ON A THEOREM DUE TO SZ.-NAGY

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B. Sz.-Nagy [4] has proved the following theorem:

**Theorem A.** Let \([T_t; t \geq 0]\) be a strongly continuous semi-group of contraction operators on a Hilbert space \(H\). Then there exists a group of unitary operators \([U_t, -\infty < t < \infty]\) on a larger Hilbert space \(H'\) such that

\[
T_t y = PUty, \quad y \in H, \, t \geq 0; 
\]

here \(P\) is the projection operator with range \(H\). Then space \(H\) can be chosen in a minimal fashion so that \([U_tH; -\infty < t < \infty]\) spans \(H\). In this case \([U_t]\) is strongly continuous and the structure \([H, U_t, H]\) is determined to within an isomorphism.\(^1\)

The infinitesimal generator \(L\) of the semi-group \([T_t]\) is defined by

\[
\lim_{\delta \to 0^+} \frac{1}{\delta} [T_\delta y - y] = Ly
\]

for all \(y \in H\) for which this limit exists. The operator \(L\) is linear and closed with dense domain, \(\mathcal{D}(L)\) (see [1]). It is shown in [2] that \(L\) is maximal dissipative in the sense that

\[
(y, Ly) + (Lx, y) \leq 0, \quad y \in \mathcal{D}(L),
\]

and \(L\) being maximal with respect to this property. Since \([U_t]\) is a semi-group as well as a group of operators, the infinitesimal generator \(L\) of \([U_t]\) also shares these properties; however in the case of a group of unitary operators \(iL\) is in addition self-adjoint.

The purpose of this note is to study the relation between \(L\) and \(L\). It turns out that \(L\) is a restriction of \(L\) only when \(L\) is maximal symmetric. In general \(L\) is neither a restriction nor a projection of \(L\); in fact \(\mathcal{D}(L) \cap H\) may contain only the zero element. Nevertheless we shall obtain \(H, L,\) and \([U_t]\) directly from \(L\), our principal tool being the discrete analogue of the above theorem, which is also due to Sz.-Nagy [4], namely

**Theorem B.** Let \(J\) be a contraction operator on a Hilbert space \(H\). Then there exists a unitary operator \(J\) on a larger Hilbert space \(H'\) such that

\[
J^n y = PJ^n y, \quad y \in H, \, n \geq 0; 
\]

here \(P\) is the projection operator with range \(H\). The space \(H\) can be

\(^1\) Two structures \([H, U_t, H]\) and \([H', U'_t, H]\) are isomorphic if there is a unitary map \(V\) of \(H\) onto \(H'\) which is the identity on \(H\) and is such that \(VU_t y = U'_t Vy\) for all \(y \in H\).
chosen in a minimal fashion in the sense that \([J^nH; -\infty < n < \infty]\) spans \(H\). In this case the structure \(\{H, J, H\}\) is determined to within an isomorphism.

For a maximal dissipative operator \(L\) with dense domain, it is shown in [2, §1.1] that \((I - L)\) is one-to-one with range \(R(I - L) = H\) and that
\[
J = (I + L)(I - L)^{-1}
\]
is a contraction operator with \(\mathcal{D}(J) = H\) and such that \((I + J)\) is one-to-one. Applying Theorem B we obtain the unitary operator \(J\) on the enlarged space \(H\) spanned by \([J^nH; -\infty < n < \infty]\) with \(J\) satisfying the property (4).

**Lemma 1.** The operator \((I + J)\) is one-to-one.

**Proof.** Let \(S\) be a contraction operator, set \(\mathcal{Z}(S) = [y; Sy + y = \theta]\), and denote the projection operator with range \(\mathcal{Z}(S)\) by \(P_S\). Then the ergodic theorem (see [3, pp. 400-406]) asserts that
\[
\text{st. lim}_{n \to \infty} (n + 1)^{-1} \sum_{n=0}^{\infty} (-S)^k = P_S
\]
and that \(SP_S = P_SP_S = -P_S\). We apply this result first to \(J\) and then to \(J\). Making use of (4) we see that
\[
PP_Jy = P_Sy, \quad y \in H.
\]
As noted above \(P_S = \Theta\), so that \(PP_JP = \Theta\). Actually \(P_JP = \Theta\); for otherwise there would exist a \(y \in H\) with \(P_Jy \neq \theta\) so that
\[
(P_PJP_Jy, y) = (P_Jy, y) = \|P_Jy\|^2 > 0,
\]
which is impossible. Thus \(P_JP = \Theta\) and hence \(\mathcal{Z}(J)\) is orthogonal to \(H\). But this means that
\[
P_JJ^nH = J^nP_JH = \theta,
\]
and we infer that \(J^nH\) is orthogonal to \(\mathcal{Z}(J)\) for all \(n\). The minimal property of \(H\) therefore requires that \(\mathcal{Z}(J) = \Theta\).

**Remark.** Associated with \(J\) is the resolution of the identity \([E(\sigma); -\pi < \sigma \leq \pi]\) and the integral representation
\[
J^n = \int_{-\pi}^{\pi} \exp(i\sigma)dE(\sigma).
\]
Setting the restriction of \(PE(\sigma)\) to \(H\) equal to \(F(\sigma)\) we see by (4) that
\[
J^n = \int_{-\pi}^{\pi} \exp(i\sigma)dF(\sigma).
\]
The argument used in Lemma 1 applied to \(S = \exp(i\mu)J\) shows that if
J has no eigenvalues of absolute value one, then neither does J and hence that both \( E(\sigma) \) and \( F(\sigma) \) are strongly continuous in \( \sigma \). Conversely, \( F(\sigma) \) is strongly continuous then as is readily verified

\[
(n + 1)^{-1} \sum_{\mu=0}^{n-1} \exp(i\mu J)y = \int_{-\pi}^{\pi} K_n(\sigma + \mu) dF(\sigma)y \to \theta , \quad y \in H ;
\]

here

\[
K_n(\sigma) = (n+1)^{-1} \exp(i\sigma/2) \sin \left( \frac{n+1}{2} \sigma \right) \sin \left( \frac{\sigma}{2} \right)^{-1}.
\]

It then follows from the ergodic theorem that \( \mathbb{1}_{\{-\exp(i\mu J)\}} = \theta \) and hence that \( J \) has no eigenvalues of absolute value one.

**Theorem.** Set

\[
L = (J - I)(J + I)^{-1}.
\]

Then \( L \) generates a strongly continuous group of unitary operators \( [U_t; -\infty < t < \infty] \) such that

\[
T_t y = PU_t y , \quad y \in H , t \geq 0
\]

and \( [U_t H; -\infty < t < \infty] \) spans \( H \).

**Proof.** It follows from the above lemma that \( (I + J) \) is one-to-one and hence that \( L \) is well-defined. Moreover \( \sigma(L) = \Re(I + J) \) is necessarily dense in \( H \) since otherwise \( (I + J^*) \) would nullify some non-zero vector and since \( J^{-1} = J^* \) the same would be true of \( (I + J) \). Further it is clear that \( iL \) is the Cayley transform of \( iJ \) and hence \( L \) generates a strongly continuous group of unitary operators which we shall denote by \( [U_t] \). In order to verify (7) we proceed to represent the resolvent \( R(\lambda, L) = (\lambda I - L)^{-1} \) in terms of \( J \) for \( \lambda > 0 \). We see from (5) that

\[
y = 2^{-1}(Ju + u) \quad \text{and} \quad Ly = 2^{-1}(Ju - u) , \quad u \in H .
\]

Suppose next that \( \lambda y - Ly = f \). Replacing \( y \) by \( u \) as in (8) we obtain

\[
2^{-1} \lambda (Ju + u) - 2^{-1} (Ju - u) = f
\]

so that

\[
u = 2(1 + \lambda)^{-1} \sum_{n=1}^{\infty} [(1 - \lambda)(1 + \lambda)^{-1}]^n J^n f , \quad \lambda > 0 .
\]

Again making use of (8) we get

\[
y = 2^{-1}(Ju + u) = \sum_{n=0}^{\infty} a_n(\lambda) J^n f
\]

where
\[ a_0(\lambda) = (1 + \lambda)^{-1} \text{ and } a_n(\lambda) = 2(1 - \lambda)^{n-1}(1 + \lambda)^{-n-1} \text{ for } n > 0. \]

Thus \( R(\lambda, L) \) can be represented by an absolutely convergent series in powers of \( J \) for \( \lambda > 0 \). Taking powers of \( R(\lambda, L) \) we see that

\[ [R(\lambda, L)]^k = \sum_{n=0}^{\infty} a_n^{(k)}(\lambda)J^n, \]

where again the series is absolutely convergent. Similarly

\[ R(\lambda, L)^k = \sum_{n=0}^{\infty} a_n^{(k)}(\lambda)J^n, \]

and it follows from (4) that

\[ (9) \quad [R(\lambda, L)]^k y = P[R(\lambda, L)]^k y, \quad y \in H, k \geq 0, \lambda > 0. \]

According to Yosida's proof of the Hille-Yosida theorem (see [1]),

\[ (10) \quad T_k = \lim_{\lambda \to -\infty} \exp(\lambda B_\lambda) \text{ and } U_k = \lim_{\lambda \to \infty} \exp(\lambda B_\lambda), \quad t \geq 0, \]

where

\[ B_\lambda = \lambda^2 R(\lambda, L) - \lambda I \text{ and } B_\lambda = \lambda^2 R(\lambda, L) - \lambda I. \]

Thus for \( y \in H \) the relation (9) implies

\[ \exp(t B_\lambda)y = P \exp(t B_\lambda)y, \quad y \in H, \lambda > 0, \]

and this together with (10) gives (7).

It remains to prove that \( H \) is the same as

\[ H_0 = \text{closed linear extension of } [U_t H; -\infty < t < \infty]. \]

Let \( P_0 \) be the projection of \( H \) onto \( H_0 \). Then clearly \( U_t H_0 \subseteq H_0 \) for all real \( t \), and since \( U_t^* = U_{-t} \) the same is true of the orthogonal complement to \( H_0 \). As a consequence \( P_0 U_t = U_t P_0 \) for all real \( t \). Hence for \( y \in \mathcal{D}(L) \)

\[ P_0 Ly = \lim_{\delta \to +0} (P_0 U_\delta y - \delta y) = \lim_{\delta \to +0} (U_\delta P_0 y - P_0 y) = LP_0 y. \]

Thus \( P_0 \) commutes with \( L \) and hence with \( J \). But since \( H \) is obviously contained in \( H_0 \) we have

\[ J^n H = J^n P_0 H = P_0 J^n H \subseteq H_0. \]

The minimal property of \( H \) asserted in Theorem B therefore implies that \( H = H_0 \). This concludes the proof of the theorem.

It should be noted that since \( iL \) is self-adjoint, the largest restriction to \( H \) of \( iL \) will be symmetric. On the other hand if \( iL \) is symmetric then it is easily verified that \( J \) is an isometry and hence that \( J \) is an extension of \( J \); in this case then \( L \) will be an extension of \( L \). However in general if \( u \in H \) and \( y = Ju + u \), then \( z = Py = Ju + u \in \mathcal{D}(L) \)
and $LPy = PLy$; each $z \in \mathcal{D}(L)$ can be so represented. A simple example shows that $\mathcal{D}(L) \cap H$ may contain only the zero element.\(^3\)

**REFERENCES**


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\(^2\) Suppose $H$ is one-dimensional and $T_t = \exp(-\ell t)$. The Sz.-Nagy construction for $H$ in Theorem B then results in $H = l_2$, the space of complex-valued sequences $y = \{\eta_n\}; -\infty < n < \infty$ with

$$
(y, \xi) = \sum_{n=\infty}^{\infty} \overline{\eta_n} \xi_n ,
$$

$J(\eta_n) = \{\eta_{n-1}\}$, and $P(\eta_n) = \{\eta'_n\} (\eta'_0 = \eta'_1 = 0 \text{ for } n \neq 0)$. Then relation (8) as applied to $J$ and $L$ asserts that for each $\{\eta_n\} \in \mathcal{D}(L)$ there is a $\{\mu_n\} \in H$ such that

$$
2\eta_n = \mu_{n-1} + \mu_n , \quad 2[L(\eta_n)]_n = \mu_{n-1} - \mu_n .
$$

If we also require that $\{\eta_n\} \in H$, then $\mu_{n-1} + \mu_n = 0$ for all $n \neq 0$ and this together with the condition $\sum |\mu_n|^2 < \infty$ implies that $\mu_n = 0$ for all $n$. It follows that $\mathcal{D}(L) \cap H = \emptyset$. 

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