

# Pacific Journal of Mathematics

**ON ONE-TO-ONE HARMONIC MAPPINGS**

ERHARD HEINZ

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In this paper we shall prove the following :

**THEOREM.** *Let  $z = z(w)$  ( $z = x + iy$ ,  $w = u + iv$ ) be a one-to-one harmonic mapping of the disc  $|w| < 1$  onto the disc  $|z| < 1$  such that  $z(0) = 0$ . Then we have for  $|w| < 1$  the estimate*

$$(1) \quad |z_u|^2 + |z_v|^2 \geq \frac{2}{\pi^2} .$$

As an improvement of an earlier result established in [1] J. C. C. Nitsche [4] showed that under the above conditions the inequality

$$(2) \quad (|z_u|^2 + |z_v|^2)_{w=0} \geq \frac{1}{2}$$

is satisfied<sup>1</sup>. In contrast to (2) the estimate (1) holds throughout the unit disc  $|w| < 1$ , but the constant involved is smaller than that of Nitsche.

In order to establish (1) we shall make use of a known result on harmonic functions (the analogue of the Schwarz Lemma)<sup>2</sup>. For the sake of completeness the proof of it will be given here.

**LEMMA.** *Let  $z = z(w) = x(w) + iy(w)$  be a complex-valued harmonic function in the disc  $|w| < 1$ . Furthermore, let  $z(0) = 0$  and  $|z(w)| < 1$  for  $|w| < 1$ . Then we have the inequality*

$$(3) \quad |z(w)| \leq \frac{4}{\pi} \arctan |w| \quad |w| < 1.$$

*Proof.* Let  $\theta$  be an arbitrary real number, and  $f(w)$  be the function, which is regular-analytic in the disc  $|w| < 1$  and satisfies the relations  $f(0) = 0$  and

$$(4) \quad \Re f(w) = x(w) \cos \theta + y(w) \sin \theta .$$

On account of our hypotheses we have

$$(5) \quad |\Re f(w)| < 1 \quad |w| < 1,$$

hence,

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<sup>1</sup> For further references see [2].

<sup>2</sup> See Polya-Szegö [5], p. 140.

$$(6) \quad \Re \left( \exp \left[ \frac{i\pi}{2} f(w) \right] \right) > 0 \quad |w| < 1.$$

Consequently the function

$$(7) \quad g(w) = \frac{\exp \left[ \frac{i\pi}{2} f(w) \right] - 1}{\exp \left[ \frac{i\pi}{2} f(w) \right] + 1}$$

satisfies the inequality

$$(8) \quad |g(w)| < 1 \quad |w| < 1,$$

and we have  $g(0) = 0$ . Applying now the Schwarz Lemma and the elementary inequality

$$(9) \quad \left| \frac{e^{i\zeta} - 1}{e^{i\zeta} + 1} \right| \geq \tan \frac{1}{2} |\Re \zeta| \quad |\Re \zeta| \leq \frac{\pi}{2}$$

we obtain the estimate

$$(10) \quad \tan \frac{\pi}{4} |\Re f(w)| \leq |g(w)| \leq |w|,$$

hence, by (4)

$$(11) \quad |x(w) \cos \theta + y(w) \sin \theta| \leq \frac{4}{\pi} \arctan |w|$$

for  $|w| < 1$ .

Since this holds for every real value of  $\theta$  the inequality (3) follows, which proves the lemma.

*Proof of the theorem.* (I) We first prove (1) under the additional hypothesis that the function  $z(w)$  and its first derivatives are continuous in the closed disc  $|w| \leq 1$ . Since the mapping  $w \rightarrow z(w)$  is one-to-one and harmonic, its Jacobian  $|z_w|^2 - |z_{\bar{w}}|^2$ <sup>3</sup> cannot vanish, in virtue of a theorem of H. Lewy [3]. Furthermore, since hypothesis and conclusion of our theorem remain unchanged, if  $z(w)$  is replaced by  $\overline{z(w)}$ , we may assume without loss of generality that

$$(12) \quad |z_w|^2 - |z_{\bar{w}}|^2 > 0 \quad |w| < 1.$$

Consequently, the function  $z_w$  does not vanish in the disc  $|w| < 1$ . Furthermore, because of  $z_{w\bar{w}} = 0$ , it is regular-analytic. From these facts it follows that for  $|w| \leq 1$  the inequality

<sup>3</sup> Here and in the following considerations  $\frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right)$  and

$\frac{\partial}{\partial \bar{w}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$  are the complex derivatives.

$$(13) \quad |z_w| \geq \min_{|w|=1} |z_w|$$

holds.

We shall now estimate the right-hand side of (13) from below by using our lemma. Let  $\varphi$  and  $r$  be two real numbers and  $0 < r < 1$ . Since by hypothesis the equation  $|z(w)| = 1$  holds for  $|w| = 1$  we have

$$(14) \quad \left| \frac{z(e^{i\varphi}) - z(re^{i\varphi})}{1 - r} \right| \geq \frac{1 - |z(re^{i\varphi})|}{1 - r} \geq \frac{1 - 4/\pi \arctan r}{1 - r}$$

If we let  $r$  tend to 1, we obtain

$$(15) \quad \left( \left| \frac{\partial z(re^{i\varphi})}{\partial r} \right| \right)_{r=1} \geq \frac{2}{\pi} \quad 0 \leq \varphi < 2\pi.$$

Furthermore, on account of (12) we have

$$(16) \quad \left| \frac{\partial z(re^{i\varphi})}{\partial r} \right| = |z_w(re^{i\varphi})e^{i\varphi} + z_{\bar{w}}(re^{i\varphi})e^{-i\varphi}| \leq |z_w| + |z_{\bar{w}}| \leq 2|z_w|$$

for  $0 < r \leq 1$ . Combining this with (15) we infer that for  $|w| = 1$  the estimate

$$(17) \quad |z_w| \geq \frac{1}{\pi}$$

holds.

Hence, by (13) we obtain for  $|w| \leq 1$  the inequality

$$(18) \quad \frac{1}{\pi} \leq |z_w| = \frac{1}{2} |z_u - iz_v| \leq 2^{-1/2}(|z_u|^2 + |z_v|^2)^{1/2},$$

which yields (1).

(II) Now let the mapping  $z = z(w)$  merely satisfy the hypotheses of our theorem. Obviously there exists a sequence of numbers  $\{R_n\}$  ( $n \geq 2$ ) such that the following conditions are satisfied:

(i) We have  $0 < R_n < 1$  for all  $n \geq 2$ , and

$$(19) \quad \lim_{n \rightarrow \infty} R_n = 1.$$

(ii) The disc  $|z| < R_n$  is mapped by the inverse transformation  $z \rightarrow w$  onto a simply-connected domain  $D_n$  such that

$$(20) \quad \left\{ |w| \leq 1 - \frac{1}{n} \right\} \subset D_n \subset \{ |w| < 1 \}.$$

Since the mapping  $z \rightarrow w$  is analytic in  $x$  and  $y$ , it follows that  $D_n$  is bounded by an analytic Jordan curve. By the Riemann mapping theorem there exists a uniquely determined function  $w = \Phi_n(\zeta)$ , which maps the disc  $|\zeta| < 1$  ( $\zeta = \xi + i\eta$ ) conformally onto  $D_n$  such that  $\Phi_n(0) = 0$  and  $\Phi_n'(0) > 0$ . Furthermore,  $\Phi_n(\zeta)$  is analytic for  $|\zeta| \leq 1$ . Consequently, the function

$$(21) \quad Z(\zeta) = \frac{z(\Phi_n(\zeta))}{R_n}$$

is harmonic for  $|\zeta| < 1 + \delta$ , where  $\delta$  is a positive number, and satisfies all the hypotheses of the above theorem. From the facts established in (I) we conclude

$$(22) \quad \frac{|\Phi'_n(\zeta)|^2}{R_n^2} (|z_u|^2 + |z_v|^2) = |Z_\xi|^2 + |Z_\eta|^2 \geq \frac{2}{\pi^2}.$$

Hence we have for  $w = \Phi_n(\zeta)$  ( $|\zeta| < 1$ ) the inequality

$$(23) \quad |z_u|^2 + |z_v|^2 \geq \frac{R_n^2}{|\Phi'_n(\zeta)|^2} \cdot \frac{2}{\pi^2}.$$

Furthermore, on account of (20) the estimates

$$(24) \quad \left(1 - \frac{1}{n}\right) |\zeta| \leq |\Phi_n(\zeta)| \leq |\zeta|$$

hold for  $n \geq 2$  and  $|\zeta| < 1$ . Applying the Schwarz Lemma it follows from (24) that there exists a sequence of integers  $\{n_k\}$  such that the relations

$$(25) \quad \Phi'_{n_k}(\zeta) \rightarrow 1 \quad (k \rightarrow \infty)$$

hold uniformly in every closed disc  $|\zeta| \leq \rho < 1$ .

Now let  $w^*$  be a fixed complex number with  $|w^*| < 1$  and let us determine two positive numbers  $k_0$  and  $\rho$  such that the inequalities

$$(26) \quad \frac{|w^*|}{1 - \frac{1}{n_k}} \leq \rho < 1$$

are satisfied for  $k \geq k_0$ . On account of (20) the point  $w^*$  belongs to  $D_{n_k}$  for  $k \geq k_0$ . Hence there exists a sequence of complex numbers  $\{\zeta_k\}$  with  $|\zeta_k| < 1$  such that the equations

$$(27) \quad w^* = \Phi_{n_k}(\zeta_k)$$

hold for  $k \geq k_0$ . By (24) we have

$$(28) \quad |\zeta_k| \leq \frac{|w^*|}{1 - \frac{1}{n_k}} \leq \rho < 1$$

for  $k \geq k_0$ . Applying now (23) and (25) we conclude

$$(29) \quad (|z_u|^2 + |z_v|^2)_{w=w^*} \geq \frac{R_{n_k}^2}{|\Phi'_{n_k}(\zeta_k)|^2} \cdot \frac{2}{\pi^2} \rightarrow \frac{2}{\pi^2}$$

for  $k \rightarrow \infty$ . Since  $w^*$  is an arbitrary point in the disc  $|w| < 1$ , our theorem is established.

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