

# Pacific Journal of Mathematics

## **A NOTE ON THE COMPUTATION OF ALDER'S POLYNOMIALS**

V. N. SINGH

# A NOTE ON THE COMPUTATION OF ALDER'S POLYNOMIALS

V. N. SINGH

In two recent papers [2, 3] I deduced and used the general transformation

$$(1) \quad 1 + \sum_{s=1}^{\infty} (-1)^s k^M s x^{\frac{1}{2} s (2M+1)s-1} (1 - kx^{2s}) \frac{(kx; s-1)}{(x; s)} \\ = \prod_{n=1}^{\infty} (1 - kx^n) \sum_{t=0}^{\infty} \frac{k^t G_{M,t}(x)}{(x; t)}, \quad (M = 2, 3, \dots)$$

to prove certain generalized identities of the type

$$(2) \quad \prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-s})(1 - x^{(2M+1)n-(2M+1-s)})(1 - x^{(2M+1)n})}{(1 - x^n)} \\ = \sum_{t=0}^{\infty} \frac{A_s(x, t) G_{M,t}(x)}{(x; t)},$$

where  $A_s(x, t)$  and  $G_{M,t}(x)$  are polynomials. For  $s = M$  and  $s = 1$  respectively in (2), we get Alder's generalizations of the well-known Rogers-Ramanujan identities

$$\prod_{n=1}^{\infty} \frac{(1 - x^{5n-2})(1 - x^{5n-3})(1 - x^{5n})}{(1 - x^n)} = \sum_{t=0}^{\infty} \frac{x^{t^2}}{(x; t)}$$

and

$$\prod_{n=1}^{\infty} \frac{(1 - x^{5n-1})(1 - x^{5n-4})(1 - x^{5n})}{(1 - x^n)} = \sum_{t=0}^{\infty} \frac{x^{t(t+1)}}{(x; t)}$$

in the form [1]

$$\prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-M})(1 - x^{(2M+1)n-M-1})(1 - x^{(2M+1)n})}{(1 - x^n)} = \sum_{t=0}^{\infty} \frac{G_{M,t}(x)}{(x; t)}$$

and

$$\prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-1})(1 - x^{(2M+1)n-2M})(1 - x^{(2M+1)n})}{(1 - x^n)} = \sum_{t=0}^{\infty} \frac{x^t G_{M,t}(x)}{(x; t)}.$$

For the Alder polynomials  $G_{M,t}(x)$  in (1), I gave the general form

$$(3) \quad G_{M,t}(x) = x^{t^2} \sum_{t_1=0}^{\lfloor \frac{M-2}{M-1} t \rfloor} \frac{(x^{t-2t_1+1}; 2t_1) x^{-2t_1(t-t_1)}}{(x; t_1)} \prod_{n=2}^{M-2} T_{n,M}$$

Received June 3, 1958.

where

$$T_{n,M} = \sum_{t_n=0}^{[M-n-1]t_{n-1}} \frac{(x^{t_{n-1}-2t_n+1}; 2t_n)x^{-2t_n(t_{n-1}-t_n)}}{(x; t_n)(x^{t_{n-2}-2t_{n-1}+1}; t_n)} \quad M \geq 2,$$

[*a*] denoting the integral part of *a*.

Alder in his paper [1] states that the polynomials  $G_{M,t}(x)$  do not seem to possess any striking properties, even for small values of  $M$  and  $t$ . In the present note, using a simple recurrence relation, I prove beside other results the interesting property that

$$G_{M,t}(x) = x^t, \quad t \leq (M - 1).$$

The form (3) is not very suitable for the actual computation of the polynomials  $G_{M,t}(x)$  for particular values of  $M$  and  $t$  since certain factor have to be cancelled each time. Therefore, moving into the following series the factor  $(x^{t-2t_1+1}; t_1)$  from the first series and the factor  $(x^{t_{n-1}-2t_n+1}; t_n)$  from each of the  $T_{n,M}$  series in (3), we put  $G_{M,t}(x)$  in the form

$$(4) \quad G_{M,t}(x) = x^{t^2} \sum_{t_1=0}^{[M-2]t} \frac{(x^{t-t_1+1}; t_1)x^{-2t_1(t-t_1)}}{(x; t_1)} \prod_{n=2}^{M-1} \bar{T}_{n,M}$$

where

$$(5) \quad \bar{T}_{n,M} = \sum_{t_n=0}^{[M-n-1]t_{n-1}} \frac{(x^{t_{n-1}-t_n+1}; t_n)x^{-2t_n(t_{n-1}-t_n)}}{(x; t_n)} \times (x^{t_{n-2}-2t_{n-1}+t_n+1}; t_{n-1} - t_n).$$

Now if we put

$$(6) \quad g_{M,t}(N, x) = \prod_{n=1}^{M-1} \bar{T}_{n,M} \quad (\text{where } t_{-1} \equiv N),$$

then, since

$$(7) \quad g_{M,t}(N, x) = \sum_{t_1=0}^{[M-2]t} \frac{(x^{t-t_1+1}; t_1)(x^{N-2t+t_1+1}; t-t_1)}{(x; t_1)} \times x^{-2t_1(t-t_1)} g_{M-1,t_1}(t, x),$$

it is easily seen by induction that for  $t \leq M - 1$ , we have

$$(8) \quad g_{M+1,t}(N, x) - g_{M,t}(N, x) = 0$$

because

$$(9) \quad \left[ \begin{matrix} M-2 \\ M-1 \end{matrix} t \right] + 1 > \left[ \begin{matrix} M-1 \\ M \end{matrix} t \right] \quad t \leq M - 1.$$

From (4) we have

$$\begin{aligned}
 & G_{M+1,t}(x) - G_{M,t}(x) \\
 (10) \quad &= x^{t^2} \sum_{t_1=0}^{\lfloor \frac{M-2}{M-1}t \rfloor} \frac{(x^{t-t_1+1}; t_1)x^{-2t_1(t-t_1)}}{(x; t_1)} \{g_{M,t_1}(t, x) - g_{M-1,t_1}(t, x)\} \\
 &+ \sum_{t_1=\lfloor \frac{M-2}{M-1}t \rfloor+1}^{\lfloor \frac{M-1}{M}t \rfloor} \frac{(x^{t-t_1+1}; t_1)x^{-2t_1(t-t_1)}}{(x; t_1)} g_{M,t_1}(t, x).
 \end{aligned}$$

Hence from (8) and (9) it follows that, for  $t \leq M - 1$ ,

$$G_{M,t}(x) = G_{M+1,t}(x)$$

that is,

$$G_{M,t}(x) = G_{M+1,t}(x) = \dots = G_{\infty,t}(x), \quad t \leq M - 1.$$

Now, for  $k = 1$  and  $M \rightarrow \infty$ , (1) gives

$$\frac{1}{\prod_{n=1}^{\infty} (1 - x^n)} = \sum_{t=0}^{\infty} \frac{G_{\infty,t}(x)}{(x; t)}$$

whence  $G_{\infty,t}(x) = x^t$ , so that we finally get

$$(11) \quad G_{M,t}(x) = x^t \quad t \leq M - 1.$$

(10) can be further used for the computation of polynomials  $G_{M,t}(x)$  as follows.

We first find the general form for  $G_{M,M}(x)$ .

From (10) we have

$$(12) \quad G_{M+1,M}(x) - G_{M,M}(x) = x_M x^{-2(M-1)} g_{M,M-1}(M, x),$$

where  $x_n \equiv (1 - x^n)/(1 - x)$  for all  $n$ .

From (7) we find

$$(13) \quad g_{M,M-1}(M, x) = (x; M - 1)x^{-(M-1)(M-2)}.$$

Using (13) in (12) we get

$$(14) \quad G_{M,M}(x) = x^M \{1 - (x^2; M - 1)\}$$

since  $G_{M+1,M}(x) = x^M$ . Thus, for example,

$$G_{5,5}(x) = x^7 + x^8 + x^9 - x^{11} - 2x^{12} - x^{13} + x^{15} + x^{16} + x^{17} - x^{19}.$$

More generally, taking  $t = M + r$  in (7), since

$$\left[ \frac{M^2 + (r - 2)M - 2r}{M - 1} \right] = M + r - 2 \quad r \leq M - 2 ,$$

and

$$\left[ \frac{M^2 + (r - 1)M - r}{M} \right] = M + r - 2 \quad 0 < r \leq M ,$$

we easily get

$$(15) \quad g_{M+1, M+r}(N, x) - g_{M, M+r}(N, x) = \prod_{n=1}^r \overline{T}_{n, M} \{g_{M-r+1, M-r}(t_{r-1}, x) - g_{M-r, M-r}(t_{r-1}, x)\} \quad 0 > r \leq M - 2 ,$$

where, in  $\overline{T}_{n, M}$ ,  $t = M + r$  and  $t_r = M - r$ . Thus for  $t \leq 2M - 2 (t \neq M)$  the second sum on the right of (10) does not exist and we may successively establish the general form of the polynomials  $G_{M, t}(x)$  for  $M < t \leq 2(M - 1)$ . We thus find that

$$G_{M+1, M+1}(x) - G_{M, M+1}(x) = x^{M+3}(x^3; M - 1)x_2 \quad M \geq 3 ,$$

so that, using (14), we get

$$G_{M, M+1}(x) = x^{M+1}\{1 - (x^3; M - 1)(1 + x^3)\} \quad M \geq 3 .$$

Similarly

$$G_{M, M+2}(x) = x^{M+2}\{1 - (x^4; M - 1)(1 + x^4 \cdot x_2)\} \quad M \geq 4 ,$$

$$G_{M, M+3}(x) = x^{M+3}\{1 - (x^5; M - 1)(1 + x^5 \cdot x_3)\} \quad M \geq 5 ,$$

The above values of the polynomials  $G_{M, t}(x)$  suggest that probably,

$$(16) \quad G_{M, t}(x) = x^t \{1 - (x^{t-M+2}; M - 1)(1 + x^{t-M+2} \cdot x_{t-M})\} ,$$

for  $t \leq 2(M - 1)$ .

But I have not been able to verify the truth of this conjecture directly.

However, I intend to investigate these interesting polynomials more thoroughly in a future communication.

I am grateful to Dr. R. P. Agarwal for his kind help in the preparation of this note.

## REFERENCES

1. H. L. Alder *Generalizations of the Rogers-Ramanujan identities*, Pacific J. Math. **4** (1954), 161-168.
2. V. N. Singh, *Certain generalized hypergeometric identities of the Rogers-Ramanujan type*, Pacific J. Math. **7** (1957), 1011-1014.
3. ———, *Certain generalized hypergeometric identities of the Rogers-Ramanujan type (II)*, Pacific J. Math. **7** (1957), 1691-99.

LUCKNOW UNIVERSITY



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DAVID GILBARG

Stanford University  
Stanford, California

R. A. BEAUMONT

University of Washington  
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California  
Los Angeles 7, California

L. J. PAIGE

University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH

C. E. BURGESS

E. HEWITT

A. HORN

V. GANAPATHY IYER

R. D. JAMES

M. S. KNEBELMAN

L. NACHBIN

I. NIVEN

T. G. OSTROM

H. L. ROYDEN

M. M. SCHIFFER

E. G. STRAUS

G. SZEKERES

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA

OREGON STATE COLLEGE

UNIVERSITY OF OREGON

OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE COLLEGE

UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY

CALIFORNIA RESEARCH CORPORATION

HUGHES AIRCRAFT COMPANY

SPACE TECHNOLOGY LABORATORIES

Printed in Japan by Kokusai Bunken Insatsusha  
(International Academic Printing Co., Ltd.), Tokyo, Japan



Julius Rubin Blum and Murray Rosenblatt, <i>On the structure of infinitely divisible distributions</i> .....	1
Robert Geroge Buschman, <i>Asymptotic expressions for <math>\sum n^a f(n) \log^r n</math></i> .....	9
Eckford Cohen, <i>A class of residue systems (mod <math>r</math>) and related arithmetical functions. I. A generalization of Möbius inversion</i> .....	13
Paul F. Conrad, <i>Non-abelian ordered groups</i> .....	25
Richard Henry Crowell, <i>On the van Kampen theorem</i> .....	43
Irving Leonard Glicksberg, <i>Convolution semigroups of measures</i> .....	51
Seymour Goldberg, <i>Linear operators and their conjugates</i> .....	69
Olof Hanner, <i>Mean play of sums of positional games</i> .....	81
Erhard Heinz, <i>On one-to-one harmonic mappings</i> .....	101
John Rolfe Isbell, <i>On finite-dimensional uniform spaces</i> .....	107
Erwin Kreyszig and John Todd, <i>On the radius of univalence of the function <math>\exp z^2 \int_0^z \exp(-t^2) dt</math></i> .....	123
Roger Conant Lyndon, <i>An interpolation theorem in the predicate calculus</i> .....	129
Roger Conant Lyndon, <i>Properties preserved under homomorphism</i> .....	143
Roger Conant Lyndon, <i>Properties preserved in subdirect products</i> .....	155
Robert Osserman, <i>A lemma on analytic curves</i> .....	165
R. S. Phillips, <i>On a theorem due to Sz.-Nagy</i> .....	169
Richard Scott Pierce, <i>A generalization of atomic Boolean algebras</i> .....	175
J. B. Roberts, <i>Analytic continuation of meromorphic functions in valued fields</i> .....	183
Walter Rudin, <i>Idempotent measures on Abelian groups</i> .....	195
M. Schiffer, <i>Fredholm eigen values of multiply-connected domains</i> .....	211
V. N. Singh, <i>A note on the computation of Alder's polynomials</i> .....	271
Maurice Sion, <i>On integration of 1-forms</i> .....	277
Elbert A. Walker, <i>Subdirect sums and infinite Abelian groups</i> .....	287
John W. Woll, <i>Homogeneous stochastic processes</i> .....	293