Pacific Journal of Mathematics

FREDHOLM EIGEN VALUES OF MULTIPLY-CONNECTED DOMAINS

M. SCHIFFER

Vol. 9, No. 1 May 1959

FREDHOLM EIGEN VALUES OF MULTIPLY-CONNECTED DOMAINS

M. SCHIFFER

Introduction. The solution of the boundary value problems of potential theory can be reduced, according to Poincaré, to an inhomogeneous integral equation of the second kind. It was the study of this particular problem which led, at the beginning of this century, to the development of the modern integral equation theory at the hands of Fredholm and Hilbert. From the beginning, attention was drawn to the eigen value problem for the homogeneous integral equation with the potential theoretical kernel [10]. The eigen functions of this problem can be extended as harmonic functions into the domain considered as well as extended into the complementary domain and give rise to interesting series developments and to a theory relating solutions of the interior and exterior boundary value problems of a closed curve or surface.

In a preceding paper [17], these Fredholm eigen functions were applied to problems of conformal mapping of simply-connected plane domains. Their connection with the dielectric Green's function of such domains was discussed and we showed the possibility of obtaining univalent functions by means of the dielectric Green's function. A variational formula for the Fredholm eigen values was established and an extremum problem for the latter was solved which permitted one to estimate the convergence of the Neumann-Liouville series solving the Dirichlet and Neumann boundary value problems.

In the present paper, the Fredholm eigen value problem is studied in the case of multiply-connected plane domains. Various new difficulties arise in this case. The complementary region of a multiply-connected domain is a domain set and the number of trivial solutions of the problem with the eigen values $|\lambda| = 1$ increases. This fact necessitates a brief restatement of the basic definitions and concepts in § 1. A certain repetition and overlap of material with the preceding paper could not be avoided; but, on the other hand, the presentation of this section makes the paper self-contained and should facilitate the understanding of it.

In § 2, the dielectric Green's functions $g_\varepsilon(z, \zeta)$ of a multiply-connected domain are discussed and their Fourier development in terms of the Fredholm eigen functions is given. The functions g_ε are of geometric-physical significance by themselves; moreover, they represent a one-parameter $(0 < \varepsilon < \infty)$ family of harmonic positive-definite kernels which

Received October 1, 1958. Prepared under contract Nonr-225 (11) for Office of Naval Research.

have also the Fredholm functions as eigen functions. For $\varepsilon = 1$, $g_\varepsilon(z, \zeta)$ reduces to the fundamental singularity $-\log|z-\zeta|$ and leads to the classical kernel of potential theory. A power series development of the dielectric Green's function in terms of $(\varepsilon - 1)/(\varepsilon + 1)$ is given; the coefficient kernels are elementary and can be calculated explicitly by integration of simple functions over the boundary curve system.

The role of the one-parameter family $g_\varepsilon(z, \zeta)$ becomes particularly interesting when one studies the limit cases $\varepsilon = 0$ and $\varepsilon = \infty$. This is done in § 3. It appears that this function family interpolates between two well-known harmonic functions which determine two important canonical mappings of the domain considered; namely the radial-slit mapping and the circular-slit mapping.

In § 4 it is proved that not only the limit cases $\varepsilon = 0$ and $\varepsilon = \infty$ of $g_\varepsilon(z, \zeta)$ give rise to univalent functions in the domain but that each dielectric Green's function does so. We obtain one-parameter families of univalent functions which connect the radial-slit mapping function continuously with the circular-slit mapping function via any prescribed univalent function in the domain. This result is applied to give a new proof for the extremum properties which characterize the above two canonical slit mappings. Another type of one-parameter sets of univalent functions is constructed which interpolates between the canonical parallel-slit mappings.

In § 5, we use the dielectric Green's functions in order to define various norms and scalar products. These are quadratic and bilinear functionals defined for harmonic functions in the multiply-connected domain $\tilde{D}$ as well as for functions harmonic in the complementary domain set D. If one pair of argument functions is defined in $\tilde{D}$, the other pair in D, and if relations between their boundary values on the separating curve system are assumed, equations between the various scalar products are obtained. It is shown that these identities yield estimates and Ritz procedures for solution of boundary value problems in $\tilde{D}$ if the corresponding boundary value problems for the complementary set D are already solved. In the special case $\varepsilon = 1$ the procedure becomes, of course, particularly easy to apply since the dielectric Green's function becomes trivial. It has, indeed, already been used in this form in order to prove interesting isoperimetric inequalities for polarization and for virtual mass [18–20]. The extension of the method to the case of general ε should increase its flexibility and clarify its significance. The various quadratic forms are used, finally, in order to characterize each Fredholm eigen value $|\lambda| > 1$ by the solution of a simple maximum problem without side conditions. This result lays the groundwork for proving the variational formula for the Fredholm eigen values in the next section. The extremum definition is also used in order to prove that all positive

Fredholm eigen values of a subsystem of curves are never less than the corresponding positive eigen values of the full curve system.

In § 6, we derive the variational formula for the dielectric Green's functions under a small deformation of the domain. Through the maximum definition of the Fredholm eigen values, we can derive from this result also the variational formula for the Fredholm eigen values under the same deformation, This formula could also have been obtained immediately from the general perturbation theory of operators. But it seems of methodological interest to utilize fully the maximum property of each eigen value in order to give an elementary proof for this formula.

In order to avoid a discussion of possible degeneration of eigen values it is convenient to deal with symmetric functions of all eigen values and their variation, instead of considering individual eigen values. For this purpose, we define in § 7 the Fredholm determinant of a domain ; this concept is rather natural when one comes from the general theory of integral equations. The variational formula for the Fredholm determinant is easily expressed in terms of a complex kernal closely connected with the dielectric Green's function which possesses, moreover, as limit case a kernel well-known in the theory of conformal mapping. Indeed, the variation of the Fredholm determinant for the particular value 1 of the argument is described by this classical kernel itself.

In § 8, at last, we apply the results of the preceding section in order to solve an extremum problem for univalent functions in a multiply-connected domain and involving the Fredholm determinant. This solution gives a new proof for the possibility to map every domain conformally onto a domain bounded by circumferences and characterizes this canonical domain as an extremum domain of a simple variational problem. The treatment of the variational problem for the Fredholm determinant seems also of interest from the methodological point of view and for the general theory of variations of domain functions. In general, one knows from the theory of normal families that a solution of an extremum problem for the family of functions, univalent in a given domain and with specified normalization, does exist ; the method of variations has only the task to characterize the extremum domain. In our present problem, we had to restrict ourselves to univalent functions which are analytic in the closed domain in order to be sure of the existence of the Fredholm determinant. In this case, the theory of normal families does not guarantee the existence of an extremum function of equal character. We do not characterize, therefore, the extremum function by our variations, but rather an extremum sequence within the function class, considered. We prove from the very extremum property of the sequence that its limit function does, indeed, belong to the same class and has, moreover, certain characterizing properties. This procedure is very general and may have numerous analogous applications.

1. The Fredholm eigen value problem. Let $\tilde{D}$ be a domain in the complex z-plane containing the point at infinity; let its boundary C consist of N closed curves C_j each of which is three times continuously differentiable. We denote the interior of each C_j by D_j and the union of the N domains D_j by D.

We define the kernel

$$k(z,\zeta)=\frac{\partial}{\partial n_\zeta}\log\frac{1}{|z-\zeta|} \qquad \zeta\in C \tag{1}$$

where $\boldsymbol{n}_\zeta$ denotes the normal of C at ζ pointing into D. It is well known that, under our assumptions about C, the kernel $k(z,\zeta)$ is continuous in both its arguments as long as they are restricted to C.

We want to discuss the eigen value problem

$$\varphi_\nu(z)=\frac{\lambda_\nu}{\pi}\int_C k(z,\zeta)\varphi_\nu(\zeta)ds_\zeta\,, \qquad z\in C \tag{2}$$

which plays an important role in many boundary value problems of potential theory with respect to the multiply-connected domain $\tilde{D}$. The $\varphi_\nu(z)$ and the λ_ν are called the Fredholm eigen functions and the Fredholm eigen values, respectively, of the curve system C. The study of the Fredholm eigen value problem is facilitated by the fact that the kernel $k(z,\zeta)$ is, for fixed $\zeta\in C$, defined and harmonic for all values $z\neq\zeta$ in the complex plane. The integral in (2) represents, therefore, a harmonic function in $\tilde{D}$ and a set of different harmonic functions in D. We shall use the notation

$$\frac{\lambda_\nu}{\pi}\int_C k(z,\zeta)\varphi_\nu(\zeta)ds_\zeta=\begin{cases}h_\nu(z) & \text{for } z\in D\\ \tilde{h}_\nu(z) & \text{for } z\in\tilde{D}\,.\end{cases} \tag{3}$$

The set of harmonic functions $\tilde{h}_\nu(z)$ and $h_\nu(z)$ can be interpreted as the potential due to a double layer of logarithmic charges, spread along C with the density $(\lambda_\nu/\pi)\varphi_\nu(\zeta)$. Hence, the well known discontinuity character of such potentials leads to the boundary relations at each point

$$\lim_{z\to z_0}h_\nu(z)=(1+\lambda_\nu)\varphi_\nu(z_0)\,,\quad \lim_{z\to z_0}\tilde{h}_\nu(z)=(1-\lambda_\nu)\varphi_\nu(z_0)\,, \tag{4}$$

and

$$\frac{\partial}{\partial n}h_\nu(z_0)=-\frac{\partial}{\partial\tilde{n}}\tilde{h}_\nu(z_0)\,, \tag{4'}$$

where $\tilde{n}$ denotes the normal of C pointing into $\tilde{D}$.

The Fredholm eigen value problem may thus be formulated as the following question of potential theory which is of interest by itself:

To determine a harmonic function $\tilde{h}$ in $\tilde{D}$ and a set of harmonic functions h in D which have equal normal derivatives and proportional boundary values on C!. It is easily seen that the two problems are completely equivalent and that the possible factors of proportionality in the second problem are simple functions of the Fredholm eigen values λ_ν.

Instead of the harmonic functions h_ν and $\tilde{h}_\nu$, we may consider their complex derivatives, i.e., the analytic functions

$$(5) \qquad v_\nu(z) = \frac{\partial}{\partial z} h_\nu(z), \quad \tilde{v}_\nu(z) = \frac{\partial}{\partial z} \tilde{h}_\nu(z).$$

In view of definition (3) and by our assumption on C it can be asserted that v_ν and $\tilde{v}_\nu$ are continuous in $D + C$ and $\tilde{D} + C$, respectively. In order to translate the relations (4) and (4′) into terms involving v_ν and $\tilde{v}_\nu$, we use the parametric representation $z = z(s)$ of C by means of the arc length s and introduce

$$(6) \qquad z' = \frac{dz}{ds},$$

the unit vector at $z(s)$ in direction of the tangent of C. We can then write (4) and (4′) in the form

$$(7) \qquad \Re\left\{\frac{\partial h_\nu}{\partial z} z'\right\} = \frac{1 + \lambda_\nu}{1 - \lambda_\nu} \Re\left\{\frac{\partial \tilde{h}_\nu}{\partial z} z'\right\}, \quad \Im\left\{\frac{\partial h_\nu}{\partial z} z'\right\} = \Im\left\{\frac{\partial \tilde{h}_\nu}{\partial z} z'\right\},$$

and combine these two equation into the one complex equation

$$(8) \qquad v_\nu(z)z' = \frac{1}{1 - \lambda_\nu} \tilde{v}_\nu(z)z' + \frac{\lambda_\nu}{1 - \lambda_\nu} \overline{\tilde{v}_\nu(z)z'}, \qquad z = z(s).$$

Introducing (8) into the Cauchy identity. We obtain for $\zeta \in D$

$$(9) \qquad v_\nu(\zeta) = \frac{1}{2\pi i} \oint_C \frac{v_\nu(z)}{z - \zeta} dz = \frac{\lambda_\nu}{1 - \lambda_\nu} \frac{1}{2\pi i} \oint_C \frac{\overline{(\tilde{v}_\nu(z)dz)}}{z - \zeta}$$

while the use of the equation conjugate to (8) leads to

$$(10) \qquad \frac{1}{2\pi i} \oint_C \frac{\overline{(v_\nu(z)dz)}}{z - \zeta} = \frac{1}{1 - \lambda_\nu} \frac{1}{2\pi i} \oint_C \frac{\overline{(\tilde{v}_\nu(z)dz)}}{z - \zeta}, \qquad \zeta \in D.$$

Combining (9) and (10), we arrive thus at the following integral equation for v_ν:

$$(11) \qquad v_\nu(\zeta) = \frac{\lambda_\nu}{2\pi i} \oint_C \frac{\overline{(v_\nu(z)dz)}}{z - \zeta}.$$

In the same way we prove the analogous equations

$$\tilde{v}_\nu(\zeta) = \frac{\lambda_\nu}{1-\lambda_\nu} \cdot \frac{1}{2\pi i} \oint_C \frac{\overline{(v_\nu(z)dz)}}{z-\zeta} \tag{9'}$$

and

$$\tilde{v}_\nu(\zeta) = \frac{\lambda_\nu}{2\pi i} \oint_C \frac{\overline{(\tilde{v}_\nu(z)dz)}}{z-\zeta} . \tag{11'}$$

In all these formulas the integration over the curve system C has to be performed in the positive sense with respect to D.

The line integrals in (9), (9') and (11), (11') can be transformed into area integrals and the integral equations take the forms

$$\frac{\lambda_\nu}{\pi} \iint_D \frac{\overline{v_\nu(z)}}{(z-\zeta)^2} d\tau_z = \begin{cases} v_\nu(\zeta) & \text{for } \zeta \in D \\ (1+\lambda_\nu)\tilde{v}_\nu(\zeta) & \text{for } \zeta \in \tilde{D} \end{cases} \tag{12}$$

and

$$-\frac{\lambda_\nu}{\pi} \iint_{\tilde{D}} \frac{\overline{\tilde{v}_\nu(z)}}{(z-\zeta)^2} d\tau_z = \begin{cases} (1-\lambda_\nu)v_\nu(\zeta) & \text{for } \zeta \in D \\ \tilde{v}_\nu(\zeta) & \text{for } \zeta \in \tilde{D} . \end{cases} \tag{13}$$

In both integrals $d\tau_z$ denotes the area element with respect to the variable z and the integrals have to be interpreted in the Cauchy principal sense whenever they become improper.

The transformation

$$F(\zeta) = \frac{1}{\pi} \iint_E \frac{\overline{f(z)}}{(z-\zeta)^2} d\tau_z \tag{14}$$

carries every L^2-integrable function $f(z)$ defined in the complex plane E into a new function $F(z)$ of the same class and with the same norm :

$$\iint_E |F(z)|^2 d\tau = \iint_E |f(z)|^2 d\tau . \tag{15}$$

This functional transformation plays a role in many problems of function theory [1, 3, 4] and is called the "Hilbert integral transformation". The integral equations (12) and (13) show the close connection between the theories of the Fredholm eigen functions and of the Hilbert transforms of analytic functions.

We introduce next the Green's functions of the domain $\tilde{D}$ and of the set of domains D. While the Green's function $\tilde{g}(z, \zeta)$ of $\tilde{D}$ is defined as usual, the Green's function $g(z, \zeta)$ of D is given by the equation

$$g(z,\zeta) = \begin{cases} g_j(z,\zeta) & \text{for } z, \zeta \in D_j \\ 0 & \text{for } z \in D_j\, \zeta \in D_l,\ l \neq j . \end{cases} \tag{16}$$

Here, $g_j(z, \zeta)$ is the usual Green's function of the domain D_j. By complex differentiation, we derive from $g(z, \zeta)$ the analytic function

$$L(z, \zeta) = -\frac{2}{\pi}\frac{\partial^2}{\partial z \partial \zeta} g(z, \zeta) = \frac{1}{\pi(z-\zeta)^2} - l(z, \zeta) . \tag{17}$$

The kernels $L(z, \zeta)$ and $l(z, \zeta)$ are well known in the case that D is a domain [3, 16]. We observe that our generalized kernel $l(z, \zeta)$ still preserves the following important property: If $f(z)$ is regular analytic in D, then

$$\frac{1}{\pi}\iint_D \frac{\overline{f(z)}}{(z-\zeta)^2} d\tau = \iint_D l(z, \zeta)\overline{f(z)} d\tau . \tag{18}$$

In fact, if $\zeta \in D_j$ then $l(z, \zeta) = l_j(z, \zeta)$ for $z \in D_j$ and $l(z, \zeta) = [\pi(z-\zeta)^2]^{-1}$ for $z \in D_l$, $l \neq j$. The identity (18) follows, therefore, directly from the corresponding property of the kernel $l_j(z, \zeta)$.

In particular, we may formulate the integral equations (12) and (13) for $v_\nu(z)$ and $\tilde{v}_\nu(z)$ as follows:

$$\lambda_\nu \iint_D l(z, \zeta)\overline{v_\nu(z)} d\tau = v_\nu(\zeta) , \qquad \zeta \in D \tag{19}$$

and

$$-\lambda_\nu \iint_{\tilde{D}} \tilde{l}(z, \zeta)\overline{\tilde{v}_\nu(\zeta)} d\tau = \tilde{v}_\nu(\zeta) , \qquad \zeta \in \tilde{D} . \tag{20}$$

From the symmetry of the kernels $l(z, \zeta)$ and $\tilde{l}(z, \zeta)$ we can conclude

$$\iint_D v_\nu \bar{v}_\mu d\tau = 0 \quad \text{if} \quad \lambda_\nu \neq \lambda_\mu \tag{21}$$

$$\iint_{\tilde{D}} \tilde{v}_\nu \bar{\tilde{v}}_\mu d\tau = 0 \quad \text{if} \quad \lambda_\nu \neq \lambda_\mu . \tag{21'}$$

Thus, using a familiar argument from theory of integral equation we may assume that any pair of different eigen functions v_ν, v_μ (or $\tilde{v}_\nu$. $\tilde{v}_\mu$) are orthogonal upon each other:

$$\iint_D v_\nu \bar{v}_\mu d\tau = 0 ; \iint_{\tilde{D}} \tilde{v}_\nu \bar{\tilde{v}}_\mu d\tau = 0 \quad \text{for} \quad \nu \neq \mu . \tag{21''}$$

There remains the question of normalizing the v_ν and the $\tilde{v}_\nu$. We have obviously the free choice of a real multiplicator in the definition of v_ν; however, this choice will already determine the function $\tilde{v}_\nu$ in a unique way, for example through equation (12). The relation between

the norms of v_ν and $\tilde{v}_\nu$ is best understood by returning to the harmonic functions $h_\nu(z)$ and $\tilde{h}_\nu(z)$ and to their boundary relations (4) and (4'). In fact, we have

$$(22)\qquad \iint_D |\, v_\nu\,|^2 d\tau = \frac{1}{4}\iint_D |\,\nabla h_\nu\,|^2 d\tau = -\frac{1}{4}\oint_C h_\nu \frac{\partial h_\nu}{\partial n} ds$$

$$= \frac{1}{4}\frac{1+\lambda_\nu}{1-\lambda_\nu}\oint_C \tilde{h}_\nu \frac{\partial \tilde{h}_\nu}{\partial \tilde{n}} ds = \frac{\lambda_\nu+1}{\lambda_\nu-1}\iint_{\tilde{D}} |\,\tilde{v}_\nu\,|^2 d\tau\,.$$

We can conclude first from (22) that

$$(23)\qquad |\,\lambda_\nu\,| \geq 1\,.$$

Let us consider the limit cases $\lambda_\nu = \pm 1$. For $\lambda_\nu = 1$ we have necessarily $\tilde{v}_\nu(z) \equiv 0$; the second equation (7) yields

$$(24)\qquad \Im\{v_\nu(z)z'\} = 0 \quad \text{for} \quad z \in C\,.$$

Thus, the eigen function $v_\nu(z)$ is a real differential for each component domain D_j. But a simply-connected domain D_j cannot have such real differentials; hence also $v_\nu(z) \equiv 0$. Thus, as far as the integral equation for v_ν and $\tilde{v}_\nu$ are concerned, $\lambda_\nu = 1$ cannot occur as an eigen value. The situation is, however, different when we return to the original integral equation (2) and to the harmonic functions h_ν and $\tilde{h}_\nu$. To $\lambda_\nu = 1$ must correspond

$$(25)\qquad h_\nu(z) \equiv 2c_j \text{ in } D_j\,,\ \tilde{h}_\nu(z) \equiv 0$$

and

$$(25')\qquad \varphi_\nu(z) = c_j \quad \text{on} \quad C_j\,.$$

In fact, it is immediately verified that for arbitrary choice of the constants c_j the function $\varphi(z) = c_j$ on C_j is a solution of the Fredholm eigen value problem (2) to the eigen value $\lambda_\nu = 1$. There exist thus N linearly independent solutions of (2) to the eigen value $\lambda = 1$. These solutions disappear when we replace the original integral equation (2) by the integral equations for v_ν and $\tilde{v}_\nu$, say, by (12) and (13). It is easy to show that the eigen value $\lambda = 1$ is the only one lost in this transition.

We consider next the case $\lambda_\nu = -1$. We conclude now from (22) that $v_\nu(z) \equiv 0$. We find therefore, in view of (8)

$$(26)\qquad \Im\{\tilde{v}_\nu(z)z'\} = 0 \quad \text{for} \quad z \in C\,,$$

i.e., $\tilde{v}_\nu(z)$ is a real differential of $\tilde{D}$. There are $N-1$ linearly independent differentials of this type in $\tilde{D}$ and we can construct a basis for them as follows. Let $\omega_j(z)$ be harmonic in $\tilde{D}$ and satisfy on C the boundary condition

$$\omega_j(z) = \delta_{jl} \quad \text{for} \quad z \in C_l \,. \tag{27}$$

$\omega_j(z)$ is called the harmonic measure of C_j with respect to z of $\tilde{D}$. Clearly, each function

$$\tilde{w}_j(z) = i\frac{\partial \omega_j}{\partial z} \tag{28}$$

is a real differential in $\tilde{D}$. Since $\sum_{j=1}^{N} \omega_j \equiv 1$, we have $\sum_{j=1}^{N} \tilde{w}_j(z) \equiv 0$. But it is easily seen that apart from this relation no other linear condition between the w_j does exist. Thus, we can select any $N-1$ of the $w_j(z)$ as a basis for all real differentials in $\tilde{D}$.

It is clear that each real differential in $\tilde{D}$ satisfies indeed the integral equations (12) and (13). However, there exists no corresponding single valued harmonic function $\tilde{h}_\nu(z)$ connected with the original Fredholm equation (2) which has this real differential as its complex derivative. Indeed, in view of (26) such function would have to satisfy the boundary condition

$$\frac{\partial \tilde{h}_\nu}{\partial n} \equiv 0 \quad \text{on} \quad C \tag{29}$$

which admits only the solution $\tilde{h}_\nu = \text{const.}$ and could not lead to a non-vanishing differential. Thus, while we lost in the transition to (12) and (13) the N eigen functions to the eigen value $\lambda = +1$, we have obtained $N-1$ new eigen functions to the eigen value $\lambda = -1$ which have no counterpart in the original Fredholm equation.

After discussing the exceptional cases $\lambda_\nu = \pm 1$, we consider now the eigen functions $v_\nu(z)$ and $\tilde{v}_\nu(z)$ which belong to eigen values $|\lambda_\nu| > 1$. Each such pair is obtained by complex differentiation from a pair of harmonic functions $h_\nu(z)$, $\tilde{h}_\nu(z)$ connected with the original Fredholm problem. Since $h_\nu(z)$ is harmonic in each of the simply-connected domains D_j, it can be completed to a set of single-valued analytic functions in the set of domains D_j:

$$V_\nu(z) = h_\nu(z) + ik_\nu(z) \,. \tag{30}$$

Similarly, we may complete $\tilde{h}_\nu$ in $\tilde{D}$ and define

$$\tilde{V}_\nu(z) = \tilde{h}_\nu(z) + i\tilde{k}_\nu(z) \,. \tag{31}$$

From the boundary conditions (4) and (4′) and from the Cauchy-Riemann equations we derive the boundary conditions for the k_ν:

$$\tilde{k}_\nu(z) = k_\nu(z), \quad \frac{\partial}{\partial n} k_\nu(z) = \frac{1+\lambda_\nu}{1-\lambda_\nu} \frac{\partial}{\partial n} \tilde{k}_\nu(z) \,, \qquad z \in C \,. \tag{32}$$

Equations (32) guarantee that $\tilde{k}_\nu(z)$ is single-valued in $\tilde{D}$ since $k_\nu(z)$ is single-valued in each D_j. We may characterize the single-valued analytic functions $V_\nu(z)$ and $\tilde{V}_\nu(z)$ as follows: Their real parts have equal normal derivatives on C while their boundary values are proportional in the ratio $(1+\lambda_\nu)/(1-\lambda_\nu)$. Their imaginary parts are equal on C but their normal derivatives are proportional with the same ratio.

Let us write $k_\nu^{(1)} = (1-\lambda_\nu)k_\nu$ and $\tilde{k}_\nu^{(1)} = (1+\lambda_\nu)\tilde{k}_\nu$; we have on C

$$k_\nu^{(1)}(z) = \frac{1-\lambda_\nu}{1+\lambda_\nu}\tilde{k}_\nu^{(1)}(z), \quad \frac{\partial k_\nu^{(1)}}{\partial n} = -\frac{\partial \tilde{k}_\nu^{(1)}}{\partial \tilde{n}} . \tag{32'}$$

Thus, $k_\nu^{(1)}$ and $\tilde{k}_\nu^{(1)}$ may be conceived as a pair of k-functions belonging to eigen functions of the Fredholm problem (2) with the eigen value $-\lambda_\nu$. With each eigen value λ_ν with $|\lambda_\nu| > 1$ there occurs also its negative $-\lambda_\nu$ as an eigen value. Their corresponding h-functions are, up to a factor, conjugate harmonic functions.

Finally, we introduce the analytic functions

$$u_\nu(z) = \sqrt{\lambda_\nu - 1}\, v_\nu(z), \quad \tilde{u}_\nu(z) = i\sqrt{\lambda_\nu + 1}\, \tilde{v}_\nu(z) . \tag{33}$$

By virtue of (21″) and (22), we may assume that these functions form orthonormalized sets in D and $\tilde{D}$; that is

$$\iint_D u_\nu \bar{u}_\mu d\tau = \delta_{\nu\mu}, \quad \iint_{\tilde{D}} \tilde{u}_\nu \bar{\tilde{u}}_\mu d\tau = \delta_{\nu\mu} . \tag{34}$$

Since the u-functions will be frequently used in this paper, we note here some formulas which follow immediately from the corresponding results for the v-functions. From (8) we derive the boundary relation

$$u_\nu(z)z' = \frac{i}{\sqrt{\lambda_\nu^2 - 1}}\tilde{u}_\nu(z)z' - \frac{\lambda_\nu i}{\sqrt{\lambda_\nu^2 - 1}}\overline{(\tilde{u}_\nu(z)z')} . \tag{35}$$

Equations (9), (9′) and (11), (11′) take on the form

$$\frac{\lambda_\nu}{2\pi i}\oint_C \frac{\overline{(\tilde{u}_\nu dz)}}{z-\zeta} = \begin{cases} i\sqrt{\lambda_\nu^2 - 1}\, u_\nu(\zeta) & \text{for } \zeta \in D \\ -\tilde{u}_\nu(\zeta) & \text{for } \zeta \in \tilde{D} . \end{cases} \tag{36}$$

and

$$\frac{\lambda_\nu}{2\pi i}\oint_C \frac{\overline{(u_\nu dz)}}{z-\zeta} = \begin{cases} u_\nu(\zeta) & \text{for } \zeta \in D \\ -i\sqrt{\lambda_\nu^2 - 1}\, \tilde{u}_\nu(\zeta) & \text{for } \zeta \in \tilde{D} . \end{cases} \tag{37}$$

From their connection with the Fredholm integral equation it can be shown that the $u_\nu(z)$ form a complete system of analytic functions in D, in the sense that every function $f(z)$ which is analytic in D and for which $\iint_D |f|^2 d\tau < \infty$ can be represented in the form

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu u_\nu(z),\ a_\nu = \iint_D f\bar{u}_\nu d\tau\ . \tag{38}$$

The series converges uniformly in each closed subdomain of D. In the same sense, the functions $\tilde{u}_\nu(z)$ form a complete orthonormal system within the class of all functions which are analytic in $\tilde{D}$, have a finite norm in $\tilde{D}$ and possess a finite single-valued integral in this multiply-connected domain. If we add to the $\{\tilde{u}_\nu\}$-set any $N-1$ linearly independent real differentials of $\tilde{D}$ we obtain a complete system for all analytic functions in $\tilde{D}$ with finite norm and vanishing at infinity [3, 21].

2. The dielectric Green's function. The theory of the Green's function of the domain $\tilde{D}$ is connected with the electrostatic problem of a point charge at a source point ζ in the presence of the system of grounded conductors C_j. We may consider also the problem to determine the electrostatic potential induced by the same point charge at ζ in the presence of N isotropic dielectric media which are spread over the domains D_j and have the dielectric constant ε. The corresponding potential $g_\varepsilon(z, \zeta)$ will now be defined in D as well as in $\tilde{D}$ and will be characterized by the following properties:

(a) $g_\varepsilon(z, \zeta)$ is a harmonic function of z in D and in $\tilde{D}$, except for $z = \zeta$ and for $z = \infty$.

(b) If $\zeta \in \tilde{D}$, the function $g_\varepsilon(z, \zeta) + \log|z - \zeta|$ is harmonic at ζ.

(b′) It $\zeta \in D$, the function $g_\varepsilon(z, \zeta) + \varepsilon \log|z - \zeta|$ is harmonic at ζ.

(c) $g_\varepsilon(z, \zeta)$ is continuous through C.

(d) $\dfrac{\partial}{\partial n_z} g_\varepsilon(z, \zeta) + \varepsilon \dfrac{\partial}{\partial \tilde{n}_z} g_\varepsilon(z, \zeta) = 0$ for $z \in C, \zeta$ in D or in $\tilde{D}$.

(e) $g_\varepsilon(z, \zeta) + \log|z| \to 0$ as $z \to \infty$ for ζ fixed.

If such a function $g_\varepsilon(z, \zeta)$ exists it must be unique and symmetric in its two arguments, as is shown by the standard argument of potential theory based on the second Green's identity. In order to construct the Green's function, we set it up in the form

$$g_\varepsilon(z, \zeta) = \log \frac{1}{|z - \zeta|} + \int_\sigma \mu(\eta, \zeta) \log|\eta - z|\, ds_\eta,\ \zeta \in \tilde{D} \tag{1}$$

and try to determine $\mu(\eta, \zeta)$ in such a way that the above requirements are fulfilled. We proceed analogously, if $\zeta \in D$; only the singularity term on the right side of (1) will now be $-\varepsilon \log|z - \zeta|$. By this formal set up, we have already fulfilled conditions (a) to (c). Condition (e) is satisfied if we require

$$\int_\sigma \mu(\eta, \zeta) ds_\eta = \begin{cases} \varepsilon - 1 & \text{for } \zeta \in D \\ 0 & \text{for } \zeta \in \tilde{D}\ . \end{cases} \tag{2}$$

Finally, we can satisfy (d) by choosing the density function μ of the line potential as solution of the integral equation

$$(3)\quad \mu(z,\zeta)+\frac{\varepsilon-1}{\varepsilon+1}\frac{1}{\pi}\int_\sigma \mu(\eta,\zeta)k(\eta,z)ds_\eta=\begin{cases}\dfrac{\varepsilon-1}{\varepsilon+1}\cdot\dfrac{\varepsilon}{\pi}k(\zeta,z) & \text{for } \zeta\in D\\[2ex] \dfrac{\varepsilon-1}{\varepsilon+1}\dfrac{1}{\pi}k(\zeta,z) & \text{for } \zeta\in \tilde{D}\,.\end{cases}$$

Here $k(\zeta,z)$ is defined by equation (1.1). We observe that

$$(4)\qquad \int_\sigma k(\eta,z)ds_z=\begin{cases}0 & \text{for } \eta\in\tilde{D}\\ \pi & \text{for } \eta\in C\\ 2\pi & \text{for } \eta\in D\,.\end{cases}$$

Hence, if $\mu(z,\zeta)$ is a solution of the integral equation (3) we may integrate this equation with respect to z over C and verify that condition (2) is fulfilled automatically. It is sufficient, therefore, to concentrate upon the inhomogeneous integral equation (3).

For physical reasons, we shall assume $\varepsilon>0$. In this case, we always have

$$(3')\qquad \left|\frac{\varepsilon-1}{\varepsilon+1}\right|<1\,.$$

Since we showed in § 1 that all eigen values of the kernel $k(z,\zeta)$ have absolute values ≥ 1, it follows that integral equation (3) can always be solved by the usual process of iteration and that the solution can be represented by a Liouville-Neumann series. The convergence of this series will be the better, the nearer ε will be to 1. We observe that

$$(5)\qquad g_1(z,\zeta)=\log\frac{1}{|z-\zeta|}$$

is trivially known.

The function

$$(6)\qquad \gamma_\varepsilon(z,\zeta)=g_\varepsilon(z,\zeta)-\log\frac{1}{|z-\zeta|}$$

is (for $\zeta\in D$ or for $\zeta\in\tilde{D}$) a regular harmonic function of z in $\tilde{D}$, vanishes if z tends to infinity and possesses a single-valued conjugate harmonic function in $\tilde{D}$. This last fact follows from the boundary condition (d) on the dielectric Green's function and the fact that each complementary domain D_j is simply-connected. Let $\tilde{\sum}$ be the class of all functions $\tilde{h}(s)$ which are harmonic in $\tilde{D}$, vanish at infinity and have a single-valued conjugate harmonic function. It is easy to show that the harmonic

functions $\tilde{h}_\nu(s)$ which belong to eigen values $|\lambda_\nu| > 1$ of the Fredholm problem in § 1 form a basis in the linear space $\tilde{\sum}$. By virtue of (1.5) and (1.21′), we have

$$(7) \qquad \iint_{\tilde{D}} \nabla\tilde{h}_\nu \cdot \nabla\tilde{h}_\mu d\tau = 4\Re\left\{\iint_D \tilde{v}_\nu \bar{\tilde{v}}_\mu d\tau\right\} = 0 \qquad \text{for } \nu \neq \mu .$$

By a trivial renormalization we can then achieve that

$$(8) \qquad \iint_{\tilde{D}} \nabla\tilde{h}_\nu \cdot \nabla\tilde{h}_\mu d\tau = \delta_{\nu\mu} .$$

We wish now to develop $\gamma_\varepsilon(z, \zeta)$ in terms of the complete orthonormal set $\{\tilde{h}_\nu\}$. In order to determine the Fourier coefficients or $\gamma_\varepsilon(z, \zeta)$, we consider the Dirichlet integrals

$$(9) \qquad j_\nu(\zeta) = \iint_D \nabla g_\varepsilon(z, \zeta) \cdot \nabla h_\nu(z) d\tau_z + \iint_{\tilde{D}} \nabla g_\varepsilon(z, \zeta) \cdot \nabla\tilde{h}_\nu(z) d\tau_z .$$

We integrate first by parts with respect to $g_\varepsilon(z, \zeta)$ and use the continuity of this function across C as well as the relation (1.4′) for the normal derivatives of h_ν and $\tilde{h}_\nu$ on C. We find

$$(10) \qquad j_\nu(\zeta) \equiv 0 .$$

Next, we integrate by parts with respect to $h_\nu(z)$ and $\tilde{h}_\nu(z)$; we use (1.4) and the condition (d) on $g_\varepsilon(z, \zeta)$. We obtain the equations

$$(11) \qquad j_\nu(\zeta) = 2\pi\varepsilon h_\nu(\zeta) - (1 + \varepsilon\rho_\nu) \int_\sigma \frac{\partial g_\varepsilon(z, \zeta)}{\partial\tilde{n}_z} \tilde{h}_\nu(z) ds_z \quad \text{for} \quad \zeta \in D$$

and

$$(11') \qquad j_\nu(\zeta) = 2\pi\tilde{h}_\nu(\zeta) - \frac{1 + \varepsilon\rho_\nu}{\varepsilon\rho_\nu} \int_\sigma \frac{\partial g_\varepsilon(z, \zeta)}{\partial n_z} h_\nu(z) ds_z \quad \text{for} \quad \zeta \in \tilde{D} .$$

Here, we have introduced the abbreviation

$$(12) \qquad \rho_\nu = \frac{\lambda_\nu + 1}{\lambda_\nu - 1} ;$$

this simple function of λ_ν will occur frequently in our developments.

From (10), (11) and (11′) we deduce immediately

$$(13) \qquad \iint_{\tilde{D}} \nabla g_\varepsilon(z, \zeta) \cdot \nabla\tilde{h}_\nu(z) d\tau_z = -\frac{2\pi\varepsilon}{1 + \varepsilon\rho_\nu} h_\nu(\zeta) \quad \text{for} \quad \zeta \in D$$

and

$$(13') \qquad \iint_{\tilde{D}} \nabla g_\varepsilon(z, \zeta) \cdot \nabla \tilde{h}_\nu(z) d\tau_z = \frac{2\pi\varepsilon\rho_\nu}{1 + \varepsilon\rho_\nu} \tilde{h}_\nu(\zeta) \quad \text{for} \quad \zeta \in \tilde{D}.$$

When we specialize $\varepsilon = 1$, we obtain because of (5) the values of the left-hand integrals with g_ε replaced by $\log 1/|z - \zeta|$. Hence, we obtain finally by subtraction

$$(14) \qquad \iint_{\tilde{D}} \nabla \gamma_\varepsilon(z, \zeta) \cdot \Delta \tilde{h}_\nu(z) d\tau_z = \frac{2\pi(1 - \varepsilon)}{(1 + \rho_\nu)(1 + \varepsilon\rho_\nu)} h_\nu(\zeta) \quad \text{for} \quad \zeta \in D .$$

and

$$(14') \qquad \iint_{\tilde{D}} \nabla \delta_\varepsilon(z, \zeta) \nabla \tilde{h}_\nu(z) d\tau_z = \frac{2\pi(\varepsilon - 1)}{(1 + \rho_\nu)(1 + \varepsilon\rho_\nu)} \tilde{h}_\nu(\zeta) \quad \text{for} \quad \zeta \in \tilde{D} .$$

Having expressed by (14) and (14') the Fourier coefficients of $\gamma_\varepsilon(z, \zeta)$ with respect to the complete orthonormal system in $\tilde{\sum}$, we obtain thus the two series development for $z \in \tilde{D}$;

$$(15) \qquad g_\varepsilon(z, \zeta) = \log \frac{1}{|z - \zeta|} + 2\pi(1 - \varepsilon) \sum_{\nu=1}^{\infty} \frac{\tilde{h}_\nu(z) h_\nu(\zeta)}{(1 + \rho_\nu)(1 + \varepsilon\rho_\nu)} \quad \text{for} \quad \zeta \in D$$

$$(16) \qquad g_\varepsilon(z, \zeta) = \log \frac{1}{|z - \zeta|} + 2\pi(\varepsilon - 1) \sum_{\nu=1}^{\infty} \frac{\rho_\nu \tilde{h}_\nu(z) \tilde{h}_\nu(\zeta)}{(1 + \rho_\nu)(1 + \varepsilon\rho_\nu)} \quad \text{for} \quad \zeta \in \tilde{D} .$$

Both series converge uniformly in each closed subdomain of $\tilde{D}$.

We wish next to expand analogously $g_\varepsilon(z, \zeta)$ for $z \in D$ in terms of the functions $h_\nu(z)$. By (1.4), (1.4') and the normalization (8), we have

$$(17) \qquad \iint_D \nabla h_\nu \cdot \nabla h_\mu d\tau = \rho_\nu \delta_{\nu\mu} .$$

Let $\omega_j(z)$ and $\tilde{g}(z, \infty)$ denote again the j-th harmonic measure and the Green's function with pole at infinity of $\tilde{D}$. We clearly have

$$(18) \qquad \int_\sigma h_\nu \frac{\partial \omega_j}{\partial n} ds = 0, \quad \int_\sigma h_\nu \frac{\partial \tilde{g}(z, \infty)}{\partial u} ds = 0 .$$

Indeed, because of (1.4) these linear conditions are equivalent to those with $\tilde{h}_\nu$ and these in turn follow from the fact that all $\tilde{h}_\nu$ have single-valued harmonic conjugates in $\tilde{D}$ and that they all vanish at infinity.

Let $\sum$ be the linear space of functions $h(z)$ which are regular hamonic in D and which satisfy the N linear conditions (18). Observe that $\sum$ does not contain any function $h_0(z)$ which has a constant value c_j in each D_j, except for $h_0(z) \equiv 0$. Indeed, the conditions (18) would yield for such a function $h_0(z)$

$$(18') \qquad \sum_{j=1}^{n} c_j p_{ij} = 0, \quad \sum_{j=1}^{n} c_j \omega_j(\infty) = 0$$

where

$$p_{ij} = \frac{1}{2\pi}\int_{\sigma_j} \frac{\partial \omega_i}{\partial n} ds \tag{18''}$$

denotes the period matrix connected with the harmonic measures. But the first system of linear equations (18') implies clearly [5, 15] $c_1 = c_2 = \cdots = c_N = c$ and the last equation yields

$$c \sum_{j=1}^{N} \omega_j(\infty) = c = 0 \,. \tag{19}$$

Thus, only the trivial function $h_0(z) \equiv 0$ of this type lies in $\sum$.

From this fact and the considerations of § 1, it follows that the functions $\{\rho_\nu^{1/2} h_\nu(z)\}$ form a complete orthonormal set in $\sum$. The function $\gamma_\varepsilon(z, \zeta)$ lies in $\sum$ if $\zeta \in \tilde{D}$; this follows at once from the conditions (c), (d) and (e) on the dielectric Green's function. If $\zeta \in D$, it is seen that $\gamma_\varepsilon(z, \zeta) + (1 - \varepsilon) g(z, \zeta)$ lies in $\sum$ where $g(z, \zeta)$ is the Green's function of D defined by (1.16). The Fourier coefficients of $\gamma_\varepsilon(z, \zeta)$ are easily determined from (9), (10), (13) and (13'). Observe that for $\zeta \in D_j$

$$\iint_D \nabla g(z, \zeta) \cdot \nabla h_\nu(z) d\tau_z = -\int_{\sigma_j} \frac{\partial h_\nu(z)}{\partial n} g(z, \zeta) ds_z = 0 \tag{20}$$

such that the correction term $(1 - \varepsilon) g(z, \zeta)$ does not affect the Fourier coefficients at all. We find without difficulty

$$g_\varepsilon(z, \zeta) = \log \frac{1}{|z - \zeta|} + (\varepsilon - 1) g(z, \zeta) + 2\pi(\varepsilon - 1) \sum_{\nu=1}^{\infty} \frac{h_\nu(z) h_\nu(\zeta)}{\rho_\nu (1 + \rho_\nu)(1 + \varepsilon \rho_\nu)} \qquad \text{for} \quad \zeta \in D \tag{21}$$

$$g_\varepsilon(z, \zeta) = \log \frac{1}{|z - \zeta|} + 2\pi(1 - \varepsilon) \sum_{\nu=1}^{\infty} \frac{h_\nu(z) \tilde{h}_\nu(\zeta)}{(1 + \rho_\nu)(1 + \varepsilon \rho_\nu)} \qquad \text{for} \quad \zeta \in \tilde{D} \,. \tag{22}$$

These series also converge uniformly in each closed subdomain of D. Equation (22) could have been derived from (15) and the property of symmetry of the dielectric Green's function in dependence of its two arguments.

The various series developments for $g_\varepsilon(z, \zeta)$ given so far are of theoretical interest and allow the derivation of numerous identities. They help little in the actual determination of the dielectric Green's function of a given domain since we know all Fredholm eigen functions and eigen values only in very few cases. In order to utilize the preceding formulas for actual calculations, we have to add the following considerations.

From the definition of the dielectric Green's functions and from Green's identity, one can derive the identity

$$(23)\qquad \frac{1}{e}\iint_D \nabla g_\varepsilon(z,\zeta)\cdot\nabla g_e(z,\eta)d\tau_z + \iint_{\tilde{D}} \nabla g_\varepsilon(z,\zeta)\cdot\nabla g_e(z,\eta)d\tau_z = 2\pi g_\varepsilon(\zeta,\eta)\ .$$

Interchanging ε and e in (23) and subtracting the new identitity, we obtain

$$(24)\qquad 2\pi[g_\varepsilon(\zeta,\eta) - g_e(\zeta,\eta)] = \left(\frac{1}{e} - \frac{1}{\varepsilon}\right)\iint_D \nabla g_\varepsilon(z,\zeta)\cdot\nabla g_e(z,\eta)d\tau_z\ .$$

In particular, passing to the limit $e \to \varepsilon$, we find

$$(25)\qquad \frac{\partial}{\partial\varepsilon} g_\varepsilon(\zeta,\eta) = \frac{1}{\varepsilon^2}\cdot\frac{1}{2\pi}\iint_D \nabla g_\varepsilon(z,\zeta)\nabla g_\varepsilon(z,\eta)d\tau_z\ .$$

We introduce the expression

$$(26)\qquad \Gamma(\zeta,\eta) = \frac{1}{2\pi}\iint_D \left(\nabla_z \log\frac{1}{|z-\zeta|}\cdot\nabla_z \log\frac{1}{|z-\eta|}\right)d\tau_z$$

which is a "geometric" functional of D, i.e., can be calculated from elementary functions by a simple process of integration and not by solving any boundary value problem of potential theory. Passing in (25) to the limit $\varepsilon = 1$, we find in view of (5)

$$(27)\qquad \frac{\partial}{\partial\varepsilon} g_\varepsilon(\zeta,\eta)\Big|_{\varepsilon=1} = \Gamma(\zeta,\eta)\ .$$

On the other hand, we can calculate this same ε-derivative directly from formulas (15), (16) and (21). Comparing results, we obtain

$$(28)\qquad \Gamma(\zeta,\eta) = -2\pi\sum_{\nu=1}^{\infty}\frac{h_\nu(\xi)\tilde{h}_\nu(\xi)}{(1+\rho_\nu)^2} \quad \text{for} \quad \zeta,\eta\in\tilde{D}$$

$$(28')\qquad \Gamma(\zeta,\eta) = 2\pi\sum_{\nu=1}^{\infty}\frac{\rho_\nu\tilde{h}_\nu(\zeta)\tilde{h}_\nu(\eta)}{(1+\rho_\nu)^2} \quad \text{for} \quad \zeta\in D,\ \eta\in\tilde{D}$$

$$(28'')\qquad \Gamma(\zeta,\eta) = g(\zeta,\eta) + 2\pi\sum_{\nu=1}^{\infty}\frac{h_\nu(\zeta)h_\nu(\eta)}{\rho_\nu(1+\rho_\nu)^2} \quad \text{for} \quad \zeta,\eta\in D\ .$$

The fact that these particular series in the h-functions have relatively elementary sums is of considerable interest. It leads to series developments for the dielectric Green's functions in terms of geometric expressions.

Let us define recursively

$$(29)\qquad \Gamma^{(n)}(z,\zeta) = \frac{1}{2\pi}\iint_D\left(\nabla_\eta\Gamma^{(n-1)}(\eta,z)\cdot\Delta_\eta\log\frac{1}{|\eta-\zeta|}\right)d\tau_\eta,\ \Gamma^{(1)}\equiv\Gamma\ .$$

Using equations (9), (10) and the Fourier formulas (13), (13'), we derive the series developments

$$(30)\qquad \Gamma^{(n)}(z,\zeta) = -2\pi \sum_{\nu=1}^{\infty} \frac{\tilde{h}_\nu(z)h_\nu(\zeta)}{(1+\rho_\nu)^{n+1}} \quad \text{for} \quad z \in \tilde{D}, \zeta \in D$$

$$(31)\qquad \Gamma^{(n)}(z,\zeta) = 2\pi \sum_{\nu=1}^{\infty} \frac{\rho_\nu \tilde{h}_\nu(z)\tilde{h}_\nu(\zeta)}{(1+\rho_\nu)^{n+1}} \quad \text{for} \quad z \in \tilde{D}, \zeta \in \tilde{D}$$

$$(32)\qquad \Gamma^{(n)}(z,\zeta) = g(z,\zeta) + 2\pi \sum_{\nu=1}^{\infty} \frac{h_\nu(z)h_\nu(\zeta)}{\rho_\nu(1+\rho_\nu)^{n+1}} \quad \text{for} \quad z \in D, \zeta \in D\,.$$

We return now to the formulas (15), (16) and (21) for $g_\varepsilon(z,\zeta)$. We use the series development

$$(33)\qquad \frac{\varepsilon-1}{1+\varepsilon\rho_\nu} = 2\sum_{k=0}^{\infty}\left(\frac{\varepsilon-1}{\varepsilon+1}\right)^{k+1}\frac{(1-\rho_\nu)^k}{(1+\rho_\nu)^{k+1}} = \frac{2}{1-\rho_\nu}\sum_{k=0}^{\infty}\left(\frac{1}{\lambda_\nu}\frac{1-\varepsilon}{1+\varepsilon}\right)^{k+1}$$

which converges absolutely since $\varepsilon > 0$ and $|\lambda_\nu| > 1$. We insert this series into the above formulas for $g_\varepsilon(z,\zeta)$; interchanging the order of summation, we obtain in each case the representation:

$$(34)\qquad g_\varepsilon(z,\zeta) = \log\frac{1}{|z-\zeta|} + \sum_{k=0}^{\infty}\left(\frac{\varepsilon-1}{\varepsilon+1}\right)^{k+1} M_k(z,\zeta)\,.$$

The kernels $M_k(z,\zeta)$ are defined as follows:

$$(35)\qquad M_k(z,\zeta) = -4\pi \sum_{\nu=1}^{\infty} \frac{(1-\rho_\nu)^k}{(1+\rho_\nu)^{k+2}} \tilde{h}_\nu(z)h_\nu(\zeta) \quad \text{for} \quad z \in \tilde{D}, \zeta \in \tilde{D}$$

$$(36)\qquad M_k(z,\zeta) = -4\pi \sum_{\nu=1}^{\infty} \frac{(1-\rho_\nu)^k\rho_\nu}{(1+\rho_\nu)^{k+2}} \tilde{h}_\nu(z)\tilde{h}_\nu(\zeta) \quad \text{for} \quad z \in \tilde{D}, \zeta \in \tilde{D}$$

$$(37)\qquad M_k(z,\zeta) = 2g(z,\zeta) + 4\pi \sum_{\nu=1}^{\infty} \frac{(1-\rho_\nu)^k}{\rho_\nu(1+\rho_\nu)^{k+2}} h_\nu(z)h_\nu(\zeta) \quad \text{for} \quad z \in D, \zeta \in D\,.$$

By use of the geometric terms (30), (31) and (32), we can express $M_k(z,\zeta)$ in a uniform way, independently of the location of their arguments. We find

$$(38)\qquad M_k(z,\zeta) = \sum_{\sigma=1}^{\infty}(-1)^k\binom{k}{\sigma}2^{k-\sigma+1}\Gamma^{(k-\sigma+1)}(z,\zeta)\,.$$

Formulas (34) and (38) allow a series development for all dielectric Green's functions in the entire plane in terms of the known iterated Dirichlet integrals $\Gamma^{(n)}(z,\zeta)$. They are closely related to similar developments for the classical Green's function of a multiply-connected domain in terms of geometric expressions [3, 21]. The formulas are convenient for $|\varepsilon - 1|$ small. Observe also that the geometrical terms $M_k(z,\zeta)$ are independent of ε and may be defined as the coefficients of the Taylor's series for $g_\varepsilon(z,\zeta)$ in terms of $(\varepsilon-1)/(\varepsilon+1)$.

3. Limit values of the dielectric Green's function. From the series developments for the dielectric Green's function, given in the preceding section, we can determine the limit values of $g_\varepsilon(z, \zeta)$ as ε converges to zero or to infinity. For this purpose, we have to introduce additional functions of the classes $\sum$ and $\tilde{\sum}$ and to develop them into series of the h-functions.

(a) We suppose $\zeta \in \tilde{D}$ and consider the analytic function $\tilde{\varphi}(z, \zeta)$ of z in $\tilde{D}$ which has a simple pole at $z = \zeta$, vanishes at infinity such that

$$\lim_{z\to\infty} z\tilde{\varphi}(z, \zeta) = 1 \tag{1}$$

and which maps $\tilde{D}$ in a one-to-one manner upon the complex plane slit along concentric circular arcs around the origin. These requirements determine $\tilde{\varphi}(z, \zeta)$ in a unique way.

Let now

$$\tilde{G}(z, \zeta) = \log |\tilde{\varphi}(z, \zeta)| . \tag{2}$$

The function $\tilde{G}(z, \zeta) + \log |z - \zeta|$ is harmonic in $\tilde{D}$, has a single-valued harmonic conjugate there and vanishes as $|z| \to \infty$. Hence, this function lies in the class $\tilde{\sum}$.

We can construct $\tilde{G}(z, \zeta)$ explicitly in terms of the Green's function $\tilde{g}(z, \zeta)$ of $\tilde{D}$. In fact, it is evident that

$$\tilde{G}(z, \zeta) = \tilde{g}(z, \zeta) - \tilde{g}(z, \infty) - \tilde{g}(\zeta, \infty) + \tilde{\gamma} - \sum_{j,k=1}^{\infty} \alpha_{jk}(\omega_j(z) - \omega_j(\infty))(\omega_k(\zeta) - \omega_k(\infty)) , \tag{3}$$

with

$$\tilde{\gamma} = \lim_{z\to\infty} (\tilde{g}(z, \infty) - \log |z|) . \tag{3'}$$

The coefficient matrix α_{jk} has to be chosen in such a way as to make the conjugate of $\tilde{G}$ single-valued along each boundary curve C_l. Hence. we obtain for it the linear equations

$$\omega_l(\zeta) - \omega_l(\infty) = \sum_{j,k=1}^{N-1} \alpha_{jk} p_{jl}[\omega_k(\zeta) - \omega_k(\infty)] \tag{4}$$

where the p_{jl} are the elements of the period matrix defined in (2.18″). Hence, we conclude

$$\sum_{j=1}^{N-1} \alpha_{jk} p_{jl} = \delta_{kl} , \tag{5}$$

i.e., the α-matrix is the inverse of the period matrix of rank $N - 1$.

We can develop $\tilde{G}(z, \zeta) + \log|z - \zeta|$ in terms of the complete orthonormal system $\{\tilde{h}_\nu\}$ in $\tilde{\sum}$. Since $\tilde{G}(z, \zeta)$ takes on each curve C_l a constant boundary value

$$\tilde{G}(z, \zeta) = c_l(\zeta) \quad \text{for} \quad z \in C_l, \zeta \in \tilde{D}\ , \tag{6}$$

we have

$$\iint_{\tilde{D}} \nabla\tilde{G}(z, \zeta) \cdot \nabla\tilde{h}_\nu(z) d\tau_z = \sum_{l=1}^{N} c_l(\zeta) \int_{C_l} \frac{\partial \tilde{h}_\nu}{\partial n} ds = 0\ . \tag{7}$$

Thus, combining (7) with (2.13′) for $\varepsilon = 1$, we obtain

$$\iint_{\tilde{D}} \nabla[\tilde{G}(z\,\zeta) + \log|z - \zeta|] \cdot \nabla\tilde{h}_\nu(z) d\tau_z = -\frac{2\pi\rho_\nu}{1 + \rho_\nu} \tilde{h}_\nu(\zeta)\ . \tag{8}$$

Consequently, we arrive at the following series development for $\tilde{G}(z,\zeta)$:

$$\tilde{G}(z, \zeta) = \log \frac{1}{|z - \zeta|} - 2\pi \sum_{\nu=1}^{\infty} \frac{\rho_\nu}{1 + \rho_\nu} \tilde{h}_\nu(z)\tilde{h}_\nu(\zeta)\ . \tag{9}$$

We may now cast (2.16) into the form

$$g_\varepsilon(z, \zeta) - \tilde{G}(z, \zeta) = 2\pi \sum_{\nu=1}^{\infty} \frac{\varepsilon\rho_\nu}{1 + \varepsilon\rho_\nu} \tilde{h}_\nu(z)\tilde{h}_\nu(\zeta)\ . \tag{10}$$

We recognize, in particular, that

$$\lim_{\varepsilon \to 0} g_\varepsilon(z, \zeta) = \tilde{G}(z, \zeta)\ . \tag{11}$$

Thus, the logarithm of the important canonical map function $\tilde{\varphi}(z, \zeta)$ is closely related to the limit of the dielectric Green's function as $\varepsilon \to 0$.

Let next $\tilde{\psi}(z, \zeta)$ be analytic for $z \in \tilde{D}$ except for a simple pole at $z = \zeta \in \tilde{D}$, vanish at infinity such that

$$\lim_{z \to \infty} z\tilde{\psi}(z, \zeta) = 1 \tag{1′}$$

and map $\tilde{D}$ univalently onto the entire plane slit along rectilinear segments which are all directed towards the origin. $\tilde{\psi}(z, \zeta)$ is uniquely determined and might be constructed explicitly in terms of the Neumann's function of $\tilde{D}$.

Let

$$\tilde{N}(z, \zeta) = \log|\tilde{\psi}(z, \zeta)|\ . \tag{12}$$

Obviously, the function $\tilde{N}(z, \zeta) + \log|z - \zeta|$ lies in the class $\tilde{\sum}$. Since $\tilde{N}(z, \zeta)$ has, by its definition, vanishing normal derivatives on C, we have

$$\iint_{\tilde{D}} \nabla \tilde{N}(z, \zeta) \cdot \nabla \tilde{h}_\nu(z) d\tau_z = 2\pi \tilde{h}_\nu(\zeta) ; \tag{13}$$

therefore, in view of (2.13′) for $\varepsilon = 1$

$$\iint_{\tilde{D}} \nabla[\tilde{N}(z, \zeta) + \log|z - \zeta|] \cdot \nabla \tilde{h}_\nu(z) d\tau_z = \frac{2\pi}{1 + \rho_\nu} \tilde{h}_\nu(\zeta) . \tag{14}$$

Thus, we arrive at the series development

$$\tilde{N}(z, \zeta) = \log \frac{1}{|z - \zeta|} + 2\pi \sum_{\nu=1}^{\infty} \frac{1}{1 + \rho_\nu} \tilde{h}_\nu(z) \tilde{h}_\nu(\zeta) . \tag{15}$$

We can transform (2.16) into

$$\tilde{N}(z, \zeta) - g_\varepsilon(z, \zeta) = \sum_{\nu=1}^{\infty} \frac{\tilde{h}_\nu(z) \tilde{h}_\nu(\zeta)}{1 + \varepsilon \rho_\nu} \tag{16}$$

and read off the limit relation

$$\lim_{\varepsilon \to \infty} g_\varepsilon(z, \zeta) = \tilde{N}(z, \zeta) . \tag{17}$$

The dielectric Green's function $g_\varepsilon(z, \zeta)$ yields thus in $\tilde{D}$ a continuous interpolation between the logarithms of two canonical map functions. The result is the more significant since we shall prove in the next section that each $g_\varepsilon(z, \zeta)$ is analogously related to a univalent function in $\tilde{D}$.

(b) From the fact that the function $\tilde{G}(z, \zeta) + \log|z - \zeta|$ lies in $\tilde{\Sigma}$, i.e., that it has a single-valued conjugate and that it vanishes at infinity, it follows by virtue of (6) that

$$\sum_{l=1}^{N} c_l(\zeta) \int_{\sigma_l} \frac{\partial \omega_j(z)}{\partial n} ds = \int_\sigma \log \frac{1}{|z - \zeta|} \frac{\partial \omega_j}{\partial n} ds \tag{18}$$

and

$$\sum_{l=1}^{N} c_l(\zeta) \int_{\sigma_l} \frac{\partial \tilde{g}(z, \infty)}{\partial n} ds = \int_\sigma \log \frac{1}{|z - \zeta|} \frac{\partial \tilde{g}(z, \infty)}{\partial n} ds . \tag{18′}$$

We define now for fixed $\zeta \in \tilde{D}$ the harmonic function $c(z, \zeta)$ of z in D by putting

$$c(z, \zeta) = c_l(\zeta) \quad \text{for} \quad z \in D_l . \tag{19}$$

By (18), (18′) and the definition (2.18) of the class Σ, the function $-\log|z - \zeta| - c(z, \zeta)$ lies in this linear space. We may develop it, therefore, into a series of the $h_\nu(z)$. By use of (2.10) and (2.13′), we obtain

$$\log \frac{1}{|z-\zeta|} = c(z,\zeta) - 2\pi \sum_{\nu=1}^{\infty} \frac{h_\nu(z)\tilde{h}_\nu(\zeta)}{1+\rho_\nu} \qquad z \in D, \zeta \in \tilde{D}\,. \tag{20}$$

We may combine (20) with (2.22) and find

$$g_\varepsilon(z,\zeta) = c(z,\zeta) - 2\pi\varepsilon \sum_{\nu=1}^{\infty} \frac{1}{1+\varepsilon\rho_\nu} h_\nu(z)\tilde{h}_\nu(\zeta)\,. \tag{21}$$

This leads to the limit relation

$$\lim_{\varepsilon\to 0} g_\varepsilon(z,\zeta) = c(z,\zeta) \quad \text{for} \quad z \in D, \zeta \in \tilde{D}\,. \tag{22}$$

The limit of $g_\varepsilon(z,\zeta)$ as $\varepsilon \to \infty$ does not seem to admit a simple geometric interpretation.

(c) Consider next the case $\zeta \in D$, say $\zeta \in D_l$. We define now the regular analytic functions $\tilde{\varphi}_l(z)$ which map $\tilde{D}$ univalently into a full circle around the origin which is slit along concentric circular arcs, such that $z = \infty$ goes into the center and that

$$\lim_{Z\to\infty} z\tilde{\varphi}_l(z) = 1\,. \tag{1''}$$

The function $\tilde{\varphi}_l(z)$ is uniquely determined by the additional requirement that the special boundary curve C_l shall correspond to the outer circumference.

Since the function

$$\tilde{G}_l(z) = \log|\tilde{\varphi}_l(z)| \tag{23}$$

is harmonic in $\tilde{D}$ except for a simple logarithmic pole at infinity and since

$$\tilde{G}_l(z) = c_{lj} \quad \text{for} \quad z \in C_j\,, \tag{24}$$

it is evident that $\tilde{G}_l(z)$ may again be expressed explicitly in terms of the Green's function $\tilde{g}(z,\zeta)$ of $\tilde{D}$ [5].

Since we assumed $\zeta \in D_l$, the function $\tilde{G}_l(z) + \log|z-\zeta|$ lies in the class $\tilde{\Sigma}$. We can develop it into a Fourier series of the system $\{\tilde{h}_\nu\}$. The same calculations as before lead to

$$\tilde{G}_l(z) = \log \frac{1}{|z-\zeta|} + 2\pi \sum_{\nu=1}^{\infty} \frac{\tilde{h}_\nu(z)h_\nu(\zeta)}{1+\rho_\nu},\ z \in \tilde{D}, \zeta \in D_l\,. \tag{25}$$

From (2.15) we obtain

$$g_\varepsilon(z,\zeta) - \tilde{G}_l(z) = -2\pi\varepsilon \sum_{\nu=1}^{\infty} \frac{1}{1+\varepsilon\rho_\nu} \tilde{h}_\nu(z)h_\nu(\zeta)\,; \tag{26}$$

hence

$$\lim_{\varepsilon\to 0} g_\varepsilon(z,\zeta) = \tilde{G}_l(z) \quad \text{for} \quad z \in \tilde{D}, \zeta \in D_l \,. \tag{27}$$

We obtain again interesting canonical mappings from the dielectric Green's function by passing to the limit $\varepsilon = 0$.

(d) The expression $\tilde{G}_l(z) + \log|z-\zeta|$ satisfies the linear relations (2.18) if $\zeta \in D_l$. It has on C the same boundary values as the function $g(z,\zeta) + \log|z-\zeta| + c_l(z)$ which is harmonic in D, with

$$c_l(z) = c_{lj} \quad \text{for} \quad z \in D_j \,. \tag{28}$$

Thus, the new combination will belong to the class Σ and can, therefore, be developed into a Fourier series in the $\{h_\nu\}$-system. An easy calculation leads to

$$g(z,\zeta) = \log\frac{1}{|z-\zeta|} - c_l(z) - 2\pi\sum_{\nu=1}^{\infty}\frac{h_\nu(z)h_\nu(\zeta)}{\rho_\nu(1+\rho_\nu)}, \; z \in D, \zeta \in D_l \,. \tag{29}$$

From (29) and (2.21) follows

$$g_\varepsilon(z,\zeta) - \varepsilon g(z,\zeta) = c_l(z) + 2\pi\varepsilon\sum_{\nu=1}^{\infty}\frac{1}{\rho_\nu(1+\varepsilon\rho_\nu)}h_\nu(z)h_\nu(\zeta) \,. \tag{30}$$

Thus, we find the limit formulas, valid for $z \in D, \zeta \in D_l$:

$$\lim_{\varepsilon\to 0} g_\varepsilon(z,\zeta) = c_l(z) \,, \quad \lim_{\varepsilon\to\infty}\frac{1}{\varepsilon}g_\varepsilon(z,\zeta) = g(z,\zeta) \,. \tag{31}$$

4. Dielectric Green's functions and conformal mapping. In this section, we shall show that the dielectric Green's function $g_\varepsilon(z,\zeta)$ leads to a univalent analytic function in $\tilde{D}$ and to a set of univalent analytic functions in D. Let us suppose, for the sake of definiteness, that the source point ζ lies in $\tilde{D}$. Let $p_\varepsilon(z,\zeta)$ be the analytic completion of $g_\varepsilon(z,\zeta)$ for z in $\tilde{D}$; that is, $p_\varepsilon(z,\zeta)$ is analytic for $z \in D$ and we have

$$g_\varepsilon(z,\zeta) = \Re\{p_\varepsilon(z,\zeta)\} \,. \tag{1}$$

$p_\varepsilon(z,\zeta)$ is regular analytic except for the two logarithmic poles at ζ and at ∞. The function has no periods with respect to the boundary curves C_j. Hence

$$\tilde{f}_\varepsilon(z,\zeta) = \exp[-p_\varepsilon(z,\zeta)] \,, \qquad z \in \tilde{D}, \zeta \in \tilde{D} \tag{2}$$

is a single-valued analytic function of $z \in \tilde{D}$ and regular in this domain except for the simple pole at infinity. Since the analytic completion of a harmonic function is only determined up to an additive imaginary constant, we may choose $\tilde{f}_\varepsilon$ in such a way that

$$\tilde{f}'_\varepsilon(\infty,\zeta) = 1 \,, \quad \tilde{f}_\varepsilon(\zeta,\zeta) = 0 \,. \tag{2'}$$

We may similarly complete $g_\varepsilon(z, \zeta)$ to analytic functions of z in D. In order to determine the additive constants for the disjoint domains D_j we proceed as follows. By condition (c) of § 2 on $g_\varepsilon(z, \zeta)$ and because of the Cauchy-Riemann equations, we have, whatever the analytic completion $p_\varepsilon(z, \zeta)$ of $g_\varepsilon(z, \zeta)$ in D:

$$(3) \qquad \Im\{p_\varepsilon(z, \zeta)\} = \varepsilon \Im\{\tilde{p}_\varepsilon(z, \zeta)\} + k_j \quad \text{for} \quad z \in C_j .$$

Here $\tilde{p}_\varepsilon$ and p_ε shall denote the limits of p_ε from $\tilde{D}$ and D, respectively; we shall use this more specific notation whenever discussing boundary relations. We dispose now of the additive constants in the domains D_j by requiring $k_j = 0$. This convention fixes $p_\varepsilon(z, \zeta)$ in D in a unique way.

In analogy to (2), we define

$$(4) \qquad f_\varepsilon(z, \zeta) = \exp\left[-\frac{1}{\varepsilon} p_\varepsilon(z, \zeta) \right] \quad \text{for} \quad z \in D, \zeta \in \tilde{D} .$$

We shall prove the

THEOREM. *The function $\tilde{f}_\varepsilon(z, \zeta)$ is univalent in $\tilde{D}$ and the set of functions $f_\varepsilon(z, \zeta)$ is univalent in D in the sense that*

$$(5) \qquad f_\varepsilon(z_1, \zeta) = f_\varepsilon(z_2, \zeta) \text{ and } z_1, z_2 \in D \text{ implies } z_1 = z_2 .$$

In order to prove this theorem, we start with the

LEMMA. *The dielectric Green's function has no critical points. That is, the equation $p'_\varepsilon(z, \zeta) = 0$ is only satisfied at $z = \infty$ and this point is a pole of the Green's function. The dash denotes differentiation of $p_\varepsilon(z, \zeta)$ with respect to its analytic argument z.*

Proof. We denote again, more precisely, the analytic completion of $g_\varepsilon(z, \zeta)$ by $\tilde{p}_\varepsilon$ or by p_ε according to the location of z in $\tilde{D}$ or D, respectively. We combine the boundary conditions (c) and (d) of § 2 on the dielectric Green's function $g_\varepsilon(z, \zeta)$ into the one complex equation

$$(6) \qquad p'_\varepsilon(z, \zeta)z' = \frac{1 + \varepsilon}{2} \tilde{p}'_\varepsilon(z, \zeta)z' + \frac{1 - \varepsilon}{2} \overline{\tilde{p}'_\varepsilon(z, \zeta)z'} .$$

Since we assume throughout this paper $\varepsilon > 0$, equation (6) yields

$$(7) \qquad \Re\mathrm{e}\{p'_\varepsilon(z, \zeta)/\tilde{p}'_\varepsilon(z, \zeta)\} > 0 \quad \text{for } z \in C .$$

This inequality implies, in particular:

$$(8) \qquad \oint_\sigma d \arg p'_\varepsilon(z, \zeta) = \oint_\sigma d \arg \tilde{p}'_\varepsilon(z, \zeta) .$$

The statement is evident if p'_ε and $\tilde{p}'_\varepsilon$ are non-zero on C but it can be upheld in the usual way even in the case that these two functions have common zeros on C

Let Z, P and $\tilde{Z}, \tilde{P}$ denote the number of zeros and poles of p'_ε and $\tilde{p}'_\varepsilon$ respectively, in their domains of definition. By the argument principle, we have

$$(9) \qquad \oint_C d \arg p'_\varepsilon(z, \zeta) = Z - P\,, \quad \oint_C d \arg \tilde{p}'_\varepsilon(z, \zeta) = \tilde{P} - \tilde{Z}$$

if z runs through C in the positive sense with respect to D. Combining (8) and (9), we obtain

$$(10) \qquad Z + \tilde{Z} = P + \tilde{P}\,.$$

But all poles of p'_ε and $\tilde{p}'_\varepsilon$ are known; clearly $P = 0, \tilde{P} = 1$ and $Z \geq 1$. Hence, we conclude from (10):

$$(11) \qquad Z = 0\,, \quad \tilde{Z} = 1\,.$$

This proves our lemma.

In order to prove the theorem, we consider the lines defined by

$$(12) \qquad \Im\{\tilde{p}_\varepsilon(z, \zeta)\} = \alpha \text{ for } z \in \tilde{D}\,, \quad \Im\left\{\frac{1}{\varepsilon} p_\varepsilon(z, \zeta)\right\} = \alpha \text{ for } z \in D\,.$$

Each such line starts from the logarithmic pole ζ and runs to ∞. By virtue of our convention on the analytic completion of $g_\varepsilon(z, \zeta)$ these lines are continuous in the entire plane and, except on C, they are even analytic. Because of our lemma, there is no intersection between different lines except at ζ and ∞. The lines have the physical interpretation as lines of force for the corresponding electrostatic problem and the lemma asserts that there are no points of equilibrium in the field. The lines form for $0 \leq \alpha < 2\pi$ a non-intersecting system which covers the entire complex plane. Along each line, $g_\varepsilon(z, \zeta)$ decreases monotonically when we pass from ζ to ∞. These facts guarantee obviously that the analytic functions $\tilde{f}_\varepsilon(z, \zeta)$ and $f_\varepsilon(z, \zeta)$ have the above stated univalency properties. Thus, the theorem is proved.

Let us assume without loss of generality that $\zeta = 0$. Using the limit theorems of § 3, we can assert:

$$(13) \qquad \tilde{f}_0(z, 0) = \tilde{\varphi}(z, 0)^{-1}\,, \quad \tilde{f}_1(z, 0) = z\,, \quad \tilde{f}_\infty(z, 0) = \tilde{\psi}(z, 0)^{-1}\,.$$

We have thus found a one-parameter family of univalent functions which connects continuously the circular slit mapping through the identity mapping with the radial slit mapping.

In order to illustrate the significance of this result, we calculate from (2.16) that

$$\log|\tilde{f}'_\varepsilon(\zeta,\zeta)| = 2\pi(\varepsilon-1)\sum_{\nu=1}^{\infty}\frac{\rho_\nu}{(1+\rho_\nu)(1+\varepsilon\rho_\nu)}\tilde{h}_\nu(\zeta)^2 . \tag{14}$$

Since all $\rho_\nu > 0$, this is a monotonically increasing function of ε in the interval $[0, \infty)$; it is negative for $0 \leq \varepsilon < 1$ and positive for $1 < \varepsilon$. In particular:

$$|\tilde{f}'_0(\zeta,\zeta)| < 1 \quad |\tilde{f}'_\infty(\zeta,\zeta)| > 1 . \tag{15}$$

We define the family $\mathfrak{F}_\zeta$ of all functions $\tilde{f}(z)$ which are analytic and univalent in $\tilde{D}$ and normalized by the requirements

$$\tilde{f}'(\infty) = 1 \qquad \tilde{f}(\zeta) = 0 . \tag{16}$$

Through the mapping $w = \tilde{f}(z)$ we obtain the new domain $\tilde{D}_w$; applying the inequalities (15) in this domain and returning to the original domain $\tilde{D}$, we obtain the inequality

$$|f'_0(\zeta,\zeta)| \leq |\tilde{f}'(\zeta)| \leq |\tilde{f}'_\infty(\zeta,\zeta)| \tag{17}$$

valid for each $f \in \mathfrak{F}_\zeta$.

Inequality (16) asserts an extremum property of the canonical slit functions $\tilde{f}_0$ and $\tilde{f}_\infty$ which is well-known [13, 15]. It is, however, not obvious that all real values between the extrema are also possible values for $|f'(\zeta)|$ in $\mathfrak{F}_\zeta$. We have now explicity constructed a one-parameter family in $\mathfrak{F}_\zeta$ which interpolates between the two extremum values.

There are various other possibilities to obtain from the dielectric Green's function one-parameter families of univalent functions. Consider, *for example, the analytic functions*

$$A_\varepsilon(z,\zeta) = \frac{\partial}{\partial\xi}p_\varepsilon(z,\zeta) , \quad B_\varepsilon(z,\zeta) = \frac{1}{i}\frac{\partial}{\partial\eta}p_\varepsilon(z,\zeta) \tag{18}$$

with $\zeta = \xi + i\eta$. Both functions are single-valued in $\tilde{D}$ and in D; they have for $z = \zeta$ simple poles with residue 1 and are else regular in D and in D. We obtain from the identity (6) by differentiation

$$A'_\varepsilon(z,\zeta)z' = \frac{1+\varepsilon}{2}\tilde{A}'_\varepsilon(z,\zeta)z' + \frac{1-\varepsilon}{2}\overline{(\tilde{A}'_\varepsilon(z,\zeta)z')} \tag{19}$$

$$B'_\varepsilon(z,\zeta)z' = \frac{1+\varepsilon}{2}\tilde{B}'_\varepsilon(z,\zeta)z' - \frac{1-\varepsilon}{2}\overline{(\tilde{B}'_\varepsilon(z,\zeta)z')} . \tag{19'}$$

Let a be an arbitrary point on C; integrating (19) along C from a to $z \in C$, we find

$$A_\varepsilon(z,\zeta) - A_\varepsilon(a,\zeta) = \frac{1+\varepsilon}{2}[\tilde{A}_\varepsilon(z,\zeta) - \tilde{A}_\varepsilon(a,\zeta)] + \frac{1-\varepsilon}{2}\overline{[\tilde{A}_\varepsilon(z,\zeta) - \tilde{A}_\varepsilon(a,\zeta)]} . \tag{20}$$

Hence, we have

$$\Re\left\{\frac{A_\varepsilon(z,\zeta)-A_\varepsilon(a,\zeta)}{\tilde{A}_\varepsilon(z,\zeta)-\tilde{A}_\varepsilon(a,\zeta)}\right\}>0 \quad \text{for} \quad z\in C\,. \tag{21}$$

Reasoning as before we can conclude by means of the argument principle that $A_\varepsilon(z,\zeta)$ takes the value $A_\varepsilon(a,\zeta)$ precisely once in $D+C$ and that $\tilde{A}_\varepsilon(z,\zeta)$, likewise, takes every boundary value precisely once. Thus, $A_\varepsilon(z,\zeta)$ and $\tilde{A}_\varepsilon(z,\zeta)$ are univalent in their respective domains of definition. The same reasoning applies to $B_\varepsilon(z,\zeta)$ and $\tilde{B}_\varepsilon(z,\zeta)$.

It is known, and easily verified, that

$$\tilde{A}_0(z,\zeta)=\frac{\partial}{\partial\xi}\log\tilde{\varphi}\,(z,\zeta)\,,\quad \tilde{B}_0(z,\zeta)=\frac{1}{i}\frac{\partial}{\partial\eta}\log\tilde{\varphi}(z,\zeta) \tag{22}$$

are univalent functions in $\tilde{D}$ with a simple pole at $z=\zeta$ and that they map $\tilde{D}$ onto the entire complex plane, slit along rectilinear segments parallel to the imaginary and the real axis, respectively [16]. Similarly, the analytic functions

$$\tilde{A}_\infty(z,\zeta)=\frac{\partial}{\partial\xi}\log\tilde{\psi}(z,\zeta)\,,\quad \tilde{B}_\infty(z,\zeta)=\frac{1}{i}\frac{\partial}{\partial\eta}\log\tilde{\psi}(z,\zeta) \tag{23}$$

are univalent in $\tilde{D}$ with the same singularity and map the domain onto the entire complex plane, slit along segments parallel to the real and the imaginary axis, respectively. Hence, by the uniqueness theorems on the canonical mappings of a domain, we must have

$$\tilde{A}_\infty(z,\zeta)=\tilde{B}_0(z,\zeta)+\kappa(\zeta)\,;\ \tilde{B}_\infty(z,\zeta)=\tilde{A}_0(z,\zeta)+\lambda(\zeta)\,. \tag{24}$$

Finally, clearly

$$\tilde{A}_1(z,\zeta)=\tilde{B}_1(z,\zeta)=\frac{1}{z-\zeta}\,. \tag{25}$$

Hence, $\tilde{A}_\varepsilon(z,\zeta)$ and $\tilde{B}_\varepsilon(z,\zeta)$ interpolate between the two parallel slit mappings through the simple rational mapping (25).

Using the series development (2.16) for $g_\varepsilon(z,\zeta)$, $\zeta\in\tilde{D}$, we may prove the well-known extremum properties of the canonical slit mappings in the same way, as we did above for the circular and the radial slit mapping.

We do not enter into a more detailed discussion of these families of univalent functions. We want to remark, however, that the dielectric Green's function is not, like the ordinary Green's function, a conformal invariant. By auxiliary mappings of $\tilde{D}$ into a domain $\tilde{D}_w$, one may obtain very different one-parameter families of univalent functions which interpolate between the canonical slit mappings.

5. Dielectric Green's functions and norms in function spaces. With each dielectric Green's function $g_\varepsilon(z, \zeta)$ we can connect a positive-definite quadratic form which may be interpreted as a norm in the linear function spaces Σ and $\tilde{\Sigma}$, defined in § 2. This norm has remarkable properties for function pairs $h \in \Sigma$ and $\tilde{h} \in \tilde{\Sigma}$ which have on C equal boundary values or equal normal derivatives. Useful inequalities and identities can be established which facilitate the solution of the boundary value problem in potential theory by utilizing auxiliary solutions in complementary domains. One can characterize the Fredholm eigen values λ_ν as solutions of certain extremum problems involving these quadratic forms. This characterization, in turn, will lead later to elegant variational formulas for the λ_ν under infinitesimal deformation of the curve system C.

Let h and $\tilde{h}$ be two arbitrary functions of the classes Σ and $\tilde{\Sigma}$, respectively. We have the Fourier developments

$$h(z) = \sum_{\nu=1}^{\infty} x_\nu h_\nu(z) , \quad \tilde{h}(z) = \sum_{\nu=1}^{\infty} \tilde{x}_\nu \tilde{h}_\nu(z) \tag{1}$$

in terms of the complete orthonomal sets $\{\rho_\nu^{-1/2} h_\nu(z)\}$ and $\{\tilde{h}_\nu(z)\}$ of these linear spaces. The Fourier coefficients are given by

$$x_\nu = \frac{1}{\rho_\nu} D(h, h_\nu) , \quad \tilde{x}_\nu = \tilde{D}(\tilde{h}, \tilde{h}_\nu) \tag{2}$$

where D and $\tilde{D}$ denote the Dirichlet integral in Σ and $\tilde{\Sigma}$:

$$D(h, H) = \iint_D \nabla h \cdot \nabla H d\tau , \quad \tilde{D}(\tilde{h}, \tilde{H}) = \iint_{\tilde{D}} \nabla \tilde{h} \cdot \nabla \tilde{H} d\tau . \tag{3}$$

Let us consider now the particular case that

$$h(z) = \tilde{h}(z) \quad \text{on} \quad C . \tag{4}$$

By Green's identity and (1.4′), we have obviously

$$D(h, h_\nu) = -\int_C h \frac{\partial h_\nu}{\partial n} ds = -\int_C \tilde{h} \frac{\partial \tilde{h}_\nu}{\partial n} ds = -\tilde{D}(\tilde{h}, \tilde{h}_\nu) \tag{5}$$

which gives

$$x_\nu = -\frac{1}{\rho_\nu} \tilde{x}_\nu . \tag{6}$$

We proceed analogously for two function $h \in \Sigma$ and $\tilde{h} \in \tilde{\Sigma}$ which satisfy on C the relation

$$\frac{\partial h}{\partial n} = \frac{\partial \tilde{h}}{\partial n} . \tag{7}$$

Now, Green's identity and (1.4) yield

$$(8) \qquad D(h, h_\nu) = -\int_\sigma h_\nu \frac{\partial h}{\partial n} ds = \rho_\nu \int_\sigma \tilde{h}_\nu \frac{\partial \tilde{h}}{\partial n} ds = \rho_\nu \tilde{D}(\tilde{h}, \tilde{h}_\nu)$$

and, consequently

$$(9) \qquad x_\nu = \tilde{x}_\nu \ .$$

Thus, both boundary relations (4) and (7) reflect themselves in a very simple manner in the relations (6) and (9) between the Fourier coefficients.

We define next the bilinear form

$$(10) \qquad \pi_\varepsilon(h, H) = \frac{1}{2\pi} \int_\sigma \int_\sigma g_\varepsilon(z, \zeta) \frac{\partial h(z)}{\partial n} \frac{\partial H(\zeta)}{\partial n} ds_z ds_\zeta$$

for any two elements of Σ and in precisely the same manner we define the bilinear from $\tilde{\pi}_\varepsilon(\tilde{h}, \tilde{H})$ for any two elements in $\tilde{\Sigma}$.

By use of the Fourier type formulas (2.13) and (2.13') we may express the bilinear forms in terms of the Fourier coefficients of the functions involved. Let us denote the Fourier coefficients of $h, \tilde{h}$ by $x_\nu, \tilde{x}_\nu$ and of $H, \tilde{H}$ by $y_\nu, \tilde{y}_\nu$; then a straightforward calculation shows that

$$(11) \qquad \pi_\varepsilon(h, H) = \sum_{\nu=1}^{\infty} \frac{\varepsilon\rho_\nu}{1 + \varepsilon\rho_\nu} x_\nu y_\nu, \ \tilde{\pi}_\varepsilon(\tilde{h}, \tilde{H}) = \sum_{\nu=1}^{\infty} \frac{\varepsilon\rho_\nu}{1 + \varepsilon\rho_\nu} \tilde{x}_\nu \tilde{y}_\nu \ .$$

We verify, first, from (11) that the quadratic forms $\pi_\varepsilon(h, h)$ and $\tilde{\pi}_\varepsilon(\tilde{h}, \tilde{h})$ are positive-definite. This fact allows us to interpret them, indeed, as norms in their corresponding function spaces.

On the other hand, we have because of the normalizations (2.8) and (2.17)

$$(12) \qquad D(h, H) = \sum_{\nu=1}^{\infty} \rho_\nu x_\nu y_\nu, \quad \tilde{D}(\tilde{h}, \tilde{H}) = \sum_{\nu=1}^{\infty} x_\nu y_\nu \ .$$

We define further the bilinear forms

$$(13) \qquad I'_\varepsilon(h, H) = D(h, H) - \frac{1}{\varepsilon}\pi_\varepsilon(h, H) \ , \quad \tilde{I}'_\varepsilon(\tilde{h}, \tilde{H}) = \tilde{D}(\tilde{h}, \tilde{H}) - \tilde{\pi}_\varepsilon(\tilde{h}, \tilde{H})$$

and obtain for them the explicit representations:

$$(14) \qquad I'_\varepsilon(h, H) = \sum_{\nu=1}^{\infty} \frac{\varepsilon\rho_\nu^2}{1 + \varepsilon\rho_\nu} x_\nu y_\nu \ , \quad \tilde{I}'_\varepsilon(\tilde{h}, \tilde{H}) = \sum_{\nu=1}^{\infty} \frac{1}{1 + \varepsilon\rho_\nu} \tilde{x}_\nu \tilde{y}_\nu \ .$$

These formulas show that I'_ε and $\tilde{I}'_\varepsilon$, too, are positive-definite and lead to norms in Σ and $\tilde{\Sigma}$. We have the estimates:

$$0 \leq \frac{1}{\varepsilon}\pi_\varepsilon(h, H) \leq D(h, h); \quad 0 \leq \tilde{\pi}_\varepsilon(\tilde{h}, \tilde{h}) \leq \tilde{D}(\tilde{h}, \tilde{h}) . \tag{15}$$

By the very definition of π_ε and $\tilde{\pi}_\varepsilon$, we have

THEOREM I. *If*

$$\frac{\partial h}{\partial n} = \frac{\partial \tilde{h}}{\partial n} \quad \textit{and} \quad \frac{\partial H}{\partial n} = \frac{\partial \tilde{H}}{\partial n} \quad \text{on} \quad C \tag{16}$$

we have

$$\pi_\varepsilon(h, H) = \tilde{\pi}_\varepsilon(\tilde{h}, \tilde{H}) . \tag{17}$$

From (4), (6) and (14), we derive :

THEOREM II. *If*

$$h = \tilde{h} \text{ and } H = \tilde{H} \text{ on } C , \tag{18}$$

we have

$$I'_\varepsilon(h, H) = \varepsilon \tilde{I}'_\varepsilon(\tilde{h}, \tilde{H}) . \tag{19}$$

Finally, we verify from the explicit representations for the bilinear forms

THEOREM III. *If*

$$h = \tilde{h} \text{ and } \frac{\partial H}{\partial n} = \frac{\partial \tilde{H}}{\partial n} \text{ on } C , \tag{20}$$

we have

$$D(h, H) = -\tilde{D}(\tilde{h}, \tilde{H}) \tag{21}$$

and

$$\pi_\varepsilon(h, H) = -\varepsilon \tilde{I}'_\varepsilon(\tilde{h}, \tilde{H}), \quad I'_\varepsilon(h, H) = -\tilde{\pi}_\varepsilon(\tilde{h}, \tilde{H}) . \tag{22}$$

Theorems I–III show a very symmetric interrelation between the various bilinear forms for elements with matching boundary data on C.

The significance of the preceding theorems lies in the fact that one has often to solve a boundary value problem, say in $\tilde{D}$, which is much easier to solve in the complementary domain D. In this case, the above theorems provide valuable information. Let us illustrate the method by the following applications.

(a) Given a function $h \in \Sigma$, to determine the function $\tilde{h} \in \tilde{\Sigma}$ which has on C the same boundary values as h. In particular, we ask for the Dirichlet integral $\tilde{D}(\tilde{h}, \tilde{h})$.

This problem arises, for example, in two-dimensional electrostatics in connection with the question of polarization of a set of conductors in a homogeneous field [19, 22].

We derive inequalities for the Dirichlet integral in question by applying Theorems I–III. We start from the fact that π_ε and Γ_ε have definite quadratic forms and that they satisfy, therefore, the Schwarz inequalities

$$(23)\quad \pi_\varepsilon(h, H)^2 \leq \pi_\varepsilon(h, h) \cdot \pi_\varepsilon(H, H)\,; \quad \Gamma_\varepsilon(h, H)^2 \leq \Gamma_\varepsilon(h, h) \cdot \Gamma_\varepsilon(H, H)\,.$$

We select a pair of test functions $H \in \Sigma$ and $\tilde{H} \in \tilde{\Sigma}$ which have equal normal derivatives on C and obtain from Theorem III and from (23)

$$(24)\qquad \Gamma_\varepsilon(h, H)^2 \leq \tilde{\pi}_\varepsilon(\tilde{h}, \tilde{h}) \cdot \tilde{\pi}_\varepsilon(\tilde{H}, \tilde{H})\,.$$

Using the definition (13) of $\tilde{\Gamma}_\varepsilon$ and Theorems I, II, we can transform (24) into

$$(25)\qquad \Gamma_\varepsilon(h, H)^2 \leq [\tilde{D}(\tilde{h}, \tilde{h}) - \frac{1}{\varepsilon}\Gamma_\varepsilon(h, h)]\pi_\varepsilon(H, H)\,.$$

This inequality contains the sought Dirichlet integral $\tilde{D}(\tilde{h}, \tilde{h})$ and else only the known function of $h \in \Sigma$ and the arbitrary test function $H \in \Sigma$. Thus:

$$(26)\qquad \tilde{D}(\tilde{h}, \tilde{h}) \geq \frac{\Gamma_\varepsilon(h, H)^2}{\pi_\varepsilon(H, H)} + \frac{1}{\varepsilon}\Gamma_\varepsilon(h, h)\,.$$

It is easily seen from our derivation that the inequality (26) is sharp if H is chosen as that function in Σ which has on C the same normal derivative as $\tilde{h}$; in fact, in this case, the Schwarz inequality leading to (24) becomes an equality. Thus, we can express (26) as follows:

$$(26')\qquad \tilde{D}(\tilde{h}, \tilde{h}) = \max \frac{\Gamma_\varepsilon(h, H)^2}{\pi_\varepsilon(H, H)} + \frac{1}{\varepsilon}\Gamma_\varepsilon(h, h) \text{ for all } H \in \Sigma\,.$$

This representation permits us to determine the desired Dirichlet integral by a Ritz procedure in Σ.

It is sometimes more convenient to renounce a precise equation in order to obtain a simple and applicable estimate. We may select, for this purpose, the test function $H(z)$ as equal to the given function $h(z)$; in this case, we have by (13) and (26)

$$(27)\qquad \tilde{D}(\tilde{h}, \tilde{h}) \geq \frac{\Gamma_\varepsilon(h, h)}{\pi_\varepsilon(h, h)} D(h, h)\,.$$

This inequality holds for all pairs of functions $h \in \Sigma, \tilde{h} \in \tilde{\Sigma}$ which have equal boundary values at the same points of C.

In order to understand better the important inequality (27), we express it in terms of the corresponding Fourier coefficients. If we denote again by x_ν the coefficients of $h(z)$, we have by (6) the values $-\rho_\nu x_\nu = \tilde{x}_\nu$ for the Fourier coefficients of $\tilde{h}(z)$. Hence, using the explicit representations (11), (12) and (14) for the quadratic forms, we may write (27) as follows:

$$(27') \qquad \sum_{\nu=1}^{\infty} \rho_\nu^2 x_\nu^2 \cdot \sum_{\nu=1}^{\infty} \frac{\varepsilon\rho_\nu}{1+\varepsilon\rho_\nu} x_\nu^2 \geq \sum_{\nu=1}^{\infty} \rho_\nu x_\nu^2 \cdot \sum_{\nu=1} \frac{\varepsilon\rho_\nu^2}{1+\varepsilon\rho_\nu} x_\nu^2 .$$

We rearrange (27') into the from

$$(27'') \qquad \sum_{\nu,\mu=1}^{\infty} \frac{\varepsilon^2\rho_\nu\rho_\mu(\rho_\nu-\rho_\mu)^2}{(1+\varepsilon\rho_\nu)(1+\varepsilon\rho_\mu)} x_\nu^2 x_\mu^2 \geq 0 .$$

Now the inequality has become evident; but, what is more important, we recognize that equality in (27'') and, hence in (27), holds if and only if all x_ν vanish except for those which belong to a fixed eigen value λ_μ. Thus, equality in (27) holds for

$$(28) \qquad h(z) = h_\nu(z) \text{ and } \tilde{h}(z) = -\rho_\nu \tilde{h}_\nu(z) , \qquad \nu = 1, 2, \cdots,$$

and only for these functions.

It is interesting that the inequality (27) becomes precise infinitely often, namely for all functions of the sets $\{h_\nu\}$, $\{\tilde{h}_\nu\}$, which are complete in Σ and $\tilde{\Sigma}$. On the other hand, this fact leads to a new characterization of the Fredholm eigen functions

(b) We deal next with the analogous question: given a function $h \in \Sigma$, to determine the function $\tilde{h} \in \tilde{\Sigma}$ which has at corresponding points of C the same normal derivative as h. In particular, to determine the Dirichlet integral of $\tilde{h}$.

This problem occurs in the theory of a steady incompressible and irrotational fluid flow in the plane around the set of obstacles C. The sought Dirichlet integral, in this case, is the virtual mass of the curve system C [19, 22].

We select now a pair of test functions $H \in \Sigma$, $\tilde{H} \in \tilde{\Sigma}$ which have equal boundary values on C. Starting again with the Schwarz inequality (23) and Theorem III, we have

$$(29) \qquad \pi_\varepsilon(h, H)^2 \leq \varepsilon^2 \tilde{\Gamma}_\varepsilon(\tilde{h}, \tilde{h}) \cdot \tilde{\Gamma}_\varepsilon(\tilde{H}, \tilde{H}) .$$

We apply equation (13), make use of Theorems I and II and find

$$(30) \qquad \pi_\varepsilon(h, H)^2 \leq \varepsilon\Gamma_\varepsilon(H, H)[\tilde{D}(\tilde{h}, \tilde{h}) - \pi_\varepsilon(h, h)] .$$

Thus finally

$$\tilde{D}(\tilde{h}, \tilde{h}) \geq \frac{\pi_\varepsilon(h, H)^2}{\varepsilon \Gamma_\varepsilon(H, H)} + \pi_\varepsilon(h, h) \,. \tag{31}$$

We obtained thus again a lower bound for the Dirichlet integral in terms of the given function h and the arbitrary test function H. If H has on C the same boundary values as $\tilde{h}$, the inequality (31) becomes an equality. This fact allows us again to approximate arbitrarily the Dirichlet integral from below by a Ritz sequence of test functions.

When we choose, on the other hand, $H(z) = h(z)$, we obtain

$$\tilde{D}(\tilde{h}, \tilde{h}) \geq \frac{\pi_\varepsilon(h, h)}{\Gamma_\varepsilon(h, h)} D(h, h) \,. \tag{32}$$

This inequality holds for every pair of functions $h \in \Sigma$, $\tilde{h} \in \tilde{\Sigma}$ with equal normal derivatives on C.

This inequality can be verified by means of the explicit Fourier representations (11), (12) and (14) as we did in the case of the inequality (27). We can further show as before that equality in (32) can hold if and only if

$$h(z) = h_\nu(z) \,, \quad \tilde{h}(z) = \tilde{h}_\nu(z) \,, \qquad \nu = 1, 2, \cdots \,. \tag{33}$$

Thus, inequality (32) leads to another characterization of the Fredholm eigen functions.

We obtain corresponding inequalities when we interchange the role of D and $\tilde{D}$; the Dirichlet integral of a function $h \in \Sigma$ can then be estimated in terms of a function $\tilde{h} \in \tilde{\Sigma}$ which has on C either the same boundary values or the same normal derivative as h.

The most convenient form in which the preceding theory can be applied is obtained by using $\varepsilon = 1$. For, in this case, the dielectric Green's function reduces to the elementary function $-\log|z - \zeta|$ and the bilinear forms can be easily evaluated. Indeed, the general method was first applied to obtain isoperimetric inequalities for polarization and virtual mass with this particular choice of ε [18, 19, 20]. However, the flexibility of the method is obviously increased by considering arbitrary positive ε-values and the significance of the procedure is clarified in this way.

We shall now utilize the quadratic forms in order to obtain estimates for the Fredholm eigen values λ_ν. Let λ_1 be the lowest positive Fredholm eigen value > 1. We have shown in § 1 that with λ_1 also $-\lambda_1$ is an eigen value. We denote $\lambda_2 = -\lambda_1$. By definition (2.12) of the ρ_ν, we have obviously

$$\frac{1}{\rho_1} = \rho_2 \leq \rho_\nu \leq \rho_1 \,, \qquad \nu = 1, 2, 3, \cdots \,. \tag{34}$$

Using now the developments (11), (12) and (14) of the various bilinear forms, we verify by inspection the following theorems:

THEOREM IV. *For every function $h \in \Sigma$ the inequalities*

$$\frac{\varepsilon}{1+\varepsilon\rho_1} \leq \frac{\pi_\varepsilon(h,h)}{D(h,h)} \leq \frac{\varepsilon\rho_1}{\rho_1+\varepsilon} \tag{35}$$

hold. The first equality sign holds only for those function $h_\nu \in \Sigma$ which belong to the eigen value λ_1; the second equality sign holds only for functions $h_\nu \in \Sigma$ which belong to the eigen value λ_2.

THEOREM V. *Every function $\tilde{h} \in \tilde{\Sigma}$ satisfies the inequalities*

$$\frac{\varepsilon}{\rho_1+\varepsilon} \leq \frac{\tilde{\pi}_\varepsilon(\tilde{h},\tilde{h})}{\tilde{D}(\tilde{h},\tilde{h})} \leq \frac{\varepsilon\rho_1}{1+\varepsilon\rho_1}\,. \tag{36}$$

Equality holds only if $\tilde{h} = \tilde{h}_\nu$ where $\tilde{h}_\nu$ belongs to the eigen values λ_2 and λ_1, respectively.

We have thus characterized the lowest positive and non-trivial Fredholm eigen value λ_1 by a minimum and a maximum problem in Σ and in $\tilde{\Sigma}$ for the ratio of two positive-definite quadratic forms. This characterization makes it possible to estimate this eigen value by the use of test functions in Σ and in $\tilde{\Sigma}$. The most convenient case for applications is, of course, the case $\varepsilon = 1$.

It is clearly desirable to find analogous extremum problems which characterize the higher eigen values λ_ν. For this purpose, we introduce the bilinear form

$$\Gamma_{\varepsilon,e}(h,H) = \frac{\Gamma_\varepsilon(h,H) - \Gamma_e(h,H)}{\varepsilon - e}\,, \qquad \varepsilon > 0,\ e > 0 \tag{37}$$

in Σ and the bilinear form

$$\tilde{\pi}_{\varepsilon,e}(\tilde{h},\tilde{H}) = \frac{\tilde{\pi}_\varepsilon(\tilde{h},\tilde{H}) - \tilde{\pi}_e(\tilde{h},\tilde{H})}{\varepsilon - e}\,, \qquad \varepsilon > 0,\ e > 0 \tag{37'}$$

in $\tilde{\Sigma}$. From (11) and (14), we obtain the Fourier representations

$$\Gamma_{\varepsilon,e}(h,H) = \sum_{\nu=1}^{\infty} \frac{\rho_\nu^2 x_\nu y_\nu}{(1+\varepsilon\rho_\nu)(1+e\rho_\nu)}\,;\quad \tilde{\pi}_{\varepsilon,e}(\tilde{h},\tilde{H}) = \sum_{\nu=1}^{\infty} \frac{\rho_\nu \tilde{x}_\nu \tilde{y}_\nu}{(1+\varepsilon\rho_\nu)(1+e\rho_\nu)} \tag{38}$$

The quadratic forms $\Gamma_{\varepsilon,e}(h,h)$ and $\tilde{\pi}_{\varepsilon,e}(\tilde{h},\tilde{h})$ are evidently positive-definite.

We observe that the function

$$f(x) = \frac{x}{(1+\varepsilon x)(1+ex)} \tag{39}$$

takes in the interval $0 \leq x < \infty$ its maximum value at the point

$$X_m = \frac{1}{\sqrt{\varepsilon e}} . \tag{40}$$

Hence, (38) yields the following theorems:

THEOREM VI. *Every function $h \in \Sigma$ satisfies the inequality*

$$\frac{\Gamma_{\varepsilon,e}(h, h)}{D(h, h)} \leq f(\rho_m) \tag{41}$$

where ρ_m is a value in the sequence of the ρ_ν which gives the largest value of f. Equality holds only for such h_ν which belong to such a value ρ_m.

THEOREM VII. *For every function $\tilde{h} \in \tilde{\Sigma}$, the inequality*

$$\frac{\tilde{\pi}_{\varepsilon,e}(\tilde{h}, \tilde{h})}{\tilde{D}(\tilde{h}, \tilde{h})} \leq f(\rho_m) \tag{42}$$

holds where ρ_m is a value in the sequence of the ρ_ν which gives the largest possible value of $f(\rho)$. Equality holds only for such $\tilde{h}_\nu$ which belong to such a ρ_m.

Given any specific ρ_ν, we can always choose $\sqrt{\varepsilon e} = \rho_\nu^{-1}$ and the corresponding maximum problem will pick out this particular eigen value. We can apply Theorems VI and VII in order to obtain estimates for the location of ρ_ν-values near any given point x_m by the use of test functions in Σ and in $\tilde{\Sigma}$. It is easily seen that Theorems IV and V are contained in Theorems VI and VII as limit cases.

We specialize in Theorem IV $\varepsilon = 1$ and obtain the particular result

$$\frac{1}{2\pi}\int_\sigma\int_\sigma \log\frac{1}{|z-\zeta|}\frac{\partial h(z)}{\partial n}\frac{\partial h(\zeta)}{\partial n}ds_z ds_\zeta \leq \frac{\rho_1}{1+\rho_1}D(h, h) \tag{35'}$$

for every $h \in \Sigma$; equality holds only if $h = h_\nu$ and h_ν belongs to λ_2.

This result permits the following application. Consider the system of curves C^* which consists of the subset $C_1, C_2, \cdots, C_{N^*}$ of C with $N^* < N$. This system of boundaries determines a connected exterior $\tilde{D}^* \supset \tilde{D}$ and the set D^* of the domains D_j, $j = 1, \cdots, N^*$. Let Σ^* be the function class in D^* which is analogous to the class Σ in D and let $h_2^*(z)$ correspond to the largest non-trivial negative eigen value λ_2^* of C^*. We determine a function $h(z) \in \Sigma$ such that

$$\frac{\partial h}{\partial n} = \frac{\partial h_2^*}{\partial n} \text{ on } C^*, \quad \frac{\partial h}{\partial n} = 0 \text{ on } C - C^* . \tag{43}$$

Since the boundary conditions (43) determine $h(z)$ in each D_j only up to an additive constant, we may adjust these constants in such a way that $h(z)$ satisfies the N conditions (2.18) and thus belongs indeed to Σ. Observe that the Dirichlet integral of h coincides in each D_j, $j \leq N^*$ with the corresponding Dirichlet integral of h_2^*, since h and h_2^* differ only by a constant in these domains. In each D_j with $j > N^*$, $h(z)$ is a constant and has the Dirichlet integral zero. Hence:

$$D^*(h_2^*, h_2^*) = D(h, h) . \tag{44}$$

By (35′) we have

$$\begin{aligned} \frac{\rho_1^*}{1+\rho_1^*} D^*(h_2^*, h_2^*) &= \frac{1}{2\pi}\int_{\sigma^*}\int_{\sigma^*} \log\frac{1}{|z-\zeta|}\frac{\partial h_2^*(z)}{\partial n}\frac{\partial h_2^*(\zeta)}{\partial n} ds_z ds_\zeta \\ &= \frac{1}{2\pi}\int_{\sigma}\int_{\sigma} \log\frac{1}{|z-\zeta|}\frac{\partial h(z)}{\partial n}\frac{\partial h(\zeta)}{\partial n} ds_z ds_\zeta \leq \frac{\rho_1}{1+\rho_1} D(h, h) . \end{aligned} \tag{45}$$

By virtue of (44), we conclude finally

$$\frac{\rho_1^*}{1+\rho_1^*} \leq \frac{\rho_1}{1+\rho_1} , \quad \rho_1^* \leq \rho_1 . \tag{46}$$

Thus, we proved:

THEOREM VIII. *The lowest positive and non-trivial eigen value λ_1 of a curve system C is never larger than the corresponding eigen value λ_1^* of any subsystem C^* of C.*

Suppose all positive eigen values of C arranged in increasing order, say $\lambda_{\nu'}$, such that $\nu' < \nu''$ implies $\lambda_{\nu'} \leq \lambda_{\nu''}$. Let us do the same with the positive eigen values $\lambda_{\nu'}^*$ of the subsystem C^*. By the above reasoning and by use of the standard methods of eigen value theory [cf. 11], it is easily shown that quite generally

$$\lambda_{\nu'} \leq \lambda_{\nu'}^* \tag{47}$$

will be fulfilled.

We consider finally the bilinear form

$$B(h, H) = \frac{1}{2\pi}\int_{\sigma}\int_{\sigma} \Gamma(\zeta, \eta)\frac{\partial h(\zeta)}{\partial n}\frac{\partial H(\eta)}{\partial n} ds_\zeta ds_\eta \tag{48}$$

where $\Gamma(\zeta, \eta)$ is the geometric kernel defined in (2.26). For $h \in \Sigma$, $H \in \Sigma$ we have, in view of (2.28) the following Fourier representation for B:

$$B(h, H) = \sum_{\nu=1}^{\infty} \frac{\rho_\nu}{(1+\rho_\nu)^2} x_\nu y_\nu \tag{49}$$

and the same expression is also valid for $\tilde{h} \in \tilde{\Sigma}$, $\tilde{H} \in \tilde{\Sigma}$,

From (11), (38) and (49) follows

$$\pi_1(h, H) - B(h, H) = \sum_{\nu=1}^{\infty} \frac{\rho_\nu^2}{(1+\rho_\nu)^2} x_\nu y_\nu = \Gamma_{1,1}(h, H) , \tag{50}$$

and

$$B(\tilde{h}, \tilde{H}) = \tilde{\pi}_{1,1}(\tilde{h}, \tilde{H}) . \tag{51}$$

The function

$$f(x) = \frac{x}{(1+x)^2} \tag{39'}$$

takes its maximum 1/4 for positive argument at the point $x_m = 1$ and we derive from (41) and (42) the inequalities

$$0 \leq \pi_1(h, h) - B(h, h) \leq \frac{1}{4} D(h, h) , \qquad h \in \Sigma \tag{52}$$

and

$$0 \leq B(\tilde{h}, \tilde{h}) \leq \frac{1}{4} \tilde{D}(\tilde{h}, \tilde{h}) , \qquad \tilde{h} \in \tilde{\Sigma} . \tag{53}$$

These inequalities are interesting since they yield estimates for the Dirichlet integrals of h and $\tilde{h}$ by means of elementary integrations over C which involve only the normal derivatives on C of these functions and geometric terms. On the other hand, given only these normal derivatives, we could calculate the precise Dirichlet integrals only after solving a Neumann boundary value problem for the domains. We gave by inequality (32) another lower bound for $\tilde{D}(\tilde{h}, \tilde{h})$; but in this estimate we have to assume as known the solution of the corresponding boundary value problem for the complimentary region D of $\tilde{D}$. The present inequalities are, therefore, often easier to apply.

The dielectric Green's functions $g_\varepsilon(z, \zeta)$ and $g_e(z, \zeta)$ which are needed in the calculation of $\tilde{\pi}_{\varepsilon,e}$ and $\Gamma_{\varepsilon,e}$ are known only for very few domains if ε and e are different from 1. We may, however, use the series developments (2.34) for these functions and utilize the partial sums in the development together with a simple estimate for the remainder terms in order to obtain estimates for ρ_m. The calculations are clearly quite laborious, but in principle feasible.

6. Variational formulas for the dielectric Green's functions and for the Fredholm eigen values. The properties (a)–(e) ennumerated in § 2 and defining the dielectric Green's functions $g_\varepsilon(z, \zeta)$ are all invariant under a conformal mapping $z^* = F(z)$ which is normalized at infinity

such that $|F'(\infty)| = 1$. Unfortunately, the only conformal mapping of this kind which is regular in the entire complex plane has the trivial form $F(z) = az + b$, $|a| = 1$. We may consider, however, functions $F(z)$ which are analytic with isolated singularities. In this way, we are led naturally to a variational theory for the dielectric Green's functions.

The simplest possible choice of $F(z)$ is evidently

$$z^* = F(z) = z + \frac{\alpha}{z - z_0} \tag{1}$$

which has the right normalization at infinity but has a simple pole at $z = z_0$. We will choose z_0 arbitrarily in D or in $\tilde{D}$ but not on the curve system C. Let $E(z_0)$ denote the entire complex plane from which a circle of radius $\sqrt{|\alpha|}$ around the center z_0 has been removed. It is easily seen that $F(z)$ is univalent in $E(z_0)$. Given, therefore, a fixed point z_0 in D or in $\tilde{D}$, we can always choose $|\alpha|$ so small that C lies in $E(z_0)$ and is mapped in a one-to-one manner into a new curve system C^*. Since $F(z)$ is regular analytic in $E(z_0)$ all differentiability properties of C are transferred to C^*. We denote the dielectric Green's functions of the new curve system C^* by $g_\varepsilon^*(z, \zeta)$. Our aim is to connect these new functions with the functions $g_\varepsilon(z, \zeta)$ of the original system C.

We introduce the function

$$d(z, \zeta) = g_\varepsilon^*(F(z)\ ,\ F(\zeta)) - g_\varepsilon(z, \zeta)\ . \tag{2}$$

By the definition of g_ε^* and of the curve system C^*, the function $d(z, \zeta)$ is symmetric and harmonic for $z, \zeta \in E(z_0)$, except along the curve set C. The function is still continuous across C but its normal derivatives satisfy the discontinuity relation

$$\frac{\partial}{\partial n_z} d(z, \zeta) + \varepsilon \frac{\partial}{\partial \tilde{n}_z} d(z, \zeta) = 0 \quad \text{for} \quad z \in C, \zeta \in E(z_0) - C\ . \tag{3}$$

Observe that $d(z, \zeta)$ is still regular harmonic for $z = \zeta$ and that

$$\lim_{z \to \infty} d(z, \zeta) = 0\ . \tag{4}$$

We consider now the integral

$$J(\zeta, \eta) = \frac{1}{2\pi} \int_C \left[d(z, \zeta) \frac{\partial}{\partial n_z} g_\varepsilon(z, \eta) - g_\varepsilon(z, \eta) \frac{\partial}{\partial n_z} d(z, \zeta) \right] ds_z\ . \tag{5}$$

We introduce the characteristic function $\delta(z)$ of D, i.e., we define

$$\delta(z) = \begin{cases} 1 & \text{if} \quad z \in D \\ 0 & \text{if} \quad z \notin D\ . \end{cases} \tag{6}$$

By Green's identity applied to D, we find

$$(7) \qquad J(\zeta, \eta) = \varepsilon d(\zeta, \eta)\delta(\eta) + T(\zeta, \eta)\delta(z_0) .$$

Here

$$(8) \qquad T(\zeta, \eta) = \frac{1}{2\pi} \int_{c(z_0)} \left[d(z, \zeta) \frac{\partial}{\partial n_z} g_\varepsilon(z, \eta) - g_\varepsilon(z, \eta) \frac{\partial}{\partial n_z} d(z, \zeta) \right] ds_z ,$$

where $c(z_0)$ is the circumference of radius $\sqrt{|\alpha|}$ around z_0 and where $\mathbf{n}$ is its interior normal.

On the other hand, we may apply Green's identity to $J(\zeta, \eta)$ with respect to the complementary domain $\tilde{D}$. Taking notice of (4) and of the known discontinuity behavior of the various terms in the integrand, we find

$$(9) \qquad J(\zeta, \eta) = -\varepsilon d(\zeta, \eta)[1 - \delta(\eta)] - \varepsilon T(\zeta, \eta)[1 - \delta(z_0)] .$$

Subtracting (9) from (7), we obtain finally

$$(10) \qquad \varepsilon d(\zeta, \eta) = -T(\zeta, \eta)[\varepsilon + (1 - \varepsilon)\delta(z_0)] .$$

The difference function (2) of g_ε^* and g_ε is thus expressed in terms of an integral over the small circle $c(z_0)$ around the singularity point z_0.

A straightfoward calculation of the type usual in such variational problems [15, 21] yields

$$(11) \qquad g_\varepsilon^*(\zeta^*, \eta^*) = g_\varepsilon(\zeta, \eta) + \left[1 + \left(\frac{1}{\varepsilon} - 1 \right) \delta(z_0) \right] \Re\{\alpha p_\varepsilon'(z_0, \zeta) p_\varepsilon'(z_0, \eta)\} + O(|\alpha|^2) ,$$

where $p_\varepsilon(z, \zeta)$ is the analytic function defined in § 4 whose real part is $g_\varepsilon(z, \zeta)$. The error term $O(|\alpha|^2)$ can be estimated uniformly for ζ and η in $E(z_0)$ and for z_0 in any fixed closed domain which does not contain points of C.

We derived in (11) an interior variational formula for the dielectric Green's function which is very similar to the well-known variational formula for the ordinary Green's function of a domain [14, 15]. Observe that in the special case $\varepsilon = 1$ formula (11) reads

$$(11') \qquad \log \frac{1}{|\zeta^* - \eta^*|} = \log \frac{1}{|\zeta - \eta|} + \Re\left\{ \frac{\alpha}{(z_0 - \zeta)(z_0 - \eta)} \right\} + O(|\alpha|^2) .$$

In view of the identity

$$(11'') \qquad \log|\zeta^* - \eta^*| = |\zeta - \eta| + \log\left| 1 - \frac{\alpha}{(z_0 - \zeta)(z_0 - \eta)} \right| ,$$

we can verify (11') directly by means of the logarithmic series.

We shall not enter into the variational theory of the dielectric

Green's functions since it is entirely analogous to that given in the case of simply-connected domains [17]. We wish to utilize (11) in order to derive analogous variational formulas for the eigen values λ_ν. For this purpose, we shall make use of the extremum principles (5.41) and (5.42) and of the method of transplanting the extremum function [6, 11].

Let us suppose that the singular point z_0 of our variation (1) lies in $\tilde{D}$; in this case, the function $F(z)$ is regular and univalent in D. If $h(z)$ is any analytic function in D, we can define by

$$h^*(z^*) = h(z) \tag{12}$$

a regular analytic function h^* in each component D_j^* of the varied domain set D^*. We call the definition (12) the transplantation of the function $h(z)$ from D into D^*.

We define now the ratios

$$R(h) = \frac{\Gamma_{\varepsilon,e}(h, h)}{D(h, h)}, \qquad R^*(h^*) = \frac{\Gamma^*_{\varepsilon,e}(h^*, h^*)}{D^*(h^*, h^*)} \tag{13}$$

which occur in the extremum problem (5.41). In view of the conformal character of the transplantation, we have clearly

$$D(h, h) = D^*(h^*, h^*) \tag{14}$$

and

$$\frac{\partial h^*(z^*)}{\partial n^*} ds^* = \frac{\partial h(z)}{\partial n} ds\,, \qquad z \in C, z^* \in C^*\,. \tag{15}$$

It is, therefore, easy to calculate the ratio $R^*(h^*)$ by referring back to the original region D. By the definitions (5.10), (5.13) and (5.37), we find

$$\Gamma^*_{\varepsilon,e}(h^*, h^*) = \frac{1}{\varepsilon - e} \cdot \frac{1}{2\pi} \int_{\sigma^*}\int_{\sigma^*} \left[\frac{1}{e} g^*_e(\zeta^*, \eta^*) - \frac{1}{\varepsilon} g^*_\varepsilon(\zeta^*, \eta^*)\right] \cdot \frac{\partial h^*(\zeta^*)}{\partial n^*} \frac{\partial h^*(\eta^*)}{\partial n^*} ds^*_\zeta ds^*_\eta\,. \tag{16}$$

Now, we use (11) and (15) in order to return to the curve system C as the path of integration. We remember that $z_0 \in \tilde{D}$ and obtain

$$\Gamma^*_{\varepsilon,e}(h^*, h^*) = \Gamma^*_{\varepsilon,e}(h, h) + 2\pi \Re\left\{\alpha \frac{e^{-1} q_e(z_0)^2 - \varepsilon^{-1} q_\varepsilon(z_0)^2}{\varepsilon - e}\right\} + O(|\alpha|^2) \tag{17}$$

with

$$q_\varepsilon(z) = \frac{1}{2\pi} \int_\sigma p'_\varepsilon(z, \zeta) \frac{\partial h(\zeta)}{\partial n} ds_\zeta\,. \tag{17'}$$

Since $z_0 \in \tilde{D}$, we can express $q_\varepsilon(z)$ as a surface integral

$$q_\varepsilon(z_0) = -\frac{1}{\partial z_0}\left[\frac{1}{\pi}\iint_D \nabla g_\zeta(z_0, \zeta)\cdot \nabla h(\zeta) d\tau_\zeta\right]. \tag{18}$$

The error term $O(|\alpha|^2)$ can be estimated uniformly for all functions $h(z)$ with bounded Dirichlet integral and for z_0 in a closed subdomain of $\tilde{D}$. We have to use the known error term in the variational formula (11) for the dielectric Green's function.

As a first result we can conclude that the eigen values of the ratio $R^*(h^*)$ depend continuously on α and converge with $|\alpha| \to 0$ to the corresponding eigen values of $R(h)$. We can, moreover, derive a precise asymptotic formula for these eigen values.

Let indeed ρ_0 be a particular ρ_ν-value of the original curve system C and let the function $f(x)$, defined in (5.39), be chosen in such a way that it takes its maximum at a point x_m which is nearer to ρ_0 than to any other ρ_ν. If $h_0 \in \Sigma$ is an eigen function which belongs to ρ_0, we will have

$$R(h_0) = f(\rho_0)\,. \tag{19}$$

We may assume as before (see (2.17)) that

$$D(h_0, h_0) = \rho_0\,. \tag{19'}$$

If h_0^* is the transplantation of h_0 into D^*, we can use (14) and (17) in order to determine its ratio $R^*(h_0^*)$. But now we can use formulas (2.9), (2.10) and (2.13') in order to express the analytic function $q_\varepsilon(z)$ by means of the analytic completion of $\tilde{h}_0(z)$ defined in (1.31). We have

$$q_\varepsilon(z_0) = \frac{\varepsilon\rho_0}{1+\varepsilon\rho_0}\tilde{V}_0'(z_0)\,, \qquad z_0 \in \tilde{D}\,. \tag{20}$$

We can now combine (14), (17) and (19) in order to express $R^*(h_0^*)$. We make also use of (19) and of the definition (5.39) of $f(x)$; thus, we arrive finally at

$$R^*(h_0^*) = f(\rho_0) - 2\pi\rho_0 f'(\rho_0)\Re\{\alpha\tilde{V}_0'(z_0)^2\} + O(|\alpha|^2)\,. \tag{21}$$

The function $h_0^*(z^*)$ defined by the transplantation of $h_0(z)$ will not, in general, belong to the class Σ^* defined with respect to D^* by linear conditions analogous to (2.18). However, we can add to every function $h^*(z^*)$ which is analytic in D^* a different constant in each component D_j^* in order to bring it into the class Σ^*. This trivial readjustment does not affect the Dirichlet integral nor the quadratic from $\Gamma_{\varepsilon,e}^*$ which depends only upon the normal derivatives of h^*. Thus, in the theory of the ratio $R^*(h^*)$ the restriction to the class Σ^* is unessential, since easily achieved.

In particular, we may use h_0^* as a competing function for the extremum problem regarding $R^*(h^*)$ and use the identity (21) in order to estimate the extremum values. Let us suppose that the value ρ_0 belongs to k different eigen functions $h_\beta(z)$ of the unperturbed curve system C; we denote their analytic completions by $V_\beta(z)$. We restrict, at first, $h^*(z^*)$ to the linear sub-space spanned by the k transplanted eigen functions $h_\beta^*(z^*)$. In this case, the ratio $R^*(h^*)$ will have precisely the k stationary values

$$\tau_\beta = f(\rho_0) + 2\pi\rho_0 f'(\rho_0)\sigma_\beta + O(|\alpha|), \qquad \beta = 1, 2, \cdots, k \tag{22}$$

where the σ_β are the eigen values of the secular equation

$$\det \| \Re\{\alpha \tilde{V}_i'(z_0)\tilde{V}_j'(z_0)\} + \sigma\delta_{ij} \|_{i,j=1,2\cdots k} = 0 . \tag{23}$$

Let us arrange the τ_β in decreasing order; likewise, we shall arrange the values $f(\rho_\beta^*)$ in decreasing order. Since the k first values $f(\rho_\beta^*)$ are the largest stationary values of $R^*(h^*)$ for unrestricted argument function h^*, it follows from standard results on quadratic forms that

$$f(\rho_\beta^*) \geq f(\rho_0) + 2\pi\rho_0 f'(\rho_0)\sigma_\beta + O(|\alpha|^2), \qquad \beta = 1, \cdots, k . \tag{24}$$

Because of the continuous dependence of the eigen values ρ_ν^* on α there exists a positive constant δ such that for small enough α all eigen values ρ_ν^* have from ρ_0 a distance larger than δ, except for k eigen values ρ_β^* which can be brought arbitrarily near to ρ_0.

Having now chosen $|\alpha|$ sufficiently small, we can select x_m to the left of ρ_0 and the k neighboring ρ_β^* but so near that all other $f(\rho_\nu^*)$ are less than any of the $f(\rho_\beta^*)$. Since $f'(\rho) < 0$ for ρ_0 and all ρ_β^*, we derive from (24)

$$\rho_\beta^* \leq \rho_0 + 2\pi\rho_0\sigma_\beta + O(|\alpha|^2), \qquad \beta = 1, 2, \cdots, k . \tag{24'}$$

Choosing, on the other hand, x_m to the right of ρ_0 and the ρ_β^* but again so near that $f(\rho_\beta^*)$ is still larger than all $f(\rho_\nu^*)$, we obtain

$$\rho_\beta^* \geq \rho_0 + 2\pi\rho_0\sigma_\beta + O(|\alpha|^2), \qquad \beta = 1, 2, \cdots, k . \tag{24''}$$

Thus, we proved:

The variation of an eigen value ρ_0 with degree of degeneracy $k-1$ is characterized by the formula

$$\rho_\beta^* = \rho_0 + 2\pi\rho_0\sigma_\beta + 0(|\alpha|^2) \tag{25}$$

where the σ_β are the eigen values of the secular equation (23).

In the case that only one eigen function $h_\nu \in \Sigma$ belongs to ρ_ν, we obtain the simpler variational formula

$$\delta\rho_\nu = -\Re\{2\pi\alpha\rho_\nu \tilde{V}_\nu'(z_0)^2\} . \tag{26}$$

By the relation (2.12) between ρ_ν and the Fredholm eigen value λ_ν, we obtain in this case finally

$$\delta\lambda_\nu = (\lambda_\nu^2 - 1)\pi\Re\{\alpha\tilde{V}_\nu'(z_0)^2\} \ . \tag{27}$$

We can proceed in analogous fashion in the case that $z_0 \in D$. We will start then with $\tilde{h}_0 \in \tilde{\Sigma}$ which belongs to ρ_0 and which satisfies by (5.42) the equation

$$\tilde{R}(\tilde{h}_0) = \frac{\tilde{\pi}_{\varepsilon,e}(\tilde{h}_0, \tilde{h}_0)}{\tilde{D}(\tilde{h}_0, \tilde{h}_0)} = f(\rho_0) \ . \tag{28}$$

We transplant $\tilde{h}_0$ by an equation (12) into a comparison function $\tilde{h}_0^*$ in $\tilde{D}^*$. We assume the usual normalization

$$\tilde{D}(\tilde{h}_0, \tilde{h}_0) = 1 \tag{29}$$

and have, therefore, also

$$\tilde{D}^*(\tilde{h}_0^*, \tilde{h}_0^*) = 1 \ . \tag{29'}$$

The same chain of calculations as before leads to the asymptotic formula

$$\tilde{R}^*(\tilde{h}_0^*) = \frac{\tilde{\pi}_{\varepsilon,e}^*(\tilde{h}_0^*, \tilde{h}_0^*)}{\tilde{D}^*(\tilde{h}_0^*, \tilde{h}_0^*)} = f(\rho_0) + 2\pi f'(\rho_0)\Re\{\alpha V_0'(z_0)^2\} + O(|\alpha|^2) \ . \tag{30}$$

Here, $V_0(z)$ is the analytic completion of $h_0(z)$ in D. This formula is very similar to (21); it differs only by the factor $-\rho_0$. We obtain, therefore, the following result:

If ρ_ν is an eigen value of degeneracy $k-1$ it will change according to the formula

$$\rho_\beta^* = \rho_\nu + 2\pi\sigma_\beta + O(|\alpha|^2) \qquad \beta = 1, 2, \cdots, k \tag{31}$$

under a variation (1) of the curve system C. The σ_β are the k eigen values of the secular equation

$$\det \| \Re\{\alpha V_i'(z_0) V_j'(z_0)\} - \sigma\delta_{ij} \|_{i,j=1,\ldots,k} = 0 \tag{32}$$

and the $V_i(z)$ are the k analytic functions whose real parts are the eigen functions $h_i(z)$ which belong to ρ_ν.

In the particular case $k = 1$, i.e., non-degeneracy, we have

$$\delta\rho_\nu = \Re\{2\pi\alpha V_\nu'(z_0)^2\} \tag{32'}$$

and hence

$$\delta\lambda_\nu = -(\lambda_\nu - 1)^2\pi\Re\{\alpha V_\nu'(z_0)^2\} \ . \tag{33}$$

There is a lack of symmetry between the variational formulas (23), (25), on the one hand, and (31), (32) on the other. This fact is due to the different normalizations

$$\iint_D \left| V_i'(z) \right|^2 d\tau = D(h_i, h_i) = \rho_i \tag{34}$$

and

$$\iint_{\tilde{D}} \left| \tilde{V}_i'(z) \right|^2 d\tau = D(\tilde{h}_i, \tilde{h}_i) = 1 \,. \tag{35}$$

We were led to these normalizations from the theory of the Fredholm eigen functions $\varphi_i(z)$ through the representation (1.3). These normalizations were also used in the series developments of §§ 2 and 3. However, the variational formulas become symmetric when we define

$$u_\nu(z) = \rho_\nu^{-1/2} V_\nu'(z) \,, \quad \tilde{u}_\nu(z) = i \tilde{V}_\nu'(z) \,. \tag{36}$$

From the definition of the $V_\nu(z)$ and $\tilde{V}_\nu(z)$, their normalizations (34) and (35) and from the definitions (1.33), (1.34) it follows at once that the functions (36) are identical with the functions $u_\nu(z)$ and $\tilde{u}_\nu(z)$ defined at the end of § 1 and normalized by (1.34).

By means of the functions $u_\nu(z)$ and $\tilde{u}_\nu(z)$ we can express the law of variations of the eigen values λ_ν as follows:

THEOREM. *Let λ_ν be a Fredholm eigen value of the curve system C and of degeneracy $k-1$; let $u_\beta(z), \tilde{u}_\beta(z) (\beta = 1, 2, \cdots, k)$ be the set of analytic eigen functions to this eigen value. If we subject the system C to a variation* (1), *we have*

$$\frac{\delta \lambda_\nu}{\lambda_\nu^2 - 1} = \pi \sigma_\beta \tag{37}$$

where σ_β is an eigen value of the secular equation

$$\det \| \Re\{\alpha u_i(z_0) u_j(z_0)\} + \sigma \delta_{ij} \| = 0 \quad \text{if} \quad z_0 \in D \tag{38}$$

or of

$$\det \| \Re\{\alpha \tilde{u}_i(z_0) \tilde{u}_j(z_0)\} + \sigma \delta_{ij} \| = 0 \quad \text{if} \quad z_0 \in \tilde{D} \,. \tag{39}$$

In particular, we have in the case of non-degeneracy

$$\frac{\delta \lambda_\nu}{\lambda_\nu^2 - 1} = -\pi \Re\{\alpha u_\nu^2(z_0)\} \quad \text{for} \quad z_0 \in D \tag{40}$$

and

$$\frac{\delta\lambda_\nu}{\lambda_\nu^2 - 1} = -\pi\Re\{\alpha\tilde{u}_\nu^2(z_0)\} \quad \text{for} \quad z_0 \in \tilde{D} . \tag{40'}$$

The preceding variational formulas can also be derived easily from the original integral equation (1.2) by means of the general theory of perturbations [17]. The above derivation is of interest since it allows a more detailed study of the error terms by means of the dielectric Green's function. It is also possible to obtain more precise statements by using the higher variational terms of these Green's functions. It is particularly easy to develop the higher variations for the lowest positive and non-trivial eigen value λ_1. Consider, for example, a variation (1) of the curve system C with $z_0 \in \tilde{D}$. Let $h(z) \in \Sigma$ and h^* its transplantation into D^*. By definition (5.10) and the identity (11''), we have

$$\pi_1^*(h^*, h^*) = \pi_1(h, h) - \frac{1}{2\pi}\int_\sigma\int_\sigma \log\left|1 - \frac{\alpha}{(z - z_0)(\zeta - z_0)}\right| \frac{\partial h(z)}{\partial n}\frac{\partial h(\zeta)}{\partial n}\, ds_z ds_\zeta . \tag{41}$$

Thus, $\pi_1(h, h)$ has a very simple transformation law under transplantation. The Dirichlet integral is invariant under transplantation. Since ρ_1 leads to the extremum values of the ratio (5.35) it is possible to determine the variations of higher order of λ_1 with relatively little labor.

We wish, finally, to add a simple algebraic remark to the variational formulas (37), (38) and (39). If λ_ν is of degeneracy $k - 1$ a variation (1) will, in general, reduce this degeneracy. It is, however, remarkable that the secular equations (38) and (39) have only two different eigen values such that even after the variation a degenerate eigen value can only split into two different eigen values, at least, up to the order $O(|\alpha|^2)$. Indeed, σ is an eigen value, say of (38) if there exist k real numbers t_j such that the linear equations

$$\sigma t_i + \sum_{j=1}^{k} \Re\{\alpha u_i u_j\} t_j = 0 , \qquad i = 1, \cdots, k \tag{42}$$

hold while

$$\sum_{j=1}^{k} t_j^2 = 1 . \tag{42'}$$

We denote

$$\sum_{j=1}^{k} u_j t_j = M \tag{43}$$

and reduce (42) to

$$\sigma t_i + \Re\{\alpha u_i M\} = 0 , \qquad i = 1, \cdots, k . \tag{44}$$

Multiplying the ith equation (44) with u_i and summing over all i-values, we find :

$$\sigma M + \frac{1}{2}\alpha M \sum_{i=1}^{k} u_i^2 + \frac{1}{2}\bar{\alpha}\bar{M} \sum_{i=1}^{k} |u_i|^2 = 0. \tag{45}$$

On the other hand, multiplying (44) with t_i and summing over i, we obtain from (42′)

$$\sigma + \Re\{\alpha M^2\} = 0 . \tag{46}$$

From (45) and (46) we derive

$$-\sigma = \Re\{\alpha M^2\} = \frac{1}{2}\alpha \sum_{i=1}^{k} u_i^2 + \frac{1}{2}\frac{|\alpha M|^2}{\alpha M^2} \sum_{i=1}^{k} |u_i|^2 . \tag{47}$$

Let us put

$$\alpha M^2 = p e^{i\gamma} . \tag{48}$$

The real part and imaginary part of (47) are:

$$\begin{aligned} p\cos\gamma &= \frac{1}{2}\Re\left\{\alpha \sum_{i=1}^{k} u_i^2\right\} + \frac{|\alpha|}{2}\cos\gamma \cdot \sum_{i=1}^{k} |u_i|^2 \\ 0 &= \frac{1}{2}\Im\left\{\alpha \sum_{i=1}^{k} u_i^2\right\} - \frac{|\alpha|}{2}\sin\gamma \cdot \sum_{i=1}^{k} |u_i|^2 \end{aligned} \tag{47′}$$

Eliminating $\cos\gamma$ form the first equation by means of the second, we find

$$\begin{aligned} \sigma = &-\frac{1}{2}\Re\left\{\alpha \sum_{i=1}^{k} u_i(z_0)^2\right\} \\ &\pm \frac{1}{2}\sqrt{|\alpha|^2\left(\sum_{i=1}^{k} |u_i(z_0)|^2\right)^2 - \left[\Im\left\{\alpha \sum_{i=1}^{k} u_i(z_0)^2\right\}\right]^2} . \end{aligned} \tag{49}$$

We see, in particular, that the first variation of each eigen value, whatever its degree of degeneracy, depends only on

$$U(z_0) = \sum_{i=1}^{k} u_i(z_0)^2 \quad \text{and} \quad \Omega(z_0) = \sum_{i=1}^{k} |u_i(z_0)|^2 . \tag{50}$$

Observe that the product of the two possible σ-values (49) is

$$\frac{1}{4}\left|\alpha \sum_{i=1}^{k} u_i^2\right|^2 - \frac{1}{4}|\alpha|^2\left(\sum_{i=1}^{k} |u_i(z_0)|^2\right)^2 \leq 0 \tag{51}$$

such that under a variation (1) at least one component of a split up multiple eigen value is non-increasing. This is the reason why many maximum problems for positive eigen values lead to degenerate eigen values in the extremum case.

7. The L_ε-kernels and the variation of the Fredholm determinants. In this section, we shall discuss certain kernels obtained by complex

differentiation of the dielectric Green's functions which will appear in certain variational formulas for important combinations of Fredholm eigen values. The significance of these kernels is best understood by considering the kernel obtained in an analogous way from the ordinary Green's function, say $\tilde{g}(z, \zeta)$ of $\tilde{D}$.

We defined already in (1.17) a kernel $L(z, \zeta)$ with respect to the Green's function $g(z, \zeta)$ of the domain set D and observed its remarkable property (1.18). Analogously, we introduce the kernel

$$(1) \qquad \tilde{L}(z, \zeta) = -\frac{2}{\pi}\frac{\partial^2}{\partial z \partial \zeta}\tilde{g}(z, \zeta) = \frac{1}{\pi(z-\zeta)^2} - \tilde{l}(z, \zeta) .$$

$\tilde{l}(z, \zeta)$ is a regular analytic function for z and ζ in $\tilde{D}$. We shall need two important facts about $\tilde{l}(z, \zeta)$ for later applications.

(a) For $\zeta \in C$ and $z \in \tilde{D}$, we have

$$(2) \qquad \frac{\partial \tilde{g}(z. \zeta)}{\partial z} \equiv 0 \quad \text{identically in } z \in \tilde{D}, \zeta \in C .$$

This identity remains even valid when z moves onto C but to a point different from ζ. Let now s be the length parameter on C, $\zeta(s)$ its parametric representation and $\zeta' = d\zeta/ds$ the local tangent unit vector. We differentiate the identity (2) with respect to s and find

$$(3) \qquad \frac{\partial^2 \tilde{g}(z, \zeta)}{\partial z \partial \zeta}\zeta' + \frac{\partial^2 \tilde{g}(z, \zeta)}{\partial z \partial \zeta}\bar{\zeta}' = 0 , \qquad z \in C, \zeta \in C .$$

We multiply this identity by z' and using the symmetry of the first term in z and ζ as well as the hermitian symmetry of the second term, we conclude:

$$(4) \qquad \tilde{L}(z, \zeta)z'\zeta' = \text{real} \quad \text{for } z \in C, \zeta \in C .$$

By use of (1), we may express this result also in the form

$$(5) \qquad \Im\{\tilde{l}(z, \zeta)z'\zeta'\} = \frac{1}{\pi}\Im\left\{\frac{z'\zeta'}{(z-\zeta)^2}\right\} .$$

This identity is of great interest since the left side expression is a differential depending on the Green's function while the right hand term depends only on the geometry of the curve system C. Moreover, it can be shown that $\tilde{l}(z, \zeta)$ is continuous in both variables in the closed domain $\tilde{D} + C$ [3, 21]. We may pass to the limit $z = \zeta$ on both sides of (5); an easy calculation yields the boundary condition

$$(6) \qquad \Im\{\tilde{l}(z, z)z'^2\} = \frac{1}{6\pi}\Im\left\{\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2\right\} .$$

Let us denote by $\kappa = \kappa(s)$ the curvature of C at $z(s)$; then (6) obtains the elegant form

$$\Im\{\tilde{l}(z, z)z'^2\} = \frac{1}{6\pi}\frac{d\kappa}{ds} . \tag{7}$$

In particular, we note that (7) and our assumptions on C yield the

THEOREM. *The function $\tilde{l}(z, z)$ is a quadratic differential of D, i.e., satisfies*

$$\tilde{l}(z, z)z'^2 = \text{real} \quad \text{on } C \tag{7'}$$

if and only if $\tilde{D}$ is a domain bounded by circumferences C_j.

(b) Let $z^* = f(z)$ be a univalent analytic function in $\tilde{D}$ which maps this domain into $\tilde{D}^*$. The conformal invariance of the Green's function is expressed by the identity

$$\tilde{g}^*(z^*, \zeta^*) = \tilde{g}(z, \zeta) \tag{8}$$

which leads by differentiation to

$$\tilde{L}^*(z^*, \zeta^*)f'(z)f'(\zeta) = \tilde{L}(z, \zeta) . \tag{9}$$

The $\tilde{l}$-kernel has, therefore, the transformation law

$$\tilde{l}^*(z^*, \zeta^*)f'(z)f'(\zeta) = \tilde{l}(z, \zeta) + \frac{1}{\pi}\left[\frac{f'(z)f'(\zeta)}{(f(z) - f(\zeta))^2} - \frac{1}{(z - \zeta)^2}\right] \tag{10}$$

and, as a simple calculation shows, in particular

$$\tilde{l}^*(z^*, z^*)f'(z)^2 = \tilde{l}(z, z) + \frac{1}{6\pi}\{f, z\} \tag{11}$$

where

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{12}$$

is the Schwarzian derivative of $f(z)$.

After these remarks on the kernel $\tilde{L}(z, \zeta)$, we introduce now a new kernel by the following formula which is modeled after (1):

$$L_s(z, \zeta) = -\frac{2}{\pi}\frac{\partial^2 g_s(z, \zeta)}{\partial z \partial \zeta} . \tag{13}$$

This kernel is regular analytic and symmetric in both its arguments in D and in $\tilde{D}$, except for a double pole for $z = \zeta$. We define further two kernels which are regular analytic for $z, \zeta \in D$ and for $z, \zeta \in \tilde{D}$, respectively:

$$l_\varepsilon(z,\zeta) = \frac{1}{\pi(z-\zeta)^2} - \frac{1}{\varepsilon} L_\varepsilon(z,\zeta) \quad \text{in } D \tag{14}$$

and

$$\tilde{l}_\varepsilon(z,\zeta) = \frac{1}{\pi(z-\zeta)^2} - L_\varepsilon(z,\zeta) \quad \text{in } \tilde{D}\,. \tag{15}$$

These kernels have elegant developments in terms of the complex eigen functions of the Fredholm integral equation. We start with the Fourier developments (2.16) and (2.21) for $g_\varepsilon(z,\zeta)$ in terms of the harmonic eigen functions $h_\nu(z)$ and $\tilde{h}_\nu(z)$. Using definition (1.17) and (2.21), we obtain by differentiation

$$l_\varepsilon(z,\zeta) = \left(1 - \frac{1}{\varepsilon}\right)\left[l(z,\zeta) + \sum_{\nu=1}^{\infty} \frac{V_\nu'(z)\,V_\nu'(\zeta)}{\rho_\nu(1+\rho_\nu)(1+\varepsilon\rho_\nu)}\right] \tag{16}$$

where the $V_\nu(z)$ are the analytic functions whose real part is $h_\nu(z)$. As pointed out in the preceding section, all $V_\nu'(z)$ have a different normalization and it is more convenient to introduce the functions $u_\nu(z)$ defined by (6.36) which have all the norm 1. Then (16) transforms to

$$l_\varepsilon(z,\zeta) = \left(1 - \frac{1}{\varepsilon}\right)\left[l(z,\zeta) + \sum_{\nu=1}^{\infty} \frac{u_\nu(z)u_\nu(\zeta)}{(1+\rho_\nu)(1+\varepsilon\rho_\nu)}\right]. \tag{17}$$

We observe next that with each eigen value $\lambda_\nu > 0$ which belongs to $u_\nu(z)$, there occurs also the eigen value $-\lambda_\nu$ and it belongs to the eigen function $iu_\nu(z)$. This assertion can be verified directly from the complex integral equations (1.36) and (1.37); it is also a consequence of the fact, noted in § 1, that if λ_ν belongs to an eigen function $h_\nu(z)$ then $-\lambda_\nu$ will be an eigen value with the conjugate harmonic eigen function $k_\nu(z)$. Thus, in formula (17), each product $u_\nu(z)\,u_\nu(\zeta)$ occurs, therefore, twice; once coupled with ρ_ν and the other time with opposite sign and coupled with $1/\rho_\nu$. We combine these pairs of terms and sum now only over those ν which correspond to the positive eigen values λ_ν. Using (2.12), we obtain finally

$$l_\varepsilon(z,\zeta) = \left(1 - \frac{1}{\varepsilon}\right)\left[l(z,\zeta) - \sum_{\nu=1}^{\infty} \frac{u_\nu(z)u_\nu(\zeta)}{\lambda_\nu}\right] + E^2 \sum_{\nu=1}^{\infty} \frac{\lambda_\nu^2 - 1}{\lambda_\nu^2 - E^2} \frac{u_\nu(z)u_\nu(\zeta)}{\lambda_\nu}\,. \tag{18}$$

with the notation

$$E = \frac{\varepsilon - 1}{\varepsilon + 1}\,. \tag{19}$$

Passing to the limit $\varepsilon = 0$ and using the limit relation (3.31), we derive first from (18)

$$l(z,\zeta)=\sum_{\nu=1}^{\infty}\frac{u_\nu(z)u_\nu(\zeta)}{\lambda_\nu} \tag{20}$$

and, hence, (18) simplifies to

$$l_\varepsilon(z,\zeta)=E^2\sum_{\nu=1}^{\infty}\frac{\lambda_\nu^2-1}{\lambda_\nu^2-E^2}\,\frac{u_\nu(z)u_\nu(\zeta)}{\lambda_\nu}\,. \tag{21}$$

Similarly, we transform (15) by differentiation of (2.16) into the identity

$$\tilde{l}_\varepsilon(z,\zeta)=(\varepsilon-1)\sum_{\nu=1}^{\infty}\frac{\rho_\nu\tilde{V}_\nu'(z)\tilde{V}_\nu'(\zeta)}{(1+\rho_\nu)(1+\varepsilon\rho_\nu)} \tag{22}$$

and replacing $\tilde{V}_\nu'(z)$ by $\tilde{u}_\nu(z)$ by means of (6.36), we find

$$\tilde{l}_\varepsilon(z,\zeta)=-(\varepsilon-1)\sum_{\nu=1}^{\infty}\frac{\rho_\nu\tilde{u}_\nu(\zeta)\tilde{u}_\nu(z)}{(1+\rho_\nu)(1+\varepsilon\rho_\nu)}\,. \tag{23}$$

We combine again terms with ρ_ν and with $1/\rho_\nu$ and sum only over the positive eigen values λ_ν; an easy calculation leads to

$$\tilde{l}_\varepsilon(z,\zeta)=E^2\sum_{\nu=1}^{\infty}\frac{\lambda_\nu^2-1}{\lambda_\nu^2-E^2}\,\frac{\tilde{u}_\nu(z)\tilde{u}_\nu(\zeta)}{\lambda_\nu}\,. \tag{24}$$

The complete symmetry between (21) and (24) is evident.

We consider the limit cases $\varepsilon=0$ and $\varepsilon=\infty$ of formula (24) which correspond both to $E^2=1$. From (3.11) and (3.17) follows

$$\tilde{\lambda}(z,\zeta)=\sum_{\nu=1}^{\infty}\frac{\tilde{u}_\nu(z)\tilde{u}_\nu(\zeta)}{\lambda_\nu}=\frac{1}{\pi(z-\zeta)^2}+\frac{2}{\pi}\frac{\partial^2\tilde{G}(z,\zeta)}{\partial z\partial\zeta} \tag{25}$$

$$=\frac{1}{\pi(z-\zeta)^2}+\frac{2}{\pi}\frac{\partial^2\tilde{N}(z,\zeta)}{\partial z\partial\zeta}\,.$$

We can, therefore, express $\tilde{\lambda}(z,\zeta)$ by means of (3.3) in the form

$$\tilde{\lambda}(z,\zeta)=\tilde{l}(z,\zeta)-\frac{1}{2\pi}\sum_{j,k=1}^{N-1}\alpha_{jk}w_j'(z)w_k'(\zeta) \tag{26}$$

where $w_j(z)$ denotes the analytic completion of the harmonic measure $\omega_j(z)$. Formula (26) is the counterpart for $\tilde{D}$ of the relation (20) in D. The kernel $\tilde{\lambda}(z,\zeta)$ is composed of functions with single-valued integral in $\tilde{D}$; the kernel $\tilde{l}(z,\zeta)$ differs from it by a kernel which is composed of a basis of $N-1$ functions in $\tilde{D}$ which do not have a single-valued integral and which are orthogonal in the Dirichlet metric to all functions in $\tilde{D}$ with single-valued integral.

For the sake of completeness, we give also the Fourier developments of the kernels

$$K_\varepsilon(z,\bar\zeta) = -\frac{2}{\pi\varepsilon}\frac{\partial^2 g_\varepsilon(z,\zeta)}{\partial z\partial\bar\zeta} \quad \text{in } D \tag{27}$$

and

$$\tilde K_\varepsilon(z,\bar\zeta) = -\frac{2}{\pi}\frac{\partial^2 g_\varepsilon(z,\zeta)}{\partial z\partial\bar\zeta} \quad \text{in } \tilde D\,. \tag{28}$$

Both kernels are analytic and have hermitian symmetry in their arguments. Putting

$$K(z,\bar\zeta) = -\frac{2}{\pi}\frac{\partial^2 g(z,\zeta)}{\partial z\partial\bar\zeta} \tag{29}$$

we obtain by differentiation of (2.21) after the above combination of terms

$$K_\varepsilon(z,\bar\zeta) = \left(1-\frac{1}{\varepsilon}\right)\left[K(z,\tilde\zeta) - \sum_{\nu=1}^{\infty} u_\nu(z)\overline{u_\nu(\zeta)}\right] + E\sum_{\nu=1}^{\infty}\frac{\lambda_\nu^2-1}{\lambda_\nu^2-E^2}u_\nu(z)\overline{u_\nu(\zeta)}\,. \tag{30}$$

Again, we obtain by passage to the limit $\varepsilon = 0$ and in view of (3.31)

$$K(z,\bar\zeta) = \sum_{\nu=1}^{\infty} u_\nu(z)\overline{u_\nu(\zeta)} \tag{31}$$

which reduces formula (30) to

$$K_\varepsilon(z,\bar\zeta) = E\sum_{\nu=1}^{\infty}\frac{\lambda_\nu^2-1}{\lambda_\nu^2-E^2}u_\nu(z)\overline{u_\nu(\zeta)}\,. \tag{32}$$

Similarly, we find by differentiation of (2.16) the identity

$$K_\varepsilon(\tilde z,\bar\zeta) = E\sum_{\nu=1}^{\infty}\frac{\lambda_\nu^2-1}{\lambda_\nu^2-E^2}\tilde u_\nu(z)\overline{\tilde u_\nu(\zeta)}\,. \tag{33}$$

Formulas (21), (24), (32) and (33) for the various kernels depend on ε only through E and this simple rational function of ε has the symmetry property $E(1/\varepsilon) = -E(\varepsilon)$. This leads to the interesting identities:

$$\frac{\partial^2 g_\varepsilon(z,\zeta)}{\partial z\partial\zeta} = \frac{\partial^2 g_{1/\varepsilon}(z,\zeta)}{\partial z\partial\zeta}, \quad \frac{\partial^2 g_\varepsilon(z,\zeta)}{\partial z\partial\bar\zeta} = -\frac{\partial^2 g_{1/\varepsilon}(z,\zeta)}{\partial z\partial\bar\zeta} \tag{34}$$

if $z, \zeta \in \tilde D$ and to a similar identity in $z, \zeta \in D$. These relations are known in the limit case $\varepsilon = 0$ where they represent differential relations between the Green's and the Neumann's function [2, 5, 21].

We define next the Fredholm determinant of the basic integral equation (1.2), Observe again that with each positive eigen value λ_ν

occurs also the eigen value $-\lambda_\nu$ in equal multiplicity. We may thus write

$$D(E) = \prod_{\nu=1}^{\infty} \left(1 - \frac{E^2}{\lambda_\nu^2}\right) \tag{35}$$

where the product is to be extended over all positive eigen values $\lambda_\nu > 1$.

By use of the variational formulas (6.38) and (6.39) and of the identities (21) and (24) one can establish readily the

THEOREM. *If the curve system C is varied according to* (6.1) *the Fredholm determinant $D(E)$ changes according to the variational formulas*

$$\delta \log D(E) = -2\pi \Re\{\alpha l_\varepsilon(z_0, z_0)\} \quad \text{for } z_0 \in D \tag{36}$$

and

$$\delta \log D(E) = -2\pi \Re\{\alpha \tilde{l}_\varepsilon(z_0, z_0)\} \quad \text{for } z_0 \in \tilde{D}\,. \tag{37}$$

$E(\varepsilon)$ is the rational function (19) *of ε.*

The elegant and symmetric variational formulas (36) and (37) show the theoretical interest of the Fredholm determinant (35). We observe that, in particular, for $\varepsilon = \infty$ and $E = 1$ we have by (20) and (25);

$$\delta \log D(1) = -2\pi \Re\{\alpha l(z_0, z_0)\} \quad \text{for } z_0 \in D \tag{38}$$

and

$$\delta \log D(1) = -2\pi \Re\{\alpha \tilde{\lambda}(z_0, z_0)\} \quad \text{for } z_0 \in \tilde{D}\,. \tag{38'}$$

The functional (35) is defined only for curve systems C which are sufficiently differentiable. This fact creates difficulties in applications of the above variational formulas to extremum problems for the Fredholm determinant since it is not sure, a priori, that the extremum system C will have the required smoothness. In many problems, however, it can be shown that the very property of being an extremum set guarantees already that the curve system C is analytic. Thus, we may restrict ourselves from the beginning to the class of analytic curve systems C and formulate the extremum problems only within this class. A first result for a general theory of extremum problems for the Fredholm determinants is the fact that $D(E)$ is semi-continuous from above in the class of all analytic curve systems C. In fact, we will prove the

THEOREM. *Let $\tilde{D}_n$ be a sequence of domains, each being bounded by an analytic curve system C_n and with the Fredholm determinant $D_n(E)$. If the domains $\tilde{D}_n$ converge in the Carathéodory sense to a domain $\tilde{D}$ with analytic boundary C and with the Fredholm determinant $D(E)$, then we have for all $E \geq 0$*

$$\overline{\lim}\, D_n(E) \leq D(E) \,. \tag{39}$$

Proof. We define the kernel

$$\tilde{\lambda}^{(2)}(z, \bar{\zeta}) = \iint_{\tilde{D}} \tilde{\lambda}(z, \eta)\overline{\tilde{\lambda}(z, \eta)}d\tau_\eta = \sum_{\nu=1}^{\infty} \frac{\tilde{u}_\nu(z)\overline{\tilde{u}_\nu(\zeta)}}{\lambda_\nu^2} \tag{40}$$

and define then recursively

$$\tilde{\lambda}^{(2j)}(z, \bar{\zeta}) = \iint_{\tilde{D}} \tilde{\lambda}^{(2j-2)}(z, \bar{\eta})\tilde{\lambda}^{(2)}(\eta, \bar{\zeta})d\tau_\eta = \sum_{\nu=1}^{\infty} \frac{\tilde{u}_\nu(z)\overline{\tilde{u}_\nu(\zeta)}}{\lambda_\nu^{2j}} \,. \tag{41}$$

We remark that

$$\iint_{\tilde{D}} \tilde{\lambda}^{(2j)}(z, \bar{z})d\tau_z = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu^{2j}} = S^{(2j)} \,. \tag{42}$$

We denote the corresponding expressions referring to the domain $\tilde{D}_n$ by the subscripts n. We assert, at first :

$$\underline{\lim}\, S_n^{(2j)} \geq S^{(2j)} \tag{43}$$

To prove this assertion, we select a number $\delta > 0$ arbitrarily small and determine a closed subdomain $\tilde{\Delta}$ of $\tilde{D}$ such that

$$\iint_{\tilde{\Delta}} \tilde{\lambda}^{(2j)}(z, \bar{z})d\tau_z > S^{(2j)} - \delta \,. \tag{44}$$

By the definitions (25), (40), (41) and in view of the continuous dependence of the Green's function $\tilde{G}(z, \zeta)$ on its domain $\tilde{D}$, the kernels $\tilde{\lambda}_n^{(2j)}(z, \bar{\zeta})$ converge to $\tilde{\lambda}^{(2j)}(z, \bar{\zeta})$ uniformly in each closed subdomain of $\tilde{D}$, in particular in $\tilde{\Delta}$. Given δ, we can choose $n(\delta)$ such that for $n > n(\delta)$ the domains $\tilde{D}_n$ contain $\tilde{\Delta}$ and that

$$\begin{aligned} S_n^{(2j)} &= \iint_{\tilde{D}_n} \tilde{\lambda}_n^{(2j)}(z, \bar{z})d\tau_z > \iint_{\tilde{\Delta}} \tilde{\lambda}_n^{(2j)}(z, \bar{z})d\tau_z \\ &\geq \iint_{\tilde{\Delta}} \tilde{\lambda}^{(2j)}(z, \bar{z})d\tau_z - \delta > S^{(2j)} - 2\delta \,. \end{aligned} \tag{45}$$

Since δ can be chosen arbitrarily small, these inequalities imply (43).

We observe next that by definition (35)

$$-\log D(E) = \sum_{j=1}^{\infty} \frac{1}{j} E^{2j} S^{(2j)} \tag{46}$$

and a corresponding representation is valid for $-\log D_n(E)$. Hence, from (43) follows immediately the asserted inequality (39) and the theorem is proved.

The significance of this theorem is the following. Let $\mathfrak{A}$ be a family of analytic curve systems C and let us ask for the maximum of $D(E)$ within the family $\mathfrak{A}$, for some fixed value E. We know that by its definition $D(E) \leq 1$ and is thus trivially bounded in $\mathfrak{A}$. Let $U \leq 1$ denote the least upper bound of $D(E)$ in $\mathfrak{A}$; we can select an extremum sequence of curve sets C_n in $\mathfrak{A}$ such that $D_n(E)$ converges to U. If it is possible to select a subsequence C_{n1} of the C_n such that the corresponding domains D_{n1} converge to a domain D_0 with analytic boundary $C_0 \in \mathfrak{A}$, then C_0 is a maximum curve system. For, by our theorem (38), we have $D_0(E) \geq U$ and, hence, $D_0(E) = U$ since no $D(E)$ in $\mathfrak{A}$ can be larger than U. This argument will be applied in the following section to an interesting problem of conformal mapping.

8. An extremum problem for Fredholm determinants and an existence proof for circular mappings. In this section, we shall utilize the variational formulas for the Fredholm determinants in order to solve a specific maximum problem. The extremum domains of this problem will be characterized by the property that their boundary C consists of circumferences. In this way, we will then prove that every plane domain can be mapped conformally upon a canonical domain whose boundaries are circumferences. This canonical mapping will appear as the solution of a simple extremum problem for the family of all univalent mappings of the given domain.

We formulate the following extremum problem:

Let $\tilde{D}$ be a domain in the complex z-plane which contains the point at infinity and which is bounded by N closed analytic curves C. Let $\mathfrak{F}$ be the family of all functions $t = f(z)$ which are analytic in $\tilde{D} + C$, normalized at infinity by $f'(\infty) = 1$ and are univalent in $\tilde{D}$. Each $f(z) \in \mathfrak{F}$ will map $\tilde{D}$ upon a domain $\tilde{\Delta}$ with analytic boundary Γ and with the Fredholm determinants $\Delta(E)$. We ask for the functions $f(z) \in \mathfrak{F}$ which lead to the maximum value of $\Delta(1)$.

The existence of such maximum functions is by no means obvious. We can assert only that all determinants $\Delta(1)$ obtained by mappings of the family $\mathfrak{F}$ have a least upper bound $U \leq 1$. Hence, we may select a sequence of mappings $f_n(z) \in \mathfrak{F}$ such that

$$\lim_{n\to\infty} \Delta_n(1) = U . \tag{1}$$

Since the $f_n(z)$ are univalent in $\tilde{D}$ we can use the well-known normality properties of these functions and assume without loss of generality that the $f_n(z)$ converge to a limit function $f(z)$, uniformly in each closed subdomain of $\tilde{D}$. The limit function $f(z)$ provides a univalent map of $\tilde{D}$ into a domain $\tilde{\Delta}$ and is normalized at infinity. The image

domains $\tilde{\Delta}_n$ converge in the Carathéodory sense to $\tilde{\Delta}$. If we could prove that $\tilde{\Delta}$ has an analytic boundary Γ, we would know that $f(z) \in \mathfrak{F}$ and the semi-continuity from above of $\Delta(1)$ would insure $\Delta(1) = U$, i.e., that $f(z)$ is a maximum function.

In order to prove the fact $f(z) \in \mathfrak{F}$ we consider the maximum sequence $f_n(z)$ which converges to $f(z)$. We want to characterize this sequence by comparing it with near-by sequences obtained by infinitesimal variations of their image domains $\tilde{\Delta}_n$. However, if we subject a multiply-connected domain $\tilde{\Delta}_n$ to an interior variation (6.1), we will, in general, obtain a domain $\tilde{\Delta}_n^*$ which is not conformally equivalent to $\tilde{\Delta}_n$ and cannot be obtained from $\tilde{D}$ by a mapping of the family $\mathfrak{F}$. Let, indeed, $\omega_l(t)$ be the harmonic measure of the boundary component Γ_l of Γ with respect to $\tilde{\Delta}$ and let $((p_{jk}))$ denote the period matrix (2.18″) of this set of harmonic measures. The period matrix $((p_{jk}))$ is a conformal invariant and if we preserve the point at infinity under the conformal mappings, the numbers $\omega_l(\infty)$ must likewise be unchanged. On the other hand, it is well-known [5, 15, 21] that under a variation of the t-plane of the type (6.1) and with the singular point $t_0 \in \tilde{\Delta}$, we have

$$p_{jk}^* = p_{jk} + \Re\{\alpha w_j'(t_0) w_k'(t_0)\} + O(|\alpha|^2) \tag{2}$$

and

$$\omega_l^*(\infty) = \omega_l(\infty) + \Re\{\alpha p'(t_0, \infty) w_l'(t_0)\} + O(|\alpha|^2) \tag{3}$$

where again $w_l(t)$ and $p(t, \tau)$ denote the analytic completions in t of the harmonic functions $\omega_l(t)$ and $g(t, \tau)$ in $\tilde{\Delta}$. We see that, in general, the numbers p_{jk} and $\omega_l(\infty)$ will change under interior variations and that the domain $\tilde{\Delta}^*$ will not be obtained from $\tilde{D}$ by a mapping of the family $\mathfrak{F}$.

Consider now m points t_μ in $\tilde{\Delta}$ and the variation

$$t^* = t + \sum_{\mu=1}^{m} \frac{\alpha_\mu}{t - t_\mu} + O(|\alpha|^2), \quad |\alpha| = \max_\mu (|\alpha_\mu|) \tag{4}$$

where the error term is estimated uniformly in $\tilde{\Delta} + \Gamma$. We may choose the α_μ and the correction term $O(|\alpha|^2)$ such that

$$\Re\left\{\sum_{\mu=1}^{m} \alpha_\mu w_j'(t_\mu) w_k'(t_\mu)\right\} = 0 \tag{5}$$

$$\Re\left\{\sum_{\mu=1}^{m} \alpha_\mu p'(t_\mu, \infty) w_l'(t_\mu)\right\} = 0 \tag{6}$$

and

$$p_{jk}^* = p_{jk}, \quad \omega_l^*(\infty) = \omega_l(\infty). \tag{7}$$

It can be shown, indeed, that given such values t_μ and α_μ, the variation (4) can be selected in such a way that $\tilde{\Delta}^*$ is conformally equivalent to $\tilde{\Delta}$ and that the points at infinity correspond [21]. Even now, we cannot assert that $\tilde{D}$ goes into $\tilde{\Delta}^*$ by a mapping of the family $\mathfrak{F}$ which is normalized at infinity. However, the Fredholm determinants do not change under a homothetic mapping of a domain and, hence, the insistence on the normalization at infinity is unnecessary in our problem. Thus, the above variations (4) will transform the domains $\tilde{\Delta}_n$ of the extremum sequence into conformally equivalent domains $\tilde{\Delta}_n^*$ whose Fredholm determinants $\Delta_n^*(1)$ may be compared with the maximum sequence $\Delta_n(1)$.

We observe that the functions $w_j'(t) \cdot w_k'(t)$ and $p'(t, \infty) \cdot w_l'(t)$ are quadratic differentials of $\tilde{\Delta}$, i.e., functions $Q_k(t)$ which are regular analytic in $\tilde{\Delta} + \Gamma$ and satisfy on Γ the boundary condition

$$Q_k(t)t'^2 = \text{real} . \tag{8}$$

At infinity all these functions satisfy the asymptotic relation

$$Q_k(t) = O(|\, t \,|^{-3}) . \tag{9}$$

All functions with the properties (8) and (9) from a linear space with real coefficients and of the dimension $3N$-3. We suppose that we have chosen from the above $N(N+1)$ quadratic differentials a fixed basis of $3N$-3 elements $Q_k(t)$, $k = 1, 2, \cdots, 3N$-3.

After these preparations, we return to our maximum sequence of domains $\tilde{\Delta}_n$; we denote by $Q_k^{(n)}(t)$ the corresponding basis of quadratic differentials of $\tilde{\Delta}_n$ and by $Q_k(t)$ the basis for their limit domain $\tilde{\Delta}$. Clearly, we can choose the basis in each $\tilde{\Delta}_n$ and in $\tilde{\Delta}$ such that

$$\lim_{n\to\infty} Q_k^{(n)}(t) = Q_k(t) , \tag{10}$$

uniformly in each closed subdomain of $\tilde{\Delta}$. The determinant

$$\det \| \Re\{Q_k(t_l)\} \| , \qquad l, k = 1, 2, \cdots, 3N\text{-}3 \tag{11}$$

does not vanish identically in $\tilde{\Delta}$ because of the supposed real independence of the $Q_k(t)$. Hence, we can determine $3N$-3 points $t_\mu \in \tilde{\Delta}$ such that

$$\det \| \Re\{Q_k^{(n)}(t_\mu)\} \| \neq 0 \qquad k, \mu = 1, 2, \cdots, 3N\text{-}3 \tag{12}$$

for large enough n; we may even assume, without loss of generality, that (12) holds for all integers n.

Let t_0 be an arbitrary point in $\tilde{\Delta}_n$ and $\alpha^{(n)}$ be an arbitrary complex number. We determine $3N$-3 real numbers $x_\mu^{(n)}$ by the linear equations

$$\Re\{\alpha^{(n)} Q_k^{(n)}(t_0)\} = \sum_{\mu=1}^{3N-3} x_\mu^{(n)} \Re\{Q_k^{(n)}(t_\mu)\} , \quad k = 1, 2, \cdots, 3N\text{-}3 \tag{13}$$

which is always posible because of (12). Observe that $x_\mu^{(n)} = O(|\alpha^{(n)}|)$ for small values of $\alpha^{(n)}$. Consider then the interior variation of $\tilde{\Delta}_n$

$$t^* = t + \frac{\alpha^{(n)}}{t - t_0} - \sum_{\mu=1}^{3N-3} \frac{x_\mu^{(n)}}{t - t_\mu} + O(|\alpha^{(n)}|^2) . \tag{14}$$

This variation is of the type (4), but by the choice (13) of the $x_\mu^{(n)}$, we are sure that the equations (5) and (6) will be fulfilled. We can, therefore, adjust the error term $O(|\alpha^{(n)}|^2)$ in such a way that the varied domain $\tilde{\Delta}_n^*$ is conformally equivalent to $\tilde{\Delta}_n$ and such that the points at infinity correspond. Hence, $\tilde{\Delta}_n^*$ may be used as a competing domain sequence to the maximum sequence $\tilde{\Delta}_n$. We apply now the variational formula (7.38′) in order to characterize the limit domain $\tilde{\Delta}$.

We derive from (7.38′) that the variation (14) of $\tilde{\Delta}_n$ yields

$$\begin{aligned} \log \Delta_n^*(1) = \log \Delta_n(1) &- 2\pi \Re\{\alpha^{(n)} \tilde{\lambda}_n(t_0, t_0)\} \\ &+ 2\pi \sum_{\mu=1}^{3N-3} x_\mu^{(n)} \Re\{\tilde{\lambda}_n(t_\mu, t_\mu)\} + O(|\alpha^{(n)}|^2) . \end{aligned} \tag{15}$$

Here, the $\tilde{\lambda}_n(t, t)$ denote the $\tilde{\lambda}$-kernels of $\tilde{\Delta}_n$. We denote

$$\delta_n = \log U - \log \Delta_n(1) . \tag{16}$$

By the definition of the maximum sequence, we have $0 < \delta_n \to 0$. Since $\log \Delta_n^*(1) \leq \log U$, we infer from (15) the inequality

$$\frac{1}{2\pi} \delta_n \geq - \Re\{\alpha^{(n)} \tilde{\lambda}_n(t_0, t_0)\} + \sum_{\mu=1}^{3N-3} x_\mu^{(n)} \Re\{\tilde{\lambda}_n(t_\mu, t_\mu)\} + O(|\alpha^{(n)}|^2) . \tag{17}$$

We choose finally

$$\alpha^{(n)} = \delta_n r e^{i\tau} , \qquad r > 0 \tag{18}$$

and define the real numbers ξ_μ by the system of linear equations

$$\sum_{\mu=1}^{3N-3} \xi_\mu \Re\{Q_k(t_\mu)\} = \Re\{e^{i\tau} Q_k(t_0)\} , \qquad k = 1, \cdots, 3N\text{-}3 . \tag{19}$$

We divide equations (13) and (17) by δ_n and pass to the limit $n \to \infty$; comparing (13) with (19), we find

$$\lim_{n\to\infty} \frac{x_\mu^{(n)}}{r\delta_n} = \xi_\mu ; \tag{20}$$

and since at $t_0, t_1, \cdots, t_{3N-3}$ holds

$$\lim_{n\to\infty} \tilde{\lambda}_n(t_\mu, t_\mu) = \tilde{\lambda}(t_\mu, t_\mu) , \tag{21}$$

we obtain from (17)

$$\frac{1}{2\pi r} \geq -\Re\{e^{i\tau}\tilde{\lambda}(t_0, t_0)\} + \sum_{\mu=1}^{3N-3} \xi_\mu \Re\{\tilde{\lambda}(t_\mu, t_\mu)\} \ . \tag{22}$$

This inequality holds for arbitrary values $r > 0$; hence, sending $r \to \infty$, we find

$$0 \geq -\Re\{e^{i\tau}\tilde{\lambda}(t_0, t_0)\} + \sum_{\mu=1}^{3N-3} \xi_\mu \Re\{\tilde{\lambda}(t_\mu, t_\mu)\} \ . \tag{23}$$

If we replace in (19) the signum $e^{i\tau}$ by $-e^{i\tau}$, the solution vector ξ_μ changes into $-\xi_\mu$. Since $e^{i\tau}$ is entirely arbitrary, the inequality (23) must also hold for inverted sign of the right hand term. Thus, we arrive finally at the equation

$$\Re\{e^{i\tau}\tilde{\lambda}(t_0, t_0)\} = \sum_{\mu=1}^{3N-3} \xi_\mu \Re\{\tilde{\lambda}(t_\mu, t_\mu)\} \ . \tag{24}$$

valid for arbitrary choice of the signum $e^{i\tau}$ and the corresponding choice (19) of the ξ_μ. The fact that, for given fixed $t_1, \cdots, t_{3N-3}$ in $\tilde{\Delta}$ and for arbitrary $t_0 \in \tilde{\Delta}$, the linear equations (19) always imply the equation (24) for arbitrary $e^{i\tau}$, guarantees the existence of $3N$-3 real numbers $\beta_\mu (\mu = 1, \cdots, 3N\text{-}3)$ such that

$$\tilde{\lambda}(t, t) = \sum_{\mu=1}^{3N-3} \beta_\mu Q_\mu(t) \ . \tag{25}$$

This identity is then the condition which characterizes the limit domain $\tilde{\Delta}$ of an extremum sequence $\tilde{\Delta}_n$.

Since, in view of (7.26), the function $\tilde{\lambda}(t, t)$ coincides with the more fundamental kernel $\tilde{l}(t, t)$ except for a quadratic differential, we may express the result (25) as follows :

THEOREM I. *If $\tilde{\Delta}$ is the limit domain of a maximum sequence $\tilde{\Delta}_n$, its $\tilde{l}$-kernel satisfies the condition*

$$\tilde{l}(t, t) = Q(t) \tag{26}$$

where $Q(t)$ is a quadratic differential of $\tilde{\Delta}$.

From Theorem I, we can deduce

THEOREM II. *All boundary curves Γ_ι of $\tilde{\Delta}$ are analytic.*

Proof. Let us express equation (26) in terms of functionals of the original domain $\tilde{D}$ which is conformally equivalent to $\tilde{\Delta}$. By (7.11) and because of the covariance character of the quadratic differentials under conformal mapping, we can express (26) in the form

$$\tilde{l}(z, z) + \frac{1}{6\pi}\{f, z\} = Q(z) \tag{27}$$

where $Q(z)$ is the quadratic differential in $\tilde{D}$ which corresponds to $Q(t)$ under the mapping $t = f(z)$ of $\tilde{D}$ into $\tilde{\Delta}$ and $\tilde{l}(z, z)$ denotes the $\tilde{l}$-kernel of $\tilde{D}$. We have assumed that $\tilde{D}$ has analytic boundaries C_ν; hence, we can assert that $\tilde{l}(z, z)$ and $Q(z)$ are analytic in the closed region $\tilde{D} + C$. By (7.12), we may now interpret the equation (27) as a linear differential equation with analytic coefficient in $\tilde{D} + C$:

$$\mu''(z) + 3\pi[Q(z) - \tilde{l}(z, z)]\mu(z) = 0 \tag{28}$$

for the unknown function

$$\mu(z) = [f'(z)]^{-1/2} . \tag{29}$$

From the general theory of ordinary differential equations we obtain that $\mu(z)$ is regular analytic in $\tilde{D} + C$ and can have only finitely many zeros on C. Hence, $f'(z)$ is analytic on C except for poles which are at least of order 2. At such singular points on C, $f(z)$ would have poles too. But $f(z)$ is univalent in $\tilde{D}$ and has already a pole at infinity. It cannot have additional poles on C; hence, $f(z)$ and $f'(z)$ are regular analytic on C and the theorem is proved.

In particular, we have now shown that the limit function $f(z)$ of the maximum sequence $f_n(z)$ belongs also to the family $\mathfrak{F}$ considered and is, therefore, a maximum function of our problem.

Since we know now that the boundary curves Γ_ι of $\tilde{\Delta}$ are analytic, we can combine (26) with (7.7) and find:

$$\Im\{\tilde{l}(t, t)t'^2\} = \Im\{Q(t)t'^2\} = \frac{1}{6\pi}\frac{d\kappa}{ds} . \tag{30}$$

But $Q(t)$ is a quadratic differential of $\tilde{\Delta}$; thus we arrive at

$$\frac{d\kappa}{ds} = 0 \text{ on each } \Gamma_\iota . \tag{31}$$

This leads to

THEOREM III. *Each boundary curve Γ_ι of the maximum domain $\tilde{\Delta}$ is a circumference.*

Since in each given domain $\tilde{D}$ there exists at least one maximum sequence $f_n(z) \in \mathfrak{F}$, we have given a new proof for the classical theorem [5, 7, 8, 9, 23]:

THEOREM IV. *Every plane domain $\tilde{D}$ can be mapped onto a domain bounded by circumferences.*

Since the domain $\tilde{\Delta}$ is the limit of a maximum sequence of domains

$\tilde{\Delta}_n$ and since it is analytically bounded, the semi-continuity of the Fredholm determinants leads to

THEOREM V. *Among all conformally equivalent domains, the circular domains have the largest value of the Fredholm determinant* $D(1)$.

BIBLIOGRAPHY

1. L. V. Ahlfors, *Conformality with respect to Riemann metrics*, Ann. Acad. Sci. Fenn., Series A **202** (1955).
2. S. Bergman, and M. Schiffer, *A representation of Green's and Neumann's functions in the theory of partial differential equations of second order*, Duke Math. J. **14**, (1947), 609-638.
3. ———, *Kernel functions and conformal mapping*, Comp. Math. **8** (1951), 205-249.
4. I. E. Block, *Kernel functions and class* L^2, Proc. Amer. Math. Soc. **4** (1953) 110-117.
5. R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Appendix by M. Schiffer, New York 1950.
6. P. R. Garabedian, and M. Schiffer, *Convexity of domain functionals*, J. Analyse Math. **2** (1953), 281-368.
7. A. Hurwitz, and R. Courant, *Funktionentheorie*, Berlin, 1929.
8. P. Koebe, *Über die konforme Abbildung mehrfach zusammenhängender Bereiche*, Jahresber. D. Math. Ver. **19** (1910), 339-348.
9. L. Lichtenstein, *Neuere Entwicklung der Potentialtheorie, Konforme Abbildung*, Enzycl. d. Math. Wissensch. II, C. 3, (1918)
10. H. Poincaré, *Théorie du Potentiel Newtonien*, Paris, 1899.
11. G. Pólya, and M. Schiffer, *Convexity of functionals by transplantation*, J. Analyse Math. **3** (1954), 245-346.
12. R. de Possel, *Sur quelques problèmes de représentation conforme*, C. R. Paris **194** (1932), 42-44.
13. E. Rengel, *Existenzbeweise für schlichte Abbildungen mehrfachzusammenhängender Bereiche auf gewisse Normalbereiche*, Jahresber. D. Math. Ver. **44** (1934), 51-55.
14. M. Schiffer, *Variation of the Green's function and theory of the p-valued functions*, Amer. J. Math. **65** (1943), 341-360.
15. ———, *Hadamard's formula and variation of domain functions*, Amer. J. Math. **68** (1946), 417-448.
16. ———, *The kernel function of an orthonormal system*, Duke Math. J. **13** (1946), 529-540.
17. ———, *The Fredholm eigen values of plane domain*, Pacific J. Math. **7** (1957), 1187-1225.
18. ———, *Sur les rapports entre les solutions des problèmes intérieurs et celles des problèmes extérieurs*. C. R. Paris **244** (1957), 2680-2683.
19. ———, *Sur la polarization et la masse virtuelle*, C. R. Paris **244** (1957), 3118-3121,
20. ———, *Problèmes aux limites et fonctions propres de l'équation intégrale de Poincaré et de Fredholm*, C. R. Paris **245** (1957), 18-21.
21. M. Schiffer, and D. C. Spencer, *Functionals of finite Riemann surfaces*, Princeton 1954.
22. M. Schiffer and G. Szegö, *Virtual mass and polarization*, Trans. Amer. Math. Soc. **67** (1949), 130-205.
23. E. Schottky, *Über die konforme Abbildung mehrfach zusammenhängender ebener Flächen*, J.f.r.u. angew. Math. **83** (1877) 300-351.

STANFORD UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by Kokusai Bunken Insatsusha
(International Academic Printing Co., Ltd.), Tokyo, Japan

Pacific Journal of Mathematics
Vol. 9, No. 1 May, 1959