pair $(T, T\,)$. A square is crossed out by its diagonals if the corresponding state is impossible without requiring X or Y to be complete. As regards the remaining squares, we place the letter Y in a square to indicate that the state cannot occur if Y is complete.

First State Diagram



Y: Cannot occur if Y is complete

4. Example of states which can occur. Excluding (I_2, III_1), all of the examples in this section will be taken in the space ℓ^2 with $X = Y = \ell^2$. The sequence space ℓ is defined to consist of all sequences $\{\xi_n\} = x$ such that $\sum_1^\infty |\xi_n|^2 < \infty$. The norm in ℓ^2 is defined by

$$\|x\| = \Big(\sum_1^\infty |\xi_n|^2\Big)^{1/2}.$$

It is well known that the conjugate space $(\ell^2)'$ of ℓ^2 is congruent to ℓ^2, whence ℓ^2 is reflexive. In fact, every element in $(\ell^2)'$ is representable in one and only one way in the form

$$(1) \qquad x'x = \sum_1^\infty \alpha_n \xi_n$$

where the sequence $a = \{\alpha_n\}$ is an element of ℓ^2. The correspondence between x' and a is a congruence between $(\ell^2)'$ and ℓ^2. We shall write $x' = a$.

The set of vectors u_k, where $u_1 = (1, 0, \cdots)$, $u_2 = (0, 1, \cdots)$, etc. will frequently be used.

As the domain $\mathscr{D}$ of each linear operator T in the examples to follow, excluding $(\mathrm{I}_2, \mathrm{III}_1)$, we take the linear combinations of the u_k. Clearly $\mathscr{D}$ is a subspace dense in $X = \ell^2$.

Taylor and Halberg [3 pp. 102–104] have shown that the seven states $(\mathrm{I}_1, \mathrm{I}_1)$, $(\mathrm{I}_3, \mathrm{III}_1)$, $(\mathrm{II}_2, \mathrm{II}_2)$, $(\mathrm{II}_3, \mathrm{III}_2)$, $(\mathrm{III}_1, \mathrm{I}_3)$, $(\mathrm{III}_2, \mathrm{II}_3)$, $(\mathrm{III}_3, \mathrm{III}_3)$ are all possible even when $Y = X = \mathscr{D} = \ell^2$ and T is continuous.

We shall now demonstrate that the conditions corresponding to the 6 blank squares still unaccounted for in the state diagram can also occur.

$(\mathrm{II}_1, \mathrm{I}_1)$: It is clear that if we let $Y = X$ and T be the identity operator on $\mathscr{D}$, then (T, T') has the state $(\mathrm{II}_1, \mathrm{I}_1)$.

$(\mathrm{II}_2, \mathrm{III}_1)$: Let $Y = X$. If $x = (\xi_1, \xi_2, \cdots, \xi_n, 0, \cdots)$,

define

$$Tx = \left(\sum_1^n j\xi_j, \xi_2, \cdots, \xi_n, 0, \cdots\right).$$

Suppose

$$y' = (\alpha_1, \alpha_2, \cdots) \in \mathscr{D}(T') \subset \ell^2 .$$

From formula (1),

$$|y'Tu_k| = |\alpha_1 k + \alpha_k| \geqq |\alpha_1|k - |\alpha_k| \geqq |\alpha_1|k - \|y'\|$$

for $k > 1$. But $\|u_k\| = 1$ and $y'T$ is continuous on $\mathscr{D}$, therefore α_1 must be zero. We now wish to determine the operator T'. If $T'y' = (\beta_1, \beta_2, \cdots) \in \ell^2$, then from formula (1),

$$\beta_k = T'y'u_k = y'Tu_k = \alpha_k$$

whence we see that $T'y' = y'$. Since $\alpha_1 = 0$, it is clear that $\overline{\mathscr{D}(T')} \neq X' = \ell^2$. Thus $T' \in \mathrm{III}_1$. Now $Tx = 0$ implies that $0 = \xi_2 = \xi_3 = \cdots = \xi_n = \xi_1 + 2\xi_2 + \cdots + n\xi_n$ or that $x = 0$, that is T^{-1} exists; furthermore $\mathscr{R}(T) \neq Y$. An inspection of the state diagram shows that T must be in II_2. For the state $(\mathrm{I}_2, \mathrm{III}_1)$, we present two examples for the cases where X is reflexive and Y is not complete or where X is complete and

Y is reflexive. We do not have an example for $(\mathrm{I}_2, \mathrm{III}_1)$ where X is reflexive and Y is complete.

$(\mathrm{I}_2, \mathrm{III}_1)$: Take $Y = \mathscr{D}$ and let T be the operator in the above example. If $y = (\eta_1, \eta_2, \cdots, \eta_p, 0, \cdots)$, then $Tx = y$ where $x = (\eta_1 - \sum_2^p k\eta_k, \eta_2, \cdots, \eta_p, 0, \cdots)$. Thus $\mathscr{R}(T) = \mathscr{D} = Y$. The above discussion now shows the existence of state $(\mathrm{I}_2, \mathrm{III}_1)$.

In [3 p. 108] it is shown that (T, T') is in state $(\mathrm{I}_2, \mathrm{III}_1)$ where T is a bounded linear operator from a normed linear space Z, which is not complete, into a reflexive normed linear sapce Y; for example, $Y = l^2$. Let X be the completion of Z. Thus $\overline{Z} = X$ and (T, T') is in state $(\mathrm{I}_2, \mathrm{III}_1)$ with respect to X and Y.

$(\mathrm{II}_3, \mathrm{III}_1)$: Let $Y = X$. if $x = (\xi_1, \xi_2, \cdots, \xi_n, 0, \cdots)$, let

$$Tx = (2\xi_2, \cdots, n\xi_n, 0, \cdots) .$$

T is clearly in II_3. Suppose $y' = (\alpha_1, \alpha_2, \cdots) \in \mathscr{D}(T')$ and that $T'y' = (\beta_1, \beta_2, \cdots) \in l^2$. Now

$$\begin{aligned} &\beta_k = T'y'u_k = y'Tu_k = y'(ku_{k-1}) = k\alpha_{k-1} \qquad \text{if } k > 1 , \\ &\beta_1 = 0 . \end{aligned}$$

Hence $\overline{\mathscr{R}(T')} \neq X' = l^2$. Moreover

$$\|T'y'\|^2 = \sum_2^\infty |k\alpha_{k-1}|^2 \geqq \sum_1^\infty |\alpha_j|^2 = \|y'\|^2 .$$

Thus $T' \in \mathrm{III}_1$.

$(\mathrm{II}_2, \mathrm{III}_2)$: Let $Y = X$. If $x = (\xi_1, \cdots, \xi_n, 0, \cdots)$, let

$$Tx = \left(\xi_1 + \frac{1}{2}\xi_2 + 3\xi_3 + \cdots + b_n\xi_n, \xi_2/2, \cdots, \xi_n/n, 0, \cdots\right)$$

where

$$b_n = \begin{cases} 1/n \text{ if } n \text{ is even} \\ n \text{ if } n \text{ in odd.} \end{cases}$$

Clearly T^{-1} exists; however T^{-1} is not bounded since

$$\|Tu_{2k}\| = \|(u_1 + u_{2k})/2k\| = 1/k$$

and $\|u_{2k}\| = 1$. Furthermore, if $y = (\gamma_1, \gamma_2, \cdots, \gamma_N, 0, \cdots)$, then $Tx = y$ where

$$x = (\gamma_1 - \sum_2^N \gamma_j, 2\gamma_2 \cdots, N\gamma_N, 0, \cdots)$$

which shows that

$\mathscr{R}(T) = \mathscr{D}$, hence $T \in \mathrm{II}_2$. Let $x_k = (1, 1/2, \cdots, 1/k, 0, \cdots)$. Then

$Tx_k = (1 + 1/2^2 + 1 + \cdots + b_k/k, 1/2^2, \cdots, 1/k^2, 0, \cdots)$.

Set $B_k = 1 + 1/2^2 + 1 + \cdots + b_k/k$. Obviously

$B_k \to \infty$. If $y' = (\alpha_1, \alpha_2, \cdots) \in \mathscr{D}(T')$, then

$$|y'Tx_k| = |\alpha_1 B_k + \sum_2^k \alpha_j/j^2| \geqq |\alpha_1| B_k - \sum_2^k |\alpha_j/j^2| .$$

But

$$||x_k|| < \sum_1^\infty \frac{1}{n^2}, \ \sum_2^\infty | \frac{\alpha_j}{j^2} | \leqq ||y'|| \sum_1^\infty \frac{1}{n^2}$$

and $y'T$ is continuous on $\mathscr{D}$. Since $B_k \to \infty$, it follows that $\alpha_1 = 0$. If $T'y' = (\beta_1, \beta_2, \cdots) \in \ell^2$, then $\beta_1 = T'y'u_1 = y'Tu_1 = 0$ whence we see that $T' \in \mathrm{III}$.

We shall now show that T' does not have a bounded inverse.

Let $y_k' = \mu_k$ for $k > 1$. If $x = (\xi_1, \xi_2, \cdots, \xi_n, 0, \cdots)$ and $||x|| \leqq 1$, then $|y_k'Tx| = |\xi_k/k| \leqq 1$. Hence $y_k' \in \mathscr{D}(T')$. Now

$$T'y_k'u_j = y_k'Tu_j = \begin{cases} 0 & \text{if } k \neq j \\ 1/k & \text{if } k = j \end{cases}$$

or $||T'y_k'|| = ||u_k/k|| = 1/k$, which shows that T' is not in state 1. This together with the fact that $T' \in \mathrm{III}$ and $T \in \mathrm{II}_2$ enable us to infer from the state diagram that (T, T') is in state $(\mathrm{II}_2, \mathrm{III}_2)$.

$(\mathrm{III}_2, \mathrm{III}_3)$: Let $Y = \ell^2$. Similar to the preceding example, we define

$$Tx = (0, \xi_1 + 1/2\xi_2 + \cdots + b_n/n, \xi_2/2, \cdots, \xi_n/n, 0, \cdots) .$$

By the same procedure as above, we see that $T \in \mathrm{III}_2$; also if $y' = (\alpha_1, \alpha_2, \cdots,) \in \mathscr{D}(T')$, it follows that $\alpha_2, = 0$ and therefore $\beta_1 = T'y'u_1 = y'u_2 = 0$ Hence $T' \in \mathrm{III}$. From an inspection of the state diagram, it is clear that $T' \in \mathrm{III}_3$.

We have now shown that twelve of the thirteen states are possible with X and Y reflexive. State $(\mathrm{I}_2, \mathrm{III}_1)$ is also possible with X complete and Y reflexive or with X reflexive and Y not complete. The state diagram assures us that no other states are possible as long as Y is complete. If X is complete and Y is not required to be complete, then it is shown in [3] p. 106 that states $(\mathrm{I}_2, \mathrm{II}_2)$, $(\mathrm{I}_2, \mathrm{III}_2)$ and $(\mathrm{I}_3, \mathrm{III}_2)$ can occur; i.e. the squares which have the letter Y become blank. Thus we have the justification of the entries is state diagram.

The question now arises as to whether in considering the same type of hypotheses on X and Y, that is reflexivity and completeness, we can show that certain additional states are impossible if we put futher

" reasonable " hypotheses on T, for example T closed. The answer to this query is in the affirmative as we show in the next section. An assumption that X be reflexive played no part in Theorems 2.1 through 2.6.

5. The State of a closed operator and its conjugate.

LEMMA. *If T' has a continuous inverse, then for each $\alpha > 0$, 0 is an interior point of $\overline{TS_\alpha}$ where $S_\alpha = \{x \mid x \in \mathscr{D}, \|x\| \leqq \alpha\}$.*

Proof. An inspection of the first part of the proof of Theorem 6 [3 p. 97] will exhibit the proof of the lemma. It is to be noted that the argument does not depend on the hypothesis that T be bounded.

THEOREM 5.1. *Suppose that X is complete. If T is closed and T' has a continuous inverse, then $\mathscr{R}(T) = Y$. Moreover, if T^{-1} exists, it is continuous.*

Proof. Define $S_n = \{x \mid x \in \mathscr{D}, \|x\| \leqq 1/2^n,\ n = 1, 2, \cdots\}$. By the lemma, we can choose a sequence of positive numbers $\{\varepsilon_n\}$ such that $\sum_1^\infty \varepsilon_n < \infty$, and $V_n = \{y \mid y \in Y, \|y\| < \varepsilon_n\} \subset \overline{TS_n}$. The existence of these V_n and the arguments used in proving Theorem 2.12.1, p. 46.2 [2] will also prove this theorem. If, in the above theorem, T were continuous on X, that is $\mathscr{D} = X$, one could conclude that Y is complete. (cf. [3 Theorem 6 p. 97]) However, we cannot conclude that Y is complete in Theorem 5.1 even if $\mathscr{D} = X$. The following example illustrates this assertion.

EXAMPLE. Let X be any complete normed linear space of infinite dimension and let H be a Hamel basis of X with all elements $h \in H$ such that $\|h\| \leqq 1$. To each $x \in X$ there corresponds a unique finite set $h_1, h_2, \cdots, h_n$ in H and unique scalars $\alpha_1, \alpha_2, \cdots, h_n$ such that $x = \sum_1^n \alpha_i h_i$. We now define another norm $\|x\|_1$ on X by letting $\|x\|_1 = \sum_1^n |\alpha_i|$. Taylor and Halberg [3, p. 109] show that X with this norm, which we designate by X_1, is not complete. Define T as the identity mapping from X onto X_1. T has a bounded inverse, since

$$\|Tx\|_1 = \|x\|_1 = \sum_1^n |\alpha_i| \geqq \sum_1^n \|\alpha_i h_i\| \geqq \|x\|.$$

In addition, T is also closed, for suppose $x_n \to x$ and $Tx_n \to y$. Since $\|x_n - y\| \leqq \|x_n - y\|_1 = \|Tx_n - y\|_1$ and $\|Tx_n - y\|_1 \to 0$, it follows that $Tx = x = y$. An inspection of the state diagram shows that T' has a bounded inverse. Thus the hypotheses of Theorem 5.1 are satisfied, but $\mathscr{R}(T) = X_1$ is not complete.

This example also serves to illustrate that in the hypotheses of the "closed graph theorem" it is essential that the closed operator map into a complete normed linear space.

DEFINITION 3. If E is a subset of X, define

$$E^\circ = \{x' \mid x' \in X' ; \; x'x = 0 \text{ for all } x \in E\} .$$

DEFINITION 4. If S is a subset of X', define

$$^\circ S = \{x \mid x'x = 0 \text{ for all } x' \in S\} .$$

The following known lemma is easy to prove.

LEMMA. *Let X be reflexive. If M is a closed linear subspace in X', then $M = ({}^\circ M)^\circ$.*

THEOREM 5.2. *Let X be reflexive. If T^{-1} exists and $\mathscr{D}(T')$ is total, then $\overline{\mathscr{R}(T')} = X'$.*

Proof. We first show that ${}^\circ\overline{\mathscr{R}(T')} = (0)$. *If* $x \in {}^\circ\overline{\mathscr{R}(T')}$, then $y'Tx = T'y'x = 0$ for all $y' \in \mathscr{D}(T')$; but then $Tx = 0$ since $\mathscr{D}(T')$ is total. The fact that T^{-1} exists implies that $x = 0$. Clearly $0 \varepsilon\ {}^\circ\overline{\mathscr{R}(T')}$, hence ${}^\circ\overline{\mathscr{R}(T')} = (0)$. Applying the preceding lemma, we see that

$$\overline{\mathscr{R}(T')} = ({}^\circ\overline{\mathscr{R}(T')})^\circ = (0)^\circ = X' .$$

COROLLARY. *Let X be reflexive. If T is closed and T^{-1} exists, then $\overline{\mathscr{R}(T')} = X'$.*

Proof. Theorems 1.2 and 5.2.

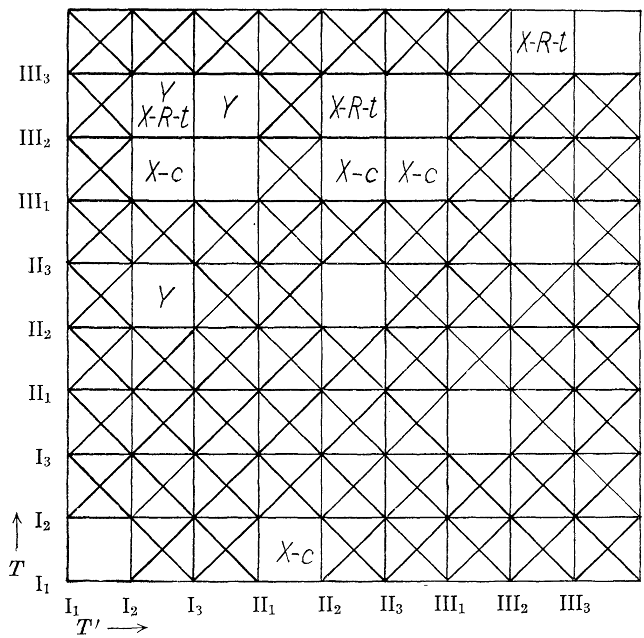
6. The second state diagram. The two theorems just proved as well as the state diagram in §3 enable us to determine the state diagram for a closed operator. We place X-R-t in a square to indicate that the state cannot occur if X is reflexive and $\mathscr{D}(T')$ total. An X-c in a square will indicate that the state cannot occur if X is complete and T is closed.

This diagram is a generalization of the Taylor-Halberg state diagram for T bounded on all of X.

7. The spectrum of an operator and its conjugate. In the present section we consider a linear transformation T, not necessarily bounded, with $\overline{\mathscr{D}} = X$ and $\mathscr{R}(T) \subset X$, where X is a normed linear space. In this case, $T_\lambda = \lambda - T$ is well defined on $\mathscr{D}$, where λ is a scalar.

DEFINITION 7.1. The values of λ for which T_λ has a bounded inverse with domain dense in X form the *resolvent set* $\rho(T)$ of T; that is $T_\lambda \in \mathrm{I}_1 \cup \mathrm{II}_1$. The values of λ for which T_λ has an unbounded inverse

Second State Diagram



Y: Can't occur if Y is complete.
X-c: Can't occur if X is complete and T is closed.
X-R-t: Can't occur if X is reflexive and $\mathscr{D}(T')$ is total (in particular if T is closed).

with domain dense in X form the *continuous spectrum* $C\sigma(T)$, that is $T_\lambda \in \mathrm{I}_2 \cup \mathrm{II}_2$. The values of λ for which T_λ has an inverse whose domain is not dense in X form the *residual spectrum* $R\sigma(T)$, that is $T_\lambda \in \mathrm{III}_1 \cup \mathrm{III}_2$. The values of λ for which no inverse exists form the *point spectrum* $P\sigma(T)$, that is $\mathrm{I}_3 \cup \mathrm{II}_3 \cup \mathrm{III}_3$. The *spectrum* $\sigma(T)$ is defined to be the set of scalars not in $\rho(T)$.

These definitions can also be applied to T'. We would like now to draw inferences about the relationships between the above defined point sets for T and T'. Since $\lambda - T' = (\lambda - T)'$, an appeal to the state diagram in §3 easily verify the following.

THEOREM 7.1. (a) $\rho(T) = \rho(T')$ *or equivalently,* $\sigma(T) = \sigma(T')$.

(b) $P\sigma(T) \subset R\sigma(T') \cup P\sigma(T')$.

(c) $P(\sigma T') \subset R\sigma(T) \cup P\sigma(T)$.

(d) $C\sigma(T) \subset R\sigma(T') \cup C\sigma(T')$.

(e) $C\sigma(T') \subset C\sigma(T)$.

(f) $R\sigma(T) \subset P\sigma(T')$.

(g) $R\sigma(T') \subset C\sigma(T) \cup P\sigma(T)$.

Suppose we now require that T be closed, in addition to the other hypotheses mentioned at the beginning of the section. It is easy to see that $\lambda - T$ is also closed for λ any scalar. Hence we can obtain the following theorem together with Theorem 7.1 by referring to the second state diagram in §6.

THEOREM 7.2. *If T is a closed operator and X is reflexive, then $C\sigma(T) = C\sigma(T')$ and $R\sigma(T') \subset P\sigma(T)$.*

REMARK. Let X and Y be Hilbert spaces. If T^* is the adjoint of T, then T^* may be put in place of T' in using the first and second state diagrams. This is easy show by considering the fact that a Hilbert space is isometric to its conjugate space.

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Printed in Japan by Kokusai Bunken Insatsusha
(International Academic Printing Co., Ltd.), Tokyo, Japan

Pacific Journal of Mathematics

Vol. 9, No. 1 May, 1959