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In two recent papers [2, 3] I deduced and used the general transformation

$$(1)\qquad 1+\sum_{s=1}^{\infty}(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}}(1-kx^{2s})\frac{(kx;s-1)}{(x;s)}$$
$$=\prod_{n=1}^{\infty}(1-kx^n)\sum_{t=0}^{\infty}\frac{k^t G_{M,t}(x)}{(x;t)}, \qquad (M=2,3,\cdots)$$

to prove certain generalized identities of the type

$$(2)\qquad \prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-s})(1-x^{(2M+1)n-(2M+1-s)})(1-x^{(2M+1)n})}{(1-x^n)}$$
$$=\sum_{t=0}^{\infty}\frac{A_s(x,t)G_{M,t}(x)}{(x;t)},$$

where $A_s(x,t)$ and $G_{M,t}(x)$ are polynomials. For $s=M$ and $s=1$ respectively in (2), we get Alder's generalizations of the well-known Rogers-Ramanujan identities

$$\prod_{n=1}^{\infty}\frac{(1-x^{5n-2})(1-x^{5n-3})(1-x^{5n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{x^{t^2}}{(x;t)}$$

and

$$\prod_{n=1}^{\infty}\frac{(1-x^{5n-1})(1-x^{5n-4})(1-x^{5n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{x^{t(t+1)}}{(x;t)}$$

in the form [1]

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-M})(1-x^{(2M+1)n-M-1})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{G_{M,t}(x)}{(x;t)}$$

and

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-1})(1-x^{(2M+1)n-2M})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{x^t G_{M,t}(x)}{(x;t)}.$$

For the Alder polynomials $G_{M,t}(x)$ in (1), I gave the general form

$$(3)\qquad G_{M,t}(x)=x^{t^2}\sum_{t_1=0}^{\left[\frac{M-2}{M-1}t\right]}\frac{(x^{t-2t_1+1};2t_1)x^{-2t_1(t-t_1)}}{(x;t_1)}\prod_{n=2}^{M-2}T_{n,M}$$

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where

$$T_{n,M} = \sum_{t_n=0}^{\left[\frac{M-n-1}{M-n}t_{n-1}\right]} \frac{(x^{t_{n-1}-2t_n+1}; 2t_n)x^{-2t_n(t_{n-1}-t_n)}}{(x; t_n)(x^{t_{n-2}-2t_{n-1}+1}; t_n)} \qquad M \geq 2 ,$$

$[a]$ denoting the integral part of a.

Alder in his paper [1] states that the polynomials $G_{M,t}(x)$ do not seem to possess any striking properties, even for small values of M and t. In the present note, using a simple recurrence relation, I prove beside other results the interesting property that

$$G_{M,t}(x) = x^t , \qquad t \leq (M-1) .$$

The form (3) is not very suitable for the actual computation of the polynomials $G_{M,t}(x)$ for particular values of M and t since certain factor have to be cancelled each time. Therefore, moving into the following series the factor $(x^{t-2t_1+1}; t_1)$ from the first series and the factor $(x^{t_{n-1}-2t_n+1}; t_n)$ from each of the $T_{n,M}$ series in (3), we put $G_{M,t}(x)$ in the form

$$(4) \qquad G_{M,t}(x) = x^{t^2} \sum_{t_1=0}^{\left[\frac{M-2}{M-1}t\right]} \frac{(x^{t-t_1+1}; t_1)x^{-2t_1(t-t_1)}}{(x; t_1)} \prod_{n=2}^{M-1} \overline{T}_{n,M}$$

where

$$(5) \qquad \overline{T}_{n,M} = \sum_{t_n=0}^{\left[\frac{M-n-1}{M-n}t_{n-1}\right]} \frac{(x^{t_{n-1}-t_n+1}; t_n)x^{-2t_n(t_{n-1}-t_n)}}{(x; t_n)} \times (x^{t_{n-2}-2t_{n-1}+t_n+1}; t_{n-1}-t_n) .$$

Now if we put

$$(6) \qquad g_{M,t}(N, x) = \prod_{n=1}^{M-1} \overline{T}_{n,M} \qquad (\text{where } t_{-1} \equiv N) ,$$

then, since

$$(7) \qquad g_{M,t}(N, x) = \sum_{t_1=0}^{\left[\frac{M-2}{M-1}t\right]} \frac{(x^{t-t_1+1}; t_1)(x^{N-2t+t_1+1}; t-t_1)}{(x; t_1)} \times x^{-2t_1(t-t_1)} g_{M-1,t_1}(t, x) ,$$

it is easily seen by induction that for $t \leq M-1$, we have

$$(8) \qquad g_{M+1,t}(N, x) - g_{M,t}(N, x) = 0$$

because

$$(9) \qquad \left[\frac{M-2}{M-1}t\right] + 1 > \left[\frac{M-1}{M}t\right] \qquad t \leq M-1 .$$

From (4) we have

$$G_{M+1,t}(x) - G_{M,t}(x)$$

$$(10) \qquad = x^{t^2} \sum_{t=0}^{\left[\frac{M-2}{M-1}t\right]} \frac{(x^{t-t_1+1}; t_1)x^{-2t_1(t-t_1)}}{(x; t_1)} \{g_{M,t_1}(t, x) - g_{M-1,t_1}(t, x)\}$$

$$+ \sum_{t_1=\left[\frac{M-2}{M-1}t\right]+1}^{\left[\frac{M-1}{M}t\right]} \frac{(x^{t-t_1+1}; t_1)x^{-2t_1(t-t_1)}}{(x; t_1)} g_{M,t_1}(t, x) .$$

Hence from (8) and (9) it follows that, for $t \leq M - 1$,

$$G_{M,t}(x) = G_{M+1,t}(x)$$

that is,

$$G_{M,t}(x) = G_{M+1,t}(x) = \cdots = G_{\infty,t}(x) , \qquad t \leq M - 1 .$$

Now, for $k = 1$ and $M \to \infty$, (1) gives

$$\frac{1}{\prod_{n=1}^{\infty}(1 - x^n)} = \sum_{t=0}^{\infty} \frac{G_{\infty,t}(x)}{(x; t)}$$

whence $G_{\infty,t}(x) = x^t$, so that we finally get

$$(11) \qquad G_{M,t}(x) = x^t \qquad t \leq M - 1 .$$

(10) can be further used for the computation of polynomials $G_{M,t}(x)$ as follows.

We first find the general form for $G_{M,M}(x)$.

From (10) we have

$$(12) \qquad G_{M+1,M}(x) - G_{M,M}(x) = x_M x^{-2(M-1)} g_{M,M-1}(M, x) ,$$

where $x_n \equiv (1 - x^n)/(1 - x)$ for all n.

From (7) we find

$$(13) \qquad g_{M,M-1}(M, x) = (x; M - 1)x^{-(M-1)(M-2)} .$$

Using (13) in (12) we get

$$(14) \qquad G_{M,M}(x) = x^M\{1 - (x^2; M - 1)\}$$

since $G_{M+1,M}(x) = x^M$. Thus, for example,

$$G_{5,5}(x) = x^7 + x^8 + x^9 - x^{11} - 2x^{12} - x^{13} + x^{15} + x^{16} + x^{17} - x^{19} .$$

More generally, taking $t = M + r$ in (7), since

$$\left[\frac{M^2+(r-2)M-2r}{M-1}\right]=M+r-2 \qquad r \leq M-2\,,$$

and

$$\left[\frac{M^2+(r-1)M-r}{M}\right]=M+r-2 \qquad 0<r \leq M\,,$$

we easily get

$$g_{M+1,M+r}(N,x)-g_{M,M+r}(N,x)$$

(15) $$=\prod_{n=1}^{r}\overline{T}_{n,M}\{g_{M-r+1,M-r}(t_{r-1},x)-g_{M-r,M-r}(t_{r-1},x)\} \quad 0>r \leq M-2\,,$$

where, in $\overline{T}_{n,M}$, $t=M+r$ and $t_r=M-r$. Thus for $t \leq 2M-2(t \neq M)$ the second sum on the right of (10) does not exist and we may successively establish the general form of the polynomials $G_{M,t}(x)$ for $M<t\leq 2(M-1)$. We thus find that

$$G_{M+1,M+1}(x)-G_{M,M+1}(x)=x^{M+3}(x^3;M-1)x_2 \qquad M \geq 3\,,$$

so that, using (14), we get

$$G_{M,M+1}(x)=x^{M+1}\{1-(x^3;M-1)(1+x^3)\} \qquad M \geq 3\,.$$

Similarly

$$G_{M,M+2}(x)=x^{M+2}\{1-(x^4;M-1)(1+x^4\cdot x_2)\} \qquad M \geq 4\,,$$

$$G_{M,M+3}(x)=x^{M+3}\{1-(x^5;M-1)(1+x^5\cdot x_3)\} \qquad M \geq 5\,,$$

The above values of the polynomials $G_{M,t}(x)$ suggest that probably,

(16) $$G_{M,t}(x)=x^t\{1-(x^{t-M+2};M-1)(1+x^{t-M+2}\cdot x_{t-M})\}\,,$$

for $$t \leq 2(M-1)\,.$$

But I have not been able to verify the truth of this conjecture directly.

However, I intend to investigate these interesting polynomials more thoroughly in a future communication.

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