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ON A THEOREM DUE TO SZ.-NAGY

R. S. PHILLIPS

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B. Sz.-Nagy [4] has proved the following theorem:

THEOREM A. *Let $[T_t; t \geq 0]$ be a strongly continuous semi-group of contraction operators on a Hilbert space H . Then there exists a group of unitary operators $[U_t, -\infty < t < \infty]$ on a larger Hilbert space \mathbf{H} such that*

$$(1) \quad T_t y = \mathbf{P} U_t y, \quad y \in H, t \geq 0;$$

here \mathbf{P} is the projection operator with range H . Then space \mathbf{H} can be chosen in a minimal fashion so that $[U_t H; -\infty < t < \infty]$ spans \mathbf{H} . In this case $[U_t]$ is strongly continuous and the structure $\{\mathbf{H}, U_t, H\}$ is determined to within an isomorphism.¹

The infinitesimal generator L of the semi-group $[T_t]$ is defined by

$$(2) \quad \lim_{\delta \rightarrow 0+} \delta^{-1} [T_\delta y - y] = Ly$$

for all $y \in H$ for which this limit exists. The operator L is linear and closed with dense domain, $\mathfrak{D}(L)$ (see [1]). It is shown in [2] that L is maximal dissipative in the sense that

$$(3) \quad (y, Ly) + (Ly, y) \leq 0, \quad y \in \mathfrak{D}(L),$$

and L being maximal with respect to this property. Since $[U_t]$ is a semi-group as well as a group of operators, the infinitesimal generator \mathbf{L} of $[U_t]$ also shares these properties; however in the case of a group of unitary operators $i\mathbf{L}$ is in addition self-adjoint.

The purpose of this note is to study the relation between L and \mathbf{L} . It turns out that L is a restriction of \mathbf{L} only when L is maximal symmetric. In general L is neither a restriction nor a projection of \mathbf{L} ; in fact $\mathfrak{D}(\mathbf{L}) \cap H$ may contain only the zero element. Nevertheless we shall obtain \mathbf{H} , \mathbf{L} , and $[U_t]$ directly from L , our principal tool being the discrete analogue of the above theorem, which is also due to Sz.-Nagy [4], namely

THEOREM B. *Let J be a contraction operator on a Hilbert space H . Then there exists a unitary operator \mathbf{J} on a larger Hilbert space \mathbf{H} such that*

$$(4) \quad J^n y = \mathbf{P} \mathbf{J}^n y, \quad y \in H, n \geq 0;$$

here \mathbf{P} is the projection operator with range H . The space \mathbf{H} can be

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¹ Two structures $\{\mathbf{H}, U_t, H\}$ and $\{\mathbf{H}', U'_t, H\}$ are isomorphic if there is a unitary map \mathbf{V} of \mathbf{H} onto \mathbf{H}' which is the identity on H and is such that $\mathbf{V} U_t y = U'_t \mathbf{V} y$ for all $y \in H$.

chosen in a minimal fashion in the sense that $[J^n H; -\infty < n < \infty]$ spans H . In this case the structure $\{H, J, H\}$ is determined to within an isomorphism.

For a maximal dissipative operator L with dense domain, it is shown in [2, §1.1] that $(I - L)$ is one-to-one with range $\Re(I - L) = H$ and that

$$(5) \quad J = (I + L)(I - L)^{-1}$$

is a contraction operator with $\mathfrak{D}(J) = H$ and such that $(I + J)$ is one-to-one. Applying Theorem B we obtain the unitary operator J on the enlarged space H spanned by $[J^n H; -\infty < n < \infty]$ with J satisfying the property (4).

LEMMA 1. *The operator $(I + J)$ is one-to-one.*

Proof. Let S be a contraction operator, set $\mathfrak{Z}(S) = [y; Sy + y = \theta]$, and denote the projection operator with range $\mathfrak{Z}(S)$ by P_S . Then the ergodic theorem (see [3, pp. 400-406]) asserts that

$$\text{st. lim}_{n \rightarrow \infty} (n + 1)^{-1} \sum_{k=0}^n (-S)^k = P_S$$

and that $SP_S = P_S S = -P_S$. We apply this result first to J and then to J . Making use of (4) we see that

$$PP_J y = P_J y, \quad y \in H.$$

As noted above $P_J = \Theta$, so that $PP_J P = \Theta$. Actually $P_J P = \Theta$; for otherwise there would exist a $y \in H$ with $P_J y \neq \theta$ so that

$$(PP_J P y, y) = (P_J y, y) = \|P_J y\|^2 > 0,$$

which is impossible. Thus $P_J P = \Theta$ and hence $\mathfrak{Z}(J)$ is orthogonal to H . But this means that

$$P_J J^n H = J^n P_J H = \theta,$$

and we infer that $J^n H$ is orthogonal to $\mathfrak{Z}(J)$ for all n . The minimal property of H therefore requires that $\mathfrak{Z}(J) = \theta$.

REMARK. Associated with J is the resolution of the identity $[E(\sigma); -\pi < \sigma \leq \pi]$ and the integral representation

$$J^n = \int_{-\pi}^{\pi} \exp(in\sigma) dE(\sigma).$$

Setting the restriction of $PE(\sigma)$ to H equal to $F(\sigma)$ we see by (4) that

$$J^n = \int_{-\pi}^{\pi} \exp(in\sigma) dF(\sigma).$$

The argument used in Lemma 1 applied to $S = \exp(i\mu)J$ shows that if

J has no eigenvalues of absolute value one, then neither does \mathbf{J} and hence that both $\mathbf{E}(\sigma)$ and $F(\sigma)$ are strongly continuous in σ . Conversely, $F(\sigma)$ is strongly continuous then as is readily verified

$$(n + 1)^{-1} \sum_{k=0}^n [\exp(i\mu)J]^k y = \int_{-\pi}^{\pi} K_n(\sigma + \mu) dF(\sigma) y \rightarrow \theta, \quad y \in H;$$

here

$$K_n(\sigma) = (n + 1)^{-1} \exp(in\sigma/2) \sin \left[\frac{n+1}{2} \sigma \right] \left[\sin \frac{\sigma}{2} \right]^{-1}.$$

It then follows from the ergodic theorem that $\mathfrak{B}\{-\exp(i\mu)J\} = \theta$ and hence that J has no eigenvalues of absolute value one.

THEOREM. *Set*

$$(6) \quad \mathbf{L} = (\mathbf{J} - \mathbf{I})(\mathbf{J} + \mathbf{I})^{-1}.$$

Then \mathbf{L} generates a strongly continuous group of unitary operators $[\mathbf{U}_t; -\infty < t < \infty]$ such that

$$(7) \quad T_t y = \mathbf{P} \mathbf{U}_t y, \quad y \in H, t \geq 0$$

and $[\mathbf{U}_t H; -\infty < t < \infty]$ spans \mathbf{H} .

Proof. It follows from the above lemma that $(\mathbf{I} + \mathbf{J})$ is one-to-one and hence that \mathbf{L} is well-defined. Moreover $\mathfrak{D}(\mathbf{L}) = \mathfrak{R}(\mathbf{I} + \mathbf{J})$ is necessarily dense in \mathbf{H} since otherwise $(\mathbf{I} + \mathbf{J}^*)$ would nullify some non-zero vector and since $\mathbf{J}^{-1} = \mathbf{J}^*$ the same would be true of $(\mathbf{I} + \mathbf{J})$. Further it is clear that $i\mathbf{L}$ is the Cayley transform of $i\mathbf{J}$ and hence \mathbf{L} generates a strongly continuous group of unitary operators which we shall denote by $[\mathbf{U}_t]$. In order to verify (7) we proceed to represent the resolvent $R(\lambda, L) = (\lambda I - L)^{-1}$ in terms of J for $\lambda > 0$. We see from (5) that

$$(8) \quad y = 2^{-1}(Ju + u) \text{ and } Ly = 2^{-1}(Ju - u), \quad u \in H.$$

Suppose next that $\lambda y - Ly = f$. Replacing y by u as in (8) we obtain

$$2^{-1}\lambda(Ju + u) - 2^{-1}(Ju - u) = f$$

so that

$$u = 2(1 + \lambda)^{-1} \sum_{n=0}^{\infty} [(1 - \lambda)(1 + \lambda)^{-1}]^n J^n f, \quad \lambda > 0.$$

Again making use of (8) we get

$$y = 2^{-1}(Ju + u) = \sum_{n=0}^{\infty} a_n(\lambda) J^n f$$

where

$$a_0(\lambda) = (1 + \lambda)^{-1} \text{ and } a_n(\lambda) = 2(1 - \lambda)^{n-1}(1 + \lambda)^{-n-1} \text{ for } n > 0 .$$

Thus $R(\lambda, L)$ can be represented by an absolutely convergent series in powers of J for $\lambda > 0$. Taking powers of $R(\lambda, L)$ we see that

$$[R(y, L)]^k = \sum_{n=0}^{\infty} a_n^{(k)}(\lambda) J^n ,$$

where again the series is absolutely convergent. Similarly

$$\mathbf{R}(\lambda, \mathbf{L})^k = \sum_{n=0}^{\infty} a_n^{(k)}(\lambda) \mathbf{J}^n ,$$

and it follows from (4) that

$$(9) \quad [R(\lambda, L)]^k y = \mathbf{P}[\mathbf{R}(\lambda, \mathbf{L})]^k y, \quad y \in H, k \geq 0, \lambda > 0 .$$

According to Yosida's proof of the Hille-Yosida theorem (see [1]),

$$(10) \quad T_t = \text{st. lim}_{\lambda \rightarrow \infty} \exp(tB_\lambda) \text{ and } U_t = \text{st. lim}_{\lambda \rightarrow \infty} \exp(t\mathbf{B}_\lambda), \quad t \geq 0 ,$$

where

$$B_\lambda = \lambda^2 R(\lambda, L) - \lambda I \text{ and } \mathbf{B}_\lambda = \lambda^2 \mathbf{R}(\lambda, \mathbf{L}) - \lambda \mathbf{I} .$$

Thus for $y \in H$ the relation (9) implies

$$\exp(tB_\lambda)y = \mathbf{P} \exp(t\mathbf{B}_\lambda)y, \quad y \in H, \lambda > 0 ,$$

and this together with (10) gives (7).

It remains to prove that \mathbf{H} is the same as

$$\mathbf{H}_0 = \text{closed linear extension of } [U_t H; -\infty < t < \infty] .$$

Let \mathbf{P}_0 be the projection of \mathbf{H} onto \mathbf{H}_0 . Then clearly $U_t \mathbf{H}_0 \subset \mathbf{H}_0$ for all real t , and since $U_t^* = U_{-t}$ the same is true of the orthogonal complement to \mathbf{H}_0 . As a consequence $\mathbf{P}_0 U_t = U_t \mathbf{P}_0$ for all real t . Hence for $y \in \mathfrak{D}(\mathbf{L})$

$$\mathbf{P}_0 \mathbf{L} y = \lim_{\delta \rightarrow 0+} \delta^{-1} (\mathbf{P}_0 U_\delta y - \mathbf{P}_0 y) = \lim_{\delta \rightarrow 0+} \delta^{-1} (U_\delta \mathbf{P}_0 y - \mathbf{P}_0 y) = \mathbf{L} \mathbf{P}_0 y .$$

Thus \mathbf{P}_0 commutes with \mathbf{L} and hence with \mathbf{J} . But since H is obviously contained in \mathbf{H}_0 we have

$$\mathbf{J}^n H = \mathbf{J}^n \mathbf{P}_0 H = \mathbf{P}_0 \mathbf{J}^n H \subset \mathbf{H}_0 .$$

The minimal property of \mathbf{H} asserted in Theorem B therefore implies that $\mathbf{H} = \mathbf{H}_0$. This concludes the proof of the theorem.

It should be noted that since $i\mathbf{L}$ is self-adjoint, the largest restriction to H of $i\mathbf{L}$ will be symmetric. On the other hand if iL is symmetric then it is easily verified that J is an isometry and hence that \mathbf{J} is an extension of J ; in this case then \mathbf{L} will be an extension of L . However in general if $u \in H$ and $y = Ju + u$, then $z = \mathbf{P}y = Ju + u \in \mathfrak{D}(L)$

and $LPy = PLy$; each $z \in \mathfrak{D}(L)$ can be so represented. A simple example shows that $\mathfrak{D}(L) \cap H$ may contain only the zero element.²

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² Suppose H is one-dimensional and $T_t = \exp(-t)$. The Sz.-Nagy construction for \mathbf{H} in Theorem B then results in $\mathbf{H} = l_2$, the space of complex-valued sequences $y = \{\gamma_n; -\infty < n < \infty\}$ with

$$(y, z) = \sum_{n=-\infty}^{\infty} \bar{\gamma}_n \bar{\zeta}_n,$$

$\mathbf{J}\{\gamma_n\} = \{\gamma_{n-1}\}$, and $\mathbf{P}\{\gamma_n\} = \{\gamma'_n\}$ ($\gamma'_0 = \gamma_0$; $\gamma'_n = 0$ for $n \neq 0$). Then relation (8) as applied to \mathbf{J} and \mathbf{L} asserts that for each $\{\gamma_n\} \in \mathfrak{D}(\mathbf{L})$ there is a $\{\mu_n\} \in \mathbf{H}$ such that

$$2\gamma_n = \mu_{n-1} + \mu_n, \quad 2[\mathbf{L}\{\gamma_n\}]_n = \mu_{n-1} - \mu_n.$$

If we also require that $\{\gamma_n\} \in \mathbf{H}$, then $\mu_{n-1} + \mu_n = 0$ for all $n \neq 0$ and this together with the condition $\sum |\mu_n|^2 < \infty$ implies that $\mu_n = 0$ for all n . It follows that $\mathfrak{D}(\mathbf{L}) \cap \mathbf{H} = \theta$.

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