ON THE RADIUS OF UNIVALENCE OF THE FUNCTION
\[ \exp z^2 \int_0^z \exp(-t^2) dt \]

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1. Introduction. We shall determine the radius of univalence \( \rho_u \) of the function

\[ E(z) = e^{z^2} \int_0^z e^{-t^2}dt . \]

We shall write \( E(z) = w = u(x, y) + iv(x, y) \). On the imaginary axis we have \( u = 0 \) and \( v \), regarded as a function of \( y \), has a single maximum at the solution \( y = \rho \) of

\[ 2yv(0, y) = 1 . \]

The value of \( \rho \) to eight decimal places has been determined by Lash Miller and Gordon \[1\] and is

\[ \rho = 0.92413887 . \]

It is evident that \( \rho_u \leq \rho \). We shall prove the following theorem.

**Theorem.** The number \( \rho \) is the radius of univalence of \( E(z) \).

Recently, the radius of univalence of the error function

\[ \text{erf}(z) = \int_0^z e^{-t^2}dt \]

was determined \[2\]. It is interesting to note that when proceeding from \( \text{erf}(z) \) to \( E(z) \) we meet an entirely different situation. In the case of \( \text{erf}(z) \), points \( z_i, z_i \) closest to the origin and such that \( \text{erf}(z_i) = \text{erf}(z_i) \) are conjugate complex and lie far apart from each other. In the case of \( E(z) \) points of that nature can be found in an arbitrarily small neighborhood of the point \( z = ip \).

The actual situation is made clear by the diagram and tables given below. In Fig. 1 we show the curves \( R = |E| = \) constant and \( \gamma = \arg E = \) constant in the square \( 0 \leq x \leq 1.5, 0 \leq y \leq 1.5 \) of the \( z \)-plane. The table shows the values of \( E \) for \( z \) on the curve \( C \) (defined below). The values given were obtained by summing an adequate number of terms of the power series on the Datatron 205 at the California Institute of Technology; some were checked by comparison with the tables of Karpov \[4, 5\] from which values of \( E(z) \) can be obtained.

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2. Idea of proof. Since

\[ E(z) = \sum_{n=0}^{\infty} \frac{2^n}{1.35 \cdots (2n+1)} z^{2n+1}, \quad |z| < \infty, \]

we have \( E(\bar{z}) = \overline{E(z)} \) and \( E(-z) = -E(z) \) and may restrict our consideration to the first quadrant \( x \geq 0, y \geq 0 \) in the \( z \)-plane.

In the subsequent section we shall prove the following lemma.

**Lemma.**

\[ E(z_1) \neq E(z_2) \]

for any two points on the boundary \( C \) of the open sector \( S \) of the circular disk \( |z| < o \) in the first quadrant.
From this it follows, since $E(z)$ is entire and thus regular in $S \cup C$ that $E(z)$ maps $S$ conformally and one-to-one onto the interior of the simple closed curve $C^*$ corresponding to $C$ in the $w$-plane [3, p. 121]. This establishes our theorem.

3. Proof of the lemma. Let $z = re^{i\phi}$. The curve $C$ consists of
the segment $S_1$: $y = 0$, $0 < x < \rho$,
the circular arc $K$: $|z| = \rho$, $0 < \phi < \pi/2$,
the segment $S_2$: $x = 0$, $0 < y < \rho$.
and the three common end points of these three arcs.

(A) On $S_1$, $E(z)$ is real and increases steadily with $x$.
(B) On $S_2$, $E(z)$ is imaginary, and $v$ increases steadily with $y$.
(C) $v \neq 0$ on $K$.
(D) On $K$, $|E(z)|$ decreases steadily with increasing $\phi$.

(A) is obvious from (2.1), and (B) follows from the definition of $\rho$.

Proof of (C). Integrating along segments parallel to the coordinate
axes we have

$$v(x, y) = e^{-\nu^2}[\cos 2xy\int_0^y e^{-\tau^2}\cos 2\tau d\tau$$
$$+ \sin 2xy\left\{\int_0^z e^{-\tau^2}d\tau + \int_0^y e^{-\tau^2}\sin 2\tau d\tau\right\}].$$

In $\{x > 0, y > 0\} \cap \{|z| \leq \rho\}$ we have $\cos 2xy > 0$, $\sin 2xy > 0$. Therefore $v > 0$ on $K$.

Proof of (D). Integrating along a radius $\phi = \text{constant}$ from 0 to $\rho$
we have

$$E(z) = e^{i\phi} \int_0^\rho e^{h(r, \phi)}dr$$

where

$$h(r, \phi) = a(r, \phi) + ib(r, \phi),$$

$$a(r, \phi) = (\rho^2 - r^2)\cos 2\phi,\ b(r, \phi) = (\rho^2 - r^2)\sin 2\phi.$$

Therefore

$$|E|^2 = \int_0^\rho e^h dr \int_0^\rho e^h dr.$$

Differentiating with respect to $\phi$ and setting

$$h^* = a^* + ib^*,\ a^* = a(r^*, \phi),\ b^* = b(r^*, \phi),$$

$$f = \cos(b^* - b) - i\sin(b^* - b)$$

we obtain
\begin{align*}
(| E |^2)_\phi = & \int_0^\rho e^{i \phi} dr \int_0^\rho e^{i \phi^*} dr^* + \int_0^\rho e^{i \phi^*} dr^* \int_0^\rho e^{i \phi} dr \\
= & \int_0^\rho \int_0^\rho e^{\alpha + \alpha^*} \{ f h_\phi + \overline{f h_\phi} \} dr dr^* .
\end{align*}

Now

\[ a_\phi = -2(\rho^2 - r^2) \sin 2\phi, \quad b_\phi = 2(\rho^2 - r^2) \cos 2\phi \]

and therefore

\[ fh_\phi + \overline{f h_\phi} = 2 \Re fh_\phi = 2 [\cos (b^* - b) a_\phi + \sin (b^* - b) b_\phi] \]

\[ = -4(\rho^2 - r^2) \sin (\alpha(\phi)) \]

where

\[ \alpha(\phi) = 2\phi + b - b^* = (r^{*2} - r^2) \sin 2\phi + 2\phi . \]

This yields

\[ (| E |^2)_\phi = -4 \int_0^\rho \int_0^\rho e^{\alpha + \alpha^*} (\rho^2 - r^2) \sin (\alpha(\phi)) dr dr^* . \]

Since from (1.2) we have \( | r^{*2} - r^2 | < 1 \), we obtain

\[ \alpha'(\phi) = 2 + 2(r^{*2} - r^2) \cos 2\phi > 0 . \]

Hence \( \alpha(\phi), 0 \leq \phi \leq \pi/2 \), has its maximum at \( \phi = \pi/2 \). Therefore

\[ 0 \leq \alpha(\phi) < \pi \text{ when } 0 \leq \phi < \pi/2 \text{ and } \sin(\alpha(\phi)) > 0 \text{ when } 0 < \phi < \pi/2 . \]

This means that the integrand in (3.1) is positive in the region \( 0 \leq r \leq \rho, 0 \leq r^* \leq \rho \) for all \( \phi \) in the interval \( 0 < \phi < \pi/2 \). Thus \( (| E |^2)_\phi < 0 \) when \( 0 < \phi < \pi/2 \). This proves (D).

We note that (D) remains true if \( K \) is replaced by quadrants of circles of radii somewhat larger than \( \rho \); this, however, is of no interest here.

For \( z_1 \in K, z_2 \in S_2 \), or \( z_1 \in K, z_2 \in K \), equation (2.2) holds, as follows from (D). For \( z_1 \in K, z_2 \in S_1 \) the same is true because of (C). In the other cases, \( z_1 \in S_1, z_2 \in S_1 \), etc., the validity of (2.2) is obvious. This proves the lemma.

\section*{References}

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\[ \rho = 0.92413 \ 88730 . \]


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