ON ONE-TO-ONE HARMONIC MAPPINGS

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In this paper we shall prove the following:

**Theorem.** Let \( z = z(w) \) (\( z = x + iy, \ w = u + iv \)) be a one-to-one harmonic mapping of the disc \( |w| < 1 \) onto the disc \( |z| < 1 \) such that \( z(0) = 0 \). Then we have for \( |w| < 1 \) the estimate

\[
|z_u|^2 + |z_v|^2 \geq \frac{2}{\pi^2}.
\]

As an improvement of an earlier result established in [1] J. C. C. Nitsche [4] showed that under the above conditions the inequality

\[
(|z_u|^2 + |z_v|^2)_{w=0} \geq \frac{1}{2}
\]

is satisfied\(^1\). In contrast to (2) the estimate (1) holds throughout the unit disc \( |w| < 1 \), but the constant involved is smaller than that of Nitsche.

In order to establish (1) we shall make use of a known result on harmonic functions (the analogue of the Schwarz Lemma)\(^2\). For the sake of completeness the proof of it will be given here.

**Lemma.** Let \( z = z(w) = x(w) + iy(w) \) be a complex-valued harmonic function in the disc \( |w| < 1 \). Furthermore, let \( z(0) = 0 \) and \( |z(w)| < 1 \) for \( |w| < 1 \). Then we have the inequality

\[
|z(w)| \leq \frac{4}{\pi} \arctan |w| \quad |w| < 1.
\]

**Proof.** Let \( \theta \) be an arbitrary real number, and \( f(w) \) be the function, which is regular-analytic in the disc \( |w| < 1 \) and satisfies the relations \( f(0) = 0 \) and

\[
\Re f(w) = x(w) \cos \theta + y(w) \sin \theta.
\]

On account of our hypotheses we have

\[
|\Re f(w)| < 1 \quad |w| < 1,
\]

\(^1\) For further references see [2].

\(^2\) See Polya-Szego [5], p. 140.

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\begin{align*}
(6) \quad \Re \left( \exp \left[ \frac{i \pi}{2} f(w) \right] \right) > 0 \quad |w| < 1.
\end{align*}

Consequently the function

\begin{align*}
(7) \quad g(w) = \frac{\exp \left[ \frac{i \pi}{2} f(w) \right] - 1}{\exp \left[ \frac{i \pi}{2} f(w) \right] + 1}
\end{align*}

satisfies the inequality

\begin{align*}
(8) \quad |g(w)| < 1 \quad |w| < 1,
\end{align*}

and we have \( g(0) = 0 \). Applying now the Schwarz Lemma and the elementary inequality

\begin{align*}
(9) \quad \left| \frac{e^{i \zeta} - 1}{e^{i \zeta} + 1} \right| \geq \tan \frac{1}{2} |\Re \zeta| \quad |\Re \zeta| \leq \frac{\pi}{2}
\end{align*}

we obtain the estimate

\begin{align*}
(10) \quad \tan \frac{\pi}{4} |\Re f(w)| \leq |g(w)| \leq |w|,
\end{align*}

hence, by (4)

\begin{align*}
(11) \quad |x(w) \cos \theta + y(w) \sin \theta| \leq \frac{4}{\pi} \arctan |w|
\end{align*}

for \( |w| < 1 \).

Since this holds for every real value of \( \theta \) the inequality (3) follows, which proves the lemma.

\textit{Proof of the theorem.} (I) We first prove (1) under the additional hypothesis that the function \( z(w) \) and its first derivatives are continuous in the closed disc \( |w| \leq 1 \). Since the mapping \( w \to z(w) \) is one-to-one and harmonic, its Jacobian \( |z_w|^2 - |z^*_w|^2 \) cannot vanish, in virtue of a theorem of H. Lewy [3]. Furthermore, since hypothesis and conclusion of our theorem remain unchanged, if \( z(w) \) is replaced by \( z(w) \), we may assume without loss of generality that

\begin{align*}
(12) \quad |z_w|^2 - |z^*_w|^2 > 0 \quad |w| < 1.
\end{align*}

Consequently, the function \( z_w \) does not vanish in the disc \( |w| < 1 \). Furthermore, because of \( z_{ww} = 0 \), it is regular-analytic. From these facts it follows that for \( |w| \leq 1 \) the inequality

\[ \frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \quad \text{and} \quad \frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \]

are the complex derivatives.
holds.

We shall now estimate the right-hand side of (13) from below by using our lemma. Let $\varphi$ and $r$ be two real numbers and $0 < r < 1$. Since by hypothesis the equation $|z(w)| = 1$ holds for $|w| = 1$ we have

$$
|z(e^{i\varphi}) - z(re^{i\varphi})| = \frac{1 - |z(re^{i\varphi})|}{1 - r} \geq 1 - \frac{4}{\pi} \arctan r
$$

If we let $r$ tend to 1, we obtain

$$
\left( \left| \frac{\partial z(re^{i\varphi})}{\partial r} \right| \right)_{r=1} \geq \frac{2}{\pi}
$$

Furthermore, on account of (12) we have

$$
|\frac{\partial z(re^{i\varphi})}{\partial r}| = |z_w(re^{i\varphi})e^{i\varphi} + z_w(re^{i\varphi})e^{-i\varphi}| \leq |z_w| + |z_{ww}| \leq 2|z_w|
$$

for $0 < r \leq 1$. Combining this with (15) we infer that for $|w| = 1$ the estimate

$$
|z_w| \geq \frac{1}{\pi}
$$

holds.

Hence, by (13) we obtain for $|w| \leq 1$ the inequality

$$
\frac{1}{\pi} \leq |z_w| = \frac{1}{2} |z_u - iz_v| \leq 2^{-1/2}(|z_u|^2 + |z_v|^2)^{1/2},
$$

which yields (1).

(II) Now let the mapping $z = z(w)$ merely satisfy the hypotheses of our theorem. Obviously there exists a sequence of numbers $\{R_n\}$ ($n \geq 2$) such that the following conditions are satisfied:

(i) We have $0 < R_n < 1$ for all $n \geq 2$, and

$$
\lim_{n \to \infty} R_n = 1.
$$

(ii) The disc $|z| < R_n$ is mapped by the inverse transformation $z \to w$ onto a simply-connected domain $D_n$ such that

$$
\left\{ |w| \leq 1 - \frac{1}{n} \right\} \subset D_n \subset \{ |w| < 1 \}.
$$

Since the mapping $z \to w$ is analytic in $x$ and $y$, it follows that $D_n$ is bounded by an analytic Jordan curve. By the Riemann mapping theorem there exists a uniquely determined function $w = \Phi_n(\zeta)$, which maps the disc $|\zeta| < 1$ ($\zeta = \xi + i\eta$) conformally onto $D_n$ such that $\Phi_n(0) = 0$ and $\Phi'_n(0) > 0$. Furthermore, $\Phi_n(\zeta)$ is analytic for $|\zeta| \leq 1$. Consequently, the function
(21) \[ Z(\zeta) = \frac{z(\phi_n(\zeta))}{R_n} \]

is harmonic for \(|\zeta| < 1 + \delta\), where \(\delta\) is a positive number, and satisfies all the hypotheses of the above theorem. From the facts established in (1) we conclude

(22) \[ \left| \frac{\phi_n'(\zeta)}{R_n^2} \right|^2 \left( |z_u|^2 + |z_v|^2 \right) = |Z_\xi|^2 + |Z_\eta|^2 \geq \frac{2}{\pi^2}. \]

Hence we have for \(w = \phi_n(\zeta) \quad (|\zeta| < 1)\) the inequality

(23) \[ |z_u|^2 + |z_v|^2 \geq -\frac{R_n^2}{|\phi_n'(\zeta)|^2} \cdot \frac{2}{\pi^2}. \]

Furthermore, on account of (20) the estimates

(24) \[ \left(1 - \frac{1}{n}\right) |\zeta| \leq |\phi_n(\zeta)| \leq |\zeta| \]

hold for \(n \geq 2\) and \(|\zeta| < 1\). Applying the Schwarz Lemma it follows from (24) that there exists a sequence of integers \(\{n_k\}\) such that the relations

(25) \[ \phi_{n_k}'(\zeta) \to 1 \quad (k \to \infty) \]

hold uniformly in every closed disc \(|\zeta| \leq \rho < 1\).

Now let \(w^*\) be a fixed complex number with \(|w^*| < 1\) and let us determine two positive numbers \(k_0\) and \(\rho\) such that the inequalities

(26) \[ \frac{|w^*|}{1 - \frac{1}{n_k}} \leq \rho < 1 \]

are satisfied for \(k \geq k_0\). On account of (20) the point \(w^*\) belongs to \(D_{n_k}\) for \(k \geq k_0\). Hence there exists a sequence of complex numbers \(\{\zeta_k\}\) with \(|\zeta_k| < 1\) such that the equations

(27) \[ w^* = \phi_{n_k}(\zeta_k) \]

hold for \(k \geq k_0\). By (24) we have

(28) \[ |\zeta_k| \leq \frac{|w^*|}{1 - \frac{1}{n_k}} \leq \rho < 1 \]

for \(k \geq k_0\). Applying now (23) and (25) we conclude

(29) \[ (|z_u|^2 + |z_v|^2)_{w^*} \geq \frac{R_{n_k}^2}{|\phi_{n_k}'(\zeta_k)|^2} \cdot \frac{2}{\pi^2} \to \frac{2}{\pi^3} \]

for \(k \to \infty\). Since \(w^*\) is an arbitrary point in the disc \(|w| < 1\), our theorem is established.
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Robert George Buschman, *Asymptotic expressions for*
\[ \sum n^a f(n) \log^f n \] ....................................................... 9

Eckford Cohen, *A class of residue systems (mod r) and related arithmetical functions. I. A generalization of Möbius inversion* .................... 13

Paul F. Conrad, *Non-abelian ordered groups* ............................................ 25

Richard Henry Crowell, *On the van Kampen theorem* .......................... 43

Irving Leonard Glicksberg, *Convolution semigroups of measures* ........ 51

Seymour Goldberg, *Linear operators and their conjugates* ..................... 69

Olof Hanner, *Mean play of sums of positional games* ............................ 81

Erhard Heinz, *On one-to-one harmonic mappings* .................................. 101

John Rolfe Isbell, *On finite-dimensional uniform spaces* ....................... 107

Erwin Kreyszig and John Todd, *On the radius of univalence of the function*
\[ \exp z^2 \int_0^z \exp(-t^2) dt \] ....................................................... 123

Roger Conant Lyndon, *An interpolation theorem in the predicate calculus* ................................................................. 129

Roger Conant Lyndon, *Properties preserved under homomorphism* ........ 143

Roger Conant Lyndon, *Properties preserved in subdirect products* ........ 155

Robert Osserman, *A lemma on analytic curves* .................................... 165

R. S. Phillips, *On a theorem due to Sz.-Nagy* ....................................... 169

Richard Scott Pierce, *A generalization of atomic Boolean algebras* ........ 175

J. B. Roberts, *Analytic continuation of meromorphic functions in valued fields* ................................................................. 183

Walter Rudin, *Idempotent measures on Abelian groups* ....................... 195

M. Schiffer, *Fredholm eigen values of multiply-connected domains* ........ 211

V. N. Singh, *A note on the computation of Alder’s polynomials* ............. 271

Maurice Sion, *On integration of 1-forms* ........................................ 277

Elbert A. Walker, *Subdirect sums and infinite Abelian groups* ............... 287

John W. Woll, *Homogeneous stochastic processes* ................................ 293