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ON INTEGRATION OF 1-FORMS

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1. Introduction. It has been noted by several people that in order to define the integral of some differential 1-form ω along a curve C , the latter need not be of bounded variation. For example, in the extreme (and trivial) case where ω is the differential of some function f , the integral can be defined as the difference of the values assumed by f at the end-points of C . No condition on C is necessary. H. Whitney [4], with J. H. Wolfe, by the introduction of certain norms, has found general abstract spaces of curves along which the integral of 1-forms satisfying certain conditions can be defined. In fact, H. Whitney considers integration of p -forms with $p \geq 1$. In a previous paper [2], we obtained rather awkward conditions for a decent integral to exist that depended on the number of higher derivatives of ω on C .

In this paper, we consider 1-forms ω possessing 'higher derivatives' on C in a sense somewhat different from that due to H. Whitney [3] which we used previously. A Lipschitz type condition on the remainders of the Taylor expansion is imposed (see 4.1.). We define the α -variation of a curve as the supremum of sums of α th powers of chords (see 2.7) and show that the integral of ω along C exists if the α -variation of C is bounded, where α is related to the number of 'higher derivatives' of ω on C . Under somewhat stronger hypotheses on C , we show that this integral is an anti-derivative of ω on C .

2. Notation and basic definitions. Throughout this paper, N is a positive integer and we use the following notation.

2.1. E denotes Euclidean $(N + 1)$ -space.

2.2. $\|x\| = \left(\sum_{i=0}^N x_i^2\right)^{1/2}$ for $x \in E$.

2.3. $\text{diam } U = \sup\{d : d = \|x - y\| \text{ for some } x \in U \text{ and } y \in U\}$

2.4. φ is a continuous function on the closed unit interval to E and $C = \text{range } \varphi$.

2.5. \mathcal{S} is the set of all subdivisions of the unit interval, i.e. functions T on $\{0, 1, \dots, k\}$ for some positive integer k such that:
 $T(0) = 0, \quad T(k) = 1, \quad T(i-1) < T(i)$ for $i = 1, \dots, k$

2.6. $[T/a, b] = \{i : a \leq T(i-1) < T(i) \leq b\}$

2.7. $V_\alpha(a, b) = \sup_{T \in \mathcal{S}} \sum_{i \in [T/a, b]} \|\varphi(T(i-1)) - \varphi(T(i))\|^\alpha$

3. Properties of V_α .

3.1. LEMMA. If $0 \leq a \leq b \leq c \leq 1$, then

$$V_\alpha(a, b) + V_\alpha(b, c) \leq \alpha(a, c) \leq V_\alpha(a, b) + V_\alpha(b, c) + (\text{diam } C)^\alpha$$

3.2. LEMMA. If $\alpha < \beta$ and $V_\alpha(a, b) < \infty$, then $V_\beta(a, b) > \infty$.

Proof. Since $V_\alpha(a, b) < \infty$, there is an integer n such that there are at most n elements $i \in [T/a, b]$ with $\|\varphi(T(i-1)) - \varphi(T(i))\| \geq 1$ for any $T \in \mathcal{S}$. For any other $i \in [T/a, b]$ we have

$$\|\varphi(T(i-1)) - \varphi(T(i))\|^\beta < \|\varphi(T(i-1)) - \varphi(T(i))\|^\alpha.$$

Hence,

$$V_\beta(a, b) < V_\alpha(a, b) + n(\text{diam } C)^\beta < \infty.$$

4. **Integration of 1-forms.** In this section, we first define the kind of differential form we shall be dealing with. Our definition is a variant of Whitney's definition of a function m times differentiable on a closed set [3]. Next, we choose a special sequence of subdivisions and proceed to define the integral of the form over the curve C by taking sums of polynomials of degree m and then passing to the limit. Under conditions involving the generalized variation V_α , we show that the integral exists and possesses, in particular, the properties of linearity and 'anti-derivative'.

Throughout this section, m is a positive integer, $\eta \geq 0$, $K > 0$.

4.1. *The Differential Form.* Let

$$\sigma k = \sum_{i=1}^N k_i \text{ for any } (N+1)\text{-tuple } k.$$

A differential 1-form ω on C is a function on the set of all $(N+1)$ -tuples k , for which k_i is a non-negative integer for $i = 0, \dots, N$ and $1 \leq \sigma k \leq m$, to the set of real-valued functions on C such that

$$\omega_k(y) = \sum_{\sigma j=0}^{m-\sigma k} \omega_{k+j}(x) \frac{(y_0 - x_0)^{j_0} \cdots (y_N - x_N)^{j_N}}{j_0! \cdots j_N!} + R_k(x, y)$$

where

$$|R_k(x, y)| < K \|x - y\|^{m+\eta-\sigma k} \text{ for } x \in C \text{ and } y \in C.$$

It is important to note that, in case $m = 1$ and $\eta > 0$, ω is a differential form on C satisfying a Hölder condition. If however $m > 1$, then ω is also a closed differential form on C , that is, $d\omega = 0$ on C .

By taking $m = 1$ and $\eta = 1$, we get the sharp forms considered by Whitney. The conditions we impose on C , however, are quite different and, we feel, in practice easier to check than those obtained in [4].

4.2. *The sequence of subdivisions.* We define first, for each $(n + 1)$ -tuple of non-negative integers (s_0, \dots, s_n) , a point $t(s_0, \dots, s_n)$ by recursion on n and on s_n . These will be the end-points of the n th subdivision of the unit interval.

4.2.1. DEFINITION. $t(0) = 0, \quad t(1) = 1,$

$$t(s_0, \dots, s_n, 0) = t(s_0, \dots, s_n),$$

$$t(s_0, \dots, s_n, j + 1) = \sup \{u : t(s_0, \dots, s_n, j) \leq u \leq t(s_0, \dots, s_n + 1)\}$$

and $\|\varphi(u') - \varphi(t(s_0, \dots, s_n, j))\| \leq \frac{1}{2^{n+1}}$ for $t(s_0, \dots, s_n, j) \leq u' \leq u$

for any non-negative integers n and j .

We shall denote by T the sequence of subdivisions of the unit interval such that:

$$\text{range } T_n = \{u : u = t(s_0, \dots, s_n) \text{ for some } n\text{-tuple } (s_0, \dots, s_n)\}.$$

4.2.2. LEMMA. *For any non-negative integers n and j , we have*

$$t(s_0, \dots, s_n) \leq t(s_0, \dots, s_n, j) \leq t(s_0, \dots, s_n + 1).$$

4.2.3. LEMMA. *For any positive integer n , $i \in [T_n/0, 1], j \in [T_{n-1}/0, 1]$ we have: T_{n+1} is a refinement of T_n , i.e. $\text{range } T_n \subset \text{range } T_{n+1}$;*

if
$$T_n(i - 1) \leq u \leq T_n(i),$$

then

$$\|\varphi(T_n(i - 1)) - \varphi(u)\| \leq \frac{1}{2^n};$$

if

$$T_{n-1}(j - 1) \leq T_n(i - 1) < T_n(i) < T_{n-1}(j),$$

then

$$\|\varphi(T_n(i - 1)) - \varphi(T_n(i))\| = \frac{1}{2^n}.$$

4.2.4 LEMMA. *If $F(x, y)$ is a real number whenever $0 \leq x \leq y \leq 1$, $a \in \text{range } T_n, b \in \text{range } T_n$, and $a \leq b$, then*

$$\sum_{i \in [T_{n+1}/a, b]} F(T_{n+1}(i - 1), T_{n+1}(i)) = \sum_{j \in [T_n/a, b]} \sum_{i \in [T_{n+1}/T_n(j-1), T_n(j)]} F(T_{n+1}(i - 1), T_{n+1}(i)).$$

4.3. *The integral of ω .* First, we define $\int_b^a \omega d\varphi$ as the limit of certain sums of polynomials.

4.3.1. *Definitions.*

$$P'(x, y) = \sum_{\sigma k=1}^m \omega_k(x) \frac{(y_0 - x_0)^{k_0} \cdots (y_N - x_N)^{k_N}}{k_0! \cdots k_N!},$$

$$P(a, b) = P'(\varphi(a), \varphi(b)),$$

$$S_n(a, b) = \sum_{i \in [T_n/a, b]} P(T_n(i-1), T_n(i)),$$

$$\int_a^b \omega d\varphi = \lim_{n \rightarrow \infty} S_n(a, b).$$

Next, in order to prove the existence of $\int_a^b \omega d\varphi$ and some of its properties under conditions involving $V_a(a, b)$ for some $\alpha < m + \eta$, we introduce the following.

4.3.2. *Definitions.*

$$R(x, y, z) = P'(x, y) + P'(y, z) - P'(x, z).$$

$$M = K \sum_{\sigma k=1}^m \frac{1}{k_0! \cdots k_N!}.$$

$$\beta = m + \eta.$$

4.3.3. LEMMA. *If $x, y, z \in C$, $\|x - y\| \leq \delta$ and $\|y - z\| \leq \delta$, then*

$$|R(x, y, z)| < M\delta^\beta.$$

Proof. Let $h(v) = P'(x, v)$ for $v \in E$. Then, h is a polynomial of degree m . Let $\mathbf{O}_r = \{k : k \text{ is an } (N+1)\text{-tuple of non-negative integers and } 1 \leq \sigma k \leq r\}$.

For $k \in \mathbf{O}_r$ and $p \in \mathbf{O}_r$, let $p \geq k$ iff $p_i \geq k_i$ for $i = 0, \dots, N$, and let

$$D_k h(v) = \frac{\partial^{\sigma k} h(v)}{\partial^{k_0} v_0 \cdots \partial^{k_N} v_N},$$

then

$$D_k h(v) = \sum_{\substack{p \in \mathbf{O}_m \\ p \geq k}} \omega_p(x) \frac{(v_0 - x_0)^{p_0 - k_0} \cdots (v_N - x_N)^{p_N - k_N}}{(p_0 - k_0)! \cdots (p_N - k_N)!}.$$

Hence, by Taylor's formula

$$h(z) = h(y) + \sum_{k \in \mathbf{O}_m} D_k h(y) \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!} = h(y) +$$

$$+ \sum_{\kappa \in \mathcal{O}_m} \left\{ \left[\sum_{\substack{p \in \mathcal{O}_m \\ p \geq \kappa}} \omega_p(x) \frac{(y_0 - x_0)^{p_0 - k_0} \cdots (y_N - x_N)^{p_N - k_N}}{(p_0 - k_0)! \cdots (p_N - k_N)!} \right] \cdot \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!} \right\}.$$

On the other hand from 4.3.1 and 4.1 we have

$$\begin{aligned} P'(y, z) &= \sum_{\kappa \in \mathcal{O}_m} \left\{ \left[\omega_\kappa(x) + \sum_{j \in \mathcal{O}_m - \sigma_\kappa} \omega_{\kappa+j}(x) \frac{(y_0 - x_0)^{j_0} \cdots (y_N - x_N)^{j_N}}{j_0! \cdots j_N!} + R_\kappa(x, y) \right] \right. \\ &\quad \left. \cdot \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!} \right\} \\ &= \sum_{\kappa \in \mathcal{O}_m} \left\{ \left[\sum_{\substack{k \in \mathcal{O}_m \\ p \geq k}} \omega_p(x) \frac{(y_0 - x_0)^{p_0 - k_0} \cdots (y_N - x_N)^{p_N - k_N}}{(p_0 - k_0)! \cdots (p_N - k_N)!} + R_k(x, y) \right] \right. \\ &\quad \left. \cdot \frac{(z_0 - x_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!} \right\} \\ &= h(z) - h(y) + \sum_{\kappa \in \mathcal{O}_m} R_\kappa(x, y) \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!}. \end{aligned}$$

Making use of the condition on $R_\kappa(x, y)$ stated in 4.1, we get

$$|P'(x, y) + P'(y, z) - P'(x, z)| < \sum_{\kappa \in \mathcal{O}_m} \frac{K \|y - x\|^{\beta - \sigma_\kappa} \|z - y\|^{\sigma_\kappa}}{k_0! \cdots k_N!} \leq M\delta^\beta.$$

4.3.4 LEMMA. *Suppose $\|x(0) - x(i)\| \leq A$ and $\|x(i-1) - x(i)\| \leq A$ for $i = 1, \dots, p$, whereas $\|x(i-1) - x(i)\| = A/r$ for $i = 1, \dots, p-1$, where all $x(i) \in C$. Then*

$$\left| \sum_{i=1}^p P'(x(i-1), x(i)) - P'(x(0), x(p)) \right| < Mr^\alpha A^{\beta-\alpha} \sum_{i=1}^p \|x(i-1) - x(i)\|^\alpha.$$

Proof.
$$\begin{aligned} &\left| \sum_{i=1}^p P'(x(i-1), x(i)) - P'(x(0), x(p)) \right| \\ &\leq \sum_{i=2}^p |P'(x(0), x(i-1)) + P'(x(i-1), x(i)) - P'(x(0), x(i))| \\ &= \sum_{i=2}^{p-1} |R(x(0), x(i-1), x(i))| < (p-1)MA^\beta = (p-1)Mr^\alpha A^{\beta-\alpha} \left(\frac{A}{r}\right)^\alpha \\ &= Mr^\alpha A^{\beta-\alpha} \sum_{i=1}^{p-1} \|x(i-1) - x(i)\|^\alpha \leq Mr^\alpha A^{\beta-\alpha} \sum_{i=1}^p \|x(i-1) - x(i)\|^\alpha. \end{aligned}$$

4.3.5 LEMMA. *Let $n > 1$, $a \in \text{range } T_n$, $b \in \text{range } T_n$, $a \leq b$,*

$$\begin{aligned} [T_{n-1}/a, b] &= 0. \quad \text{Then} \\ |S_n(a, b) - P(a, b)| &< M5^\beta V_\beta(a, b). \end{aligned}$$

Proof. Let

$$\begin{aligned} a' &= \sup\{u : u \in \text{range } T_{n-1} \text{ and } u \leq a\} \\ b' &= \sup\{u : u \in \text{range } T_{n-1} \text{ and } u \leq b\}. \end{aligned}$$

First, suppose $a \leq b' \leq b$. Then $a' < a$ and, by 4.2.3

$$\|\varphi(u) - \varphi(a')\| \leq \frac{1}{2^{n-1}} \quad \text{for } a' \leq u \leq b'$$

$$\|\varphi(u) - \varphi(b')\| \leq \frac{1}{2^{n-1}} \quad \text{for } b' \leq u \leq b.$$

Hence

$$\|\varphi(T_n(i)) - \varphi(a)\| \leq \frac{2}{2^{n-1}} \quad \text{for } i \in [T_n/a, b],$$

$$\|\varphi(T_n(i)) - \varphi(b')\| \leq \frac{1}{2^{n-1}} \quad \text{for } i \in [T_n/b', b],$$

$$\|\varphi(T_n(i-1)) - \varphi(T_n(i))\| = \frac{1}{2^n} \quad \text{for } i \in [T_n/a, b], T_n(i) \neq b', T_n(i) \neq b.$$

Replacing α by β in 4.3.4 and using 4.3.3 and 3.1, we see that

$$\begin{aligned} |S_n(a, b) - P(a, b)| &= |S_n(a, b') + S_n(b', b) - P(a, b)| \\ &\leq |S_n(a, b') - P(a, b')| + |S_n(b', b) - P(b', b)| + |P(a, b') + P(b', b) - P(a, b)| \\ &< M4^\beta V_\beta(a, b') + M2^\beta V_\beta(b', b) + MV_\beta(a, b) \leq M5^\beta V_\beta(a, b). \end{aligned}$$

Next suppose $b' < a$. Then, for $i \in [T_n/a, b]$,

$$\|\varphi(T_n(i)) - \varphi(a)\| \leq \frac{2}{2^{n-1}},$$

$$\|\varphi(T_n(i-1)) - \varphi(T_n(i))\| = \frac{1}{2^n}.$$

Hence, by 4.3.4,

$$|S_n(a, b) - P(a, b)| < M4^\beta V_\beta(a, b).$$

4.3.6 LEMMA. Let $a \in \text{range } T_n$, $b \in \text{range } T_n$, $a < b$. Then,

$$|S_{n+1}(a, b) - S_n(a, b)| < M2^\alpha V_\alpha(a, b) \left(\frac{1}{2^{\beta-\alpha}}\right)^n.$$

Proof. Using 4.2.4, 4.2.3 and 4.3.4, we see that

$$\begin{aligned} &|S_{n+1}(a, b) - S_n(a, b)| \\ &= \left| \sum_{j \in [T_n/a, b]} \left[\sum_{i \in [T_{n+1}/T_n(j-1), T_n(j)]} P(T_{n+1}(i-1), T_{n+1}(i)) - P(T_n(j-1), T_n(j)) \right] \right| \\ &< \sum_{j \in [T_n/a, b]} \left[M2^\alpha \left(\frac{1}{2^n}\right)^{\beta-\alpha} \sum_{i \in [T_{n+1}/T_n(j-1), T_n(j)]} \|\varphi(T_{n+1}(i-1)) - \varphi(T_{n+1}(i))\|^\alpha \right] \\ &= M2^\alpha \left(\frac{1}{2^n}\right)^{\beta-\alpha} \sum_{i \in [T_{n+1}/a, b]} \|\varphi(T_{n+1}(i-1)) - \varphi(T_{n+1}(i))\|^\alpha \leq M2^\alpha V_\alpha(a, b) \left(\frac{1}{2^{\beta-\alpha}}\right)^n. \end{aligned}$$

4.3.7. THEOREM. If $0 \leq a \leq b \leq 1$, $\alpha < \beta$, $V_\alpha(a, b) < \infty$, then

$$\left| \int_n^b \omega d\varphi \right| < \infty .$$

Proof. Let

$$\begin{aligned} a'_n &= \inf\{u : u \in \text{range } T_n \text{ and } a \leq u\} , \\ b'_n &= \sup\{u : u \in \text{range } T_n \text{ and } u \leq b\} . \end{aligned}$$

If $a = b$, the theorem is trivial. If $a < b$, for n sufficiently large, we have

$$\begin{aligned} a &\leq a'_{n+1} \leq a'_n \leq b'_n \leq b'_{n+1} \leq b , \\ [T_n/a, a'_n] &= 0 \quad \text{and} \quad [T_n/b'_n, b] = 0 , \\ \|\varphi(a'_{n+1})\varphi - (a'_n)\| &\leq \frac{2}{2^n} \quad \text{and} \quad \|\varphi(b'_n) - \varphi(b'_{n+1})\| \leq \frac{1}{2^n} . \end{aligned}$$

Hence

$$\begin{aligned} |S_{n+1}(a, b) - S_n(a, b)| &= |S_{n+1}(a'_{n+1}, b'_{n+1}) - S_n(a'_n, b'_n)| \\ &= |S_{n+1}(a'_{n+1}, a'_n) + S_{n+1}(a'_n, b'_n) + S_{n+1}(b'_n, b'_{n+1}) - S_n(a'_n, b'_n)| \\ &\leq |S_{n+1}(a'_{n+1}, a'_n) - P(a'_{n+1}, a'_n)| + |S_{n+1}(a'_n, b'_n) - S_n(a'_n, b'_n)| \\ &+ |S_{n+1}(b'_n, b'_{n+1}) - P(b'_n, b'_{n+1})| + |P(a'_{n+1}, a'_n)| + 1P(b'_n, b'_{n+1}) < \text{(by 4.3.5, 4.3.6)} \\ &< M5^\beta V_\beta(a'_{n+1}, a'_n) + M2^x V_\alpha(a'_n, b'_n) \left(\frac{1}{2^{\beta-\alpha}}\right)^n + M5^\beta V_\beta(b'_n, b'_{n+1}) + M' \frac{2}{2^n} + M' \frac{1}{2^n} , \end{aligned}$$

where

$$M' = \sup_{\substack{x \in \mathcal{O} \\ 1 \leq \sigma_k < m}} |\omega_k(x)| \sum_{\sigma_k=1}^m \frac{1}{k_0! \cdots k_N!} .$$

Therefore, for any positive integer p we have

$$\begin{aligned} |S_{n+p}(a, b) - S_n(a, b)| &\leq \sum_{q=0}^{p-1} |S_{n+q+1}(a, b) - S_{n+q}(a, b)| \\ &< M5^\beta \sum_{q=0}^\infty [V_\beta(a'_{n+q+1}, a'_{n+q}) + V_\beta(b'_{n+q}, b'_{n+n+q+1})] + M2^x V_\alpha(a, b) \sum_{q=0}^\infty \left(\frac{1}{2^{\beta-\alpha}}\right)^{n+q} \\ &+ 3M' \sum_{q=0}^\infty \frac{1}{2^{n+q}} < M5^\beta (V_\beta(a, a'_n) + V_\beta(b'_n, b)) + M \frac{2^3}{2^{\beta-\alpha-1}} V_\alpha(a, b) \left(\frac{1}{2^{\beta-\alpha}}\right)^n + \frac{6M'}{2^n} . \end{aligned}$$

Since, by 3.2, $V_\beta(a, b) < \infty$, with the help of 3.1 we see that $V_\beta(a, a'_n) \rightarrow 0$ and $V_\beta(b'_n, b) \rightarrow 0$ as $n \rightarrow \infty$. Thus, the $S_n(a, b)$ form a Cauchy sequence and $\left| \int_a^b \omega d\varphi \right| < \infty$.

4.3.8. THEOREM. *Suppose $\delta > 0$, $\alpha < \beta$, $L < \infty$, $\|\varphi(a) - \varphi(b)\| < 1$, and*

$$V_\alpha(a, b) < L \|\varphi(a) - \varphi(b)\|^\alpha$$

whenever $0 \leq a \leq b \leq 1$ and $b - a < \delta$. Then, for some $M' < \infty$,

$$\left| \int_a^b \omega d\varphi - P(a, b) \right| < M' \|\varphi(a) - \varphi(b)\|^\alpha$$

whenever $0 \leq a \leq b \leq 1$ and $b - a < \delta$.

Proof. Given $0 \leq a \leq b \leq 1$ and $b - a < \delta$, let

$$\begin{aligned} a'_q &= \inf\{u : u \in \text{range } T_q \text{ and } a \leq u\}, \\ b'_q &= \sup\{u : u \in \text{range } T_q \text{ and } u \leq b\}; \end{aligned}$$

and let n be the integer such that $[T_{n-1}/a, b] = 0$, $[T_n/a, b] \neq 0$.

Given $\varepsilon > 0$, we can choose p so that

$$\left| \int_a^b \omega d\varphi - S_{n+p}(a'_{n+p}, b'_{n+p}) \right| < \varepsilon$$

and

$$|P(a, b) - P(a'_{n+p}, b'_{n+p})| < \varepsilon$$

and

$$|\|\varphi(a) - \varphi(b)\| - \|\varphi(a'_{n+p}) - \varphi(b'_{n+p})\|| < \varepsilon.$$

Hence we need only to show that

$$|S_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p})| < M' \|\varphi(a'_{n+p}) - \varphi(b'_{n+p})\|^\alpha$$

for some $M' < \infty$ and all positive integers p .

We can check that

$$\begin{aligned} & |S_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p})| \\ & \leq |S_n(a'_n, b'_n) - P(a'_n, b'_n)| + |P(a'_{n+p}, a'_n) + P(a'_n, b'_n) - P(a'_{n+p}, b'_n)| \\ & + |P(a'_{n+p}, b'_n) + P(b'_n, b'_{n+p}) - P(a'_{n+p}, b'_{n+p})| \\ & + \sum_{k=0}^{p-1} \{|P(a'_{n+p}, a'_{n+k+1}) + P(a'_{n+k+1}, a'_{n+k}) - P(a'_{n+p}, a'_{n+k})| \\ & + |P(b'_{n+k}, b'_{n+k+1}) + P(b'_{n+k+1}, b'_{n+p}) - P(b'_{n+k}, b'_{n+p})| \\ & + |S_{n+k+1}(a'_{n+k+1}, a'_{n+k}) - P(a'_{n+k+1}, a'_{n+k})| \\ & + |S_{n+k+1}(b'_{n+k}, b'_{n+k+1}) - P(b'_{n+k+1}, b'_{n+k})| \\ & + |S'_{n+k+1}(a'_{n+k}, b'_{n+k}) - S_{n+k}(a'_{n+k}, b'_{n+k})|\}. \end{aligned}$$

Now, we observe that

$$\begin{aligned} \|\varphi(u) - \varphi(v)\| &\leq \frac{2}{2^{n+k}} && \text{for } a'_{n+p} \leq u \leq v \leq a'_{n+k}, \\ \|\varphi(u) - \varphi(v)\| &\leq \frac{1}{2^{n+k}} && \text{for } b'_{n+k} \leq u \leq v \leq b'_{n+p}, \end{aligned}$$

$$[T_{n+k}/a'_{n+k+1}, a'_{n+k}] = 0,$$

$$[T_{n+k}/b'_{n+k}, b'_{n+k+1}] = 0.$$

Hence by 4.3.5, 4.3.3, 4.3.6 we have

$$\begin{aligned}
 & |S'_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p})| \\
 & < M5^\beta V_\beta(a'_n, b'_n) + MV_\beta(a'_{n+p}, b'_n) + MV_\beta(a'_{n+p}, b'_{n+p}) \\
 & + M \sum_{k=0}^{p-1} \left\{ V_\alpha(a'_{n+p}, a'_{n+k}) \left(\frac{2}{2^{n+k}}\right)^{\beta-\alpha} + V_\alpha(b'_{n+k}, b'_{n+p}) \left(\frac{1}{2^{n+k}}\right)^{\beta-\alpha} \right. \\
 & \left. + 5^\beta V_\beta(a'_{n+k+1}, a'_{n+k}) + 5^\beta V_\beta(b'_{n+k}, b'_{n+k+1}) + 2^\alpha V_\alpha(a'_{n+k}, b'_{n+k}) \left(\frac{1}{2^{\beta-\alpha}}\right)^{n+k} \right\} \\
 & < M5^\beta V_\beta(a'_{n+p}, b'_{n+p}) + 2MV_\beta(a'_{n+p}, b'_{n+p}) \\
 & + MV_\alpha(a'_{n+p}, b'_{n+p})(2^{\beta-\alpha} + 1 + 2^\alpha) \sum_{k=0}^{\infty} \left(\frac{1}{2^{\beta-\alpha}}\right)^{n+k} \\
 & < MV_\alpha(a'_{n+p}, b'_{n+p}) \left[5^\beta + 2 + (2^{\beta-\alpha} + 1 + 2^\alpha) \sum_{k=0}^{\infty} \left(\frac{1}{2^{\beta-\alpha}}\right)^{n+k} \right] \\
 & < M' \|\varphi(a'_{n+p}) - \varphi(b'_{n+p})\|^\alpha
 \end{aligned}$$

where

$$M' = ML \left[5^\beta + 2 + (2^{\beta-\alpha} + 1 + 2^\alpha) \sum_{k=0}^{\infty} \left(\frac{1}{2^{\beta-\alpha}}\right)^{n+k} \right] < \infty .$$

4.3.9. THEOREM. If $0 \leq a \leq b \leq c \leq 1$, $\left| \int_a^b \omega d\varphi + \int_b^c \omega d\varphi \right| < \infty$, then

$$\int_a^c \omega d\varphi = \int_a^b \omega d\varphi + \int_b^c \omega d\varphi .$$

Proof. Let

$$\begin{aligned}
 a'_n &= \sup\{u : u \in \text{range } T_n \text{ and } u \leq b\} \\
 b'_n &= \inf\{u : u \in \text{range } T_n \text{ and } b \leq u\} .
 \end{aligned}$$

We have $\lim_{n \rightarrow \infty} P(a'_n, b'_n) = 0$ and for sufficiently large n

$$S_n(a, c) = S_n(a, b) + P(a'_n, b'_n) + S_n(b, c) .$$

Taking the limit on both sides we get the desired result.

4.3.10. REMARK. If ω and ω' are both 1-forms in the sense of 4.1, then so is $(\omega + \omega')$ and

$$\int_a^b (\omega + \omega') d\varphi = \int_a^b \omega d\varphi + \int_a^b \omega' d\varphi$$

provided the right hand side is bounded. This is an immediate consequence of the definitions.

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