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ON THE RADIUS OF UNIVALENCE OF THE FUNCTION $\exp z^2 \int_0^z \exp(-t^2)\,dt$

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1. Introduction. We shall determine the radius of univalence ρ_u of the function

$$E(z) = e^{z^2}\int_0^z e^{-t^2}dt \ . \tag{1.1}$$

We shall write $E(z) = w = u(x, y) + iv(x, y)$. On the imaginary axis we have $u = 0$ and v, regarded as a function of y, has a single maximum at the solution $y = \rho$ of

$$2yv(0, y) = 1 \ .$$

The value of ρ to eight decimal places has been determined by Lash Miller and Gordon [1] and is

$$\rho = 0.92413887 \ . \tag{1.2}$$

It is evident that $\rho_u \leqq \rho$. We shall prove the following theorem.

THEOREM. *The number ρ is the radius of univalence of $E(z)$.*

Recently, the radius of univalence of the error function

$$erf(z) = \int_0^z e^{-t^2}dt$$

was determined [2]. It is interesting to note that when proceeding from $erf(z)$ to $E(z)$ we meet an entirely different situation. In the case of $erf(z)$, points z_1, z_2 closest to the origin and such that $erf(z_1)=erf(z_2)$ are conjugate complex and lie far apart from each other. In the case of $E(z)$ points of that nature can be found in an arbitrarily small neigborhood of the point $z = i\rho$.

The actual situation is made clear by the diagram and tables given below. In Fig. 1 we show the curves $R = |E| =$ constant and $\gamma = arg\, E =$ constant in the square $0 \leqq x \leqq 1.5$, $0 \leqq y \leqq 1.5$ of the z-plane. The table shows the values of E for z on the curve C (defined below). The values given were obtained by summing an adequate number of terms of the power series on the Datatron 205 at the California Institute of Technology ; some were checked by comparison with the tables of Karpov [4, 5] from which values of $E(z)$ can be obtained.

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2. Idea of proof. Since

$$E(z) = \sum_{n=0}^{\infty} \frac{2^n}{1.3.5\cdots(2n+1)} z^{2n+1}, \qquad |z| < \infty, \tag{2.1}$$

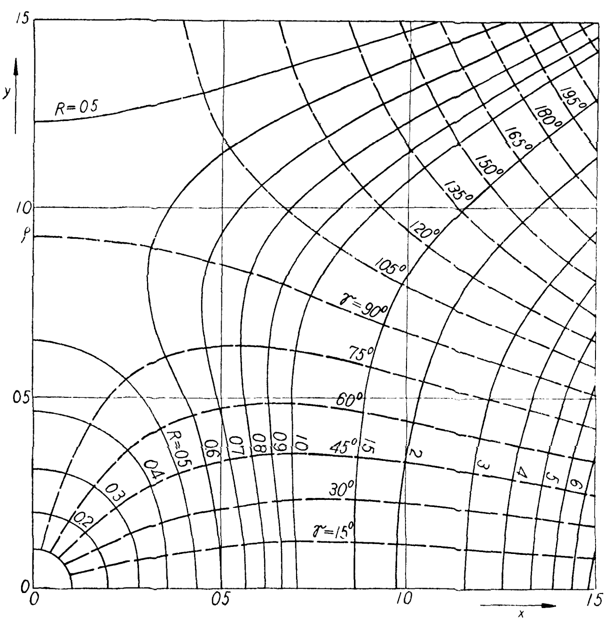

Fig. 1. Curves $R = |E| =$ const. and $\gamma = arg\, E =$ const. in the z-plane.

x	$E(x)$
0	0
0.1	0.1007
0.2	0.2054
0.3	0.3187
0.4	0.4455
0.5	0.5923
0.6	0.7671
0.7	0.9805
0.8	1.2473
0.9	1.5876

ϕ	$E(\rho e^{i\phi})$
0°	1.6837
10°	1.4957+0.6121i
20°	1.0573+0.9759i
30°	0.6079+1.0473i
40°	0.2919+0.9463i
50°	0.1189+0.8024i
60°	0.0401+0.6817i
70°	0.0099+0.6003i
80°	0.0011+0.5553i
90°	0.5410i

y	$E(iy)$
0	0
0.1	0.0993i
0.2	0.1948i
0.3	0.2826i
0.4	0.3599i
0.5	0.4244i
0.6	0.4748i
0.7	0.5105i
0.8	0.5321i
0.9	0.5407i

we have $E(\bar{z}) = \overline{E(z)}$ and $E(-z) = -E(z)$ and may restrict our consideration to the first quadrant $x \geqq 0, y \geqq 0$ in the z-plane.

In the subsequent section we shall prove the following lemma.

LEMMA.

$$E(z_1) \neq E(z_2) \tag{2.2}$$

for any two points on the boundary C of the open sector S of the circular disk $|z| < \rho$ in the first quadrant.

From this it follows, since $E(z)$ is entire and thus regular in $S \cup C$ that $E(z)$ maps S conformally and one-to-one onto the interior of the simple closed curve C^* corresponding to C in the w-plane [3, p. 121]. This establishes our theorem.

3. Proof of the lemma. Let $z = re^{i\phi}$. The curve C consists of

the segment S_1: $y = 0\,, \quad 0 < x < \rho\,,$

the circular arc K: $|z| = \rho\,, \quad 0 < \phi < \pi/2\,,$

the segment S_2: $x = 0\,, \quad 0 < y < \rho\,.$

and the three common end points of these three arcs.

(A) On S_1, $E(z)$ is real and increases steadily with x.

(B) On S_2, $E(z)$ is imaginary, and v increases steadily with y.

(C) $v \neq 0$ on K.

(D) On K, $|E(z)|$ decreases steadily with increasing ϕ.

(A) is obvious from (2.1), and (B) follows from the definition of ρ.

Proof of (C). Integrating along segments parallel to the coordinate axes we have

$$\begin{aligned} v(x, y) = e^{-y^2}\Big[&\cos 2xy\int_0^y e^{\tau^2}\cos 2x\tau d\tau \\ &+ \sin 2xy\left\{e^{x^2}\int_0^x e^{-t^2}dt + \int_0^y e^{\tau^2}\sin 2x\tau d\tau\right\}\Big]\,. \end{aligned}$$

In $\{x > 0, y > 0\} \cap \{|z| \leqq \rho\}$ we have $\cos 2xy > 0$, $\sin 2xy > 0$. Therefore $v > 0$ on K.

Proof of (D). Integrating along a radius $\phi =$ constant from 0 to ρ we have

$$E(z) = e^{i\phi}\int_0^\rho e^{h(r,\phi)}dr$$

where

$$\begin{gathered} h(r, \phi) = a(r, \phi) + ib(r, \phi)\,, \\ a(r, \phi) = (\rho^2 - r^2)\cos 2\phi,\ b(r, \phi) = (\rho^2 - r^2)\sin 2\phi\,. \end{gathered}$$

Therefore

$$|E|^2 = \int_0^\rho e^h dr\int_0^\rho e^{\bar{h}}dr\,.$$

Differentiating with respect to ϕ and setting

$$\begin{gathered} h^* = a^* + ib^*,\ a^* = a(r^*, \phi),\ b^* = b(r^*, \phi), \\ f = \cos(b^* - b) - i\sin(b^* - b) \end{gathered}$$

we obtain

$$(|E|^2)_\phi = \int_0^\rho e^h h_\phi dr \int_0^\rho e^{\overline{h^*}} dr^* + \int_0^\rho e^{h^*} dr^* \int_0^\rho e^{\bar{h}} \bar{h}_\phi dr$$
$$= \int_0^\rho \int_0^\rho e^{a+a^*} \{f h_\phi + \bar{f} \bar{h}_\phi\} dr dr^* .$$

Now

$$a_\phi = -2(\rho^2 - r^2)\sin 2\phi, \; b_\phi = 2(\rho^2 - r^2)\cos 2\phi$$

and therefore

$$\begin{aligned} fh_\phi + \overline{fh_\phi} &= 2\Re f h_\phi = 2[\cos(b^* - b)a_\phi + \sin(b^* - b)b_\phi] \\ &= -4(\rho^2 - r^2)\sin(\alpha(\phi)) \end{aligned}$$

where

$$\alpha(\phi) = 2\phi + b - b^* = (r^{*2} - r^2)\sin 2\phi + 2\phi .$$

This yields

$$(|E|^2)_\phi = -4\int_0^\rho \int_0^\rho e^{a+a^*}(\rho^2 - r^2)\sin(\alpha(\phi)) dr dr^* . \tag{3.1}$$

Since from (1.2) we have $|r^{*2} - r^2| < 1$, we obtain

$$\alpha'(\phi) = 2 + 2(r^{*2} - r^2)\cos 2\phi > 0 .$$

Hence $\alpha(\phi), 0 \leqq \phi \leqq \pi/2$, has its maximum at $\phi = \pi/2$. Therefore $0 \leqq \alpha(\phi) < \pi$ when $0 \leqq \phi < \pi/2$ and $\sin(\alpha(\phi)) > 0$ when $0 < \phi < \pi/2$. This means that the integrand in (3.1) is positive in the region $0 \leqq r \leqq \rho$, $0 \leqq r^* \leqq \rho$ for all ϕ in the interval $0 < \phi < \pi/2$. Thus $(|E|^2)_\phi < 0$ when $0 < \phi < \pi/2$. This proves (D).

We note that (D) remains true if K is replaced by quadrants of circles of radii somewhat larger than ρ; this, however, is of no interest here.

For $z_1 \in K$, $z_2 \in S_2$, or $z_1 \in K$, $z_2 \in K$, equation (2.2) holds, as follows from (D). For $z_1 \in K$, $z_2 \in S_1$ the same is true because of (C). In the other cases, $z_1 \in S_1$, $z_2 \in S_1$, etc., the validity of (2.2) is obvious. This proves the lemma.

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$$\rho = 0.92413\ 88730.$$

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