

Pacific Journal of Mathematics

SUBDIRECT SUMS AND INFINITE ABELIAN GROUPS

ELBERT A. WALKER

SUBDIRECT SUMS AND INFINITE ABELIAN GROUPS

ELBERT A. WALKER

1. **Definitions.** Let G be a group, and suppose G is a subgroup of the direct sum $\sum_{a \in I} \oplus H_a$ of the collection of groups $\{H_a\}_{a \in I}$. If the projection of G into H_a is onto H_a for each $a \in I$, then G is said to be a *subdirect sum* of the groups $\{H_a\}_{a \in I}$. (Only weak direct and subdirect sums are considered here.) If a group G is isomorphic to a subdirect sum of the groups $\{H_a\}_{a \in I}$, then G is said to be *represented* as a subdirect sum of the groups $\{H_a\}_{a \in I}$. A group is called a *rational group* if it is a subgroup of a $Z(p^\infty)$ group or a subgroup of the additive group of rational numbers.

2. **THEOREM.** *Every Abelian group can be represented as a subdirect sum of rational groups where the subdirect sum intersects each of the rational groups non-trivially.*

Proof. G is isomorphic to a subgroup of some divisible group, and thus can be represented as a subdirect sum G' of rational group $\{H_a\}_{a \in I}$. Let $(h_1, h_2, \dots, h_a, \dots)$ be an element of G' . Let $(h_1, h_2, \dots, h_a, \dots)\beta_1 = (k_1, h_2, \dots, h_a, \dots)$, where $k_1 = h_1$ if $G' \cap H_1 \neq 0$, and $k_1 = 0$ if $G' \cap H_1 = 0$. Assume β_c has been defined for $c < b$. Define

$$(h_1, h_2, \dots, h_a, \dots)\beta_b = (k_1, k_2, \dots, k_b, h_{b+1}, \dots)$$

where $k_b = h_b$ if $H_b \cap (\bigcup_{c < b} G'\beta_c) \neq 0$, and $k_b = 0$ otherwise. Each β_a preserves addition because each is a projection. Let $(h_1, h_2, \dots, h_a, \dots) \neq (0, 0, \dots, 0, \dots)$ and let

$$(h_1, h_2, \dots, h_a, \dots)\beta_a = (k_1, k_2, \dots, k_a, h_{a+1}, h_{a+2}, \dots).$$

Only a finite number of the coordinates of $(h_1, h_2, \dots, h_a, \dots)$ are not 0. Let them be $h_{a_1}, h_{a_2}, \dots, h_{a_n}$, where $a_1 < a_2 < \dots < a_n$. If $a < a_n$, then

$$\begin{aligned} & (h_1, h_2, \dots, h_a, \dots)\beta_a \\ &= (k_1, k_2, \dots, k_a, h_{a+1}, \dots, h_{a_n}, h_{a_n+1}, \dots) \neq (0, 0, \dots, 0, \dots) \end{aligned}$$

since $h_{a_n} \neq 0$. Assume $a \geq a_n$. If $n=1$ and $a_1=1$, then $(h_1, h_2, \dots, h_a, \dots) = (h_{a_1}, 0, 0, \dots, 0, \dots) \in G'$ and $G' \cap H_1 \neq 0$ so that $(h_{a_1}, 0, 0, \dots, 0, \dots)\beta_1 = (h_{a_1}, 0, 0, \dots, 0, \dots)$. That is, $k_{a_1} = h_{a_1} \neq 0$, and hence $(h_1, h_2, \dots, h_a, \dots)\beta_a \neq (0, 0, \dots, 0, \dots)$. If $n=1$ and $a_n \neq 1$, then $(0, 0, \dots, h_{a_1}, 0, 0, \dots) \in G'$ and also in $G'\beta_c$ for all $c < a_1$. Thus $H_{a_1} \cap (\bigcup_{c < a_1} G'\beta_c) \neq 0$, and

Received September 19, 1958, in revised form October 20, 1958.

$$\begin{aligned} (h_1, h_2, \dots, h_a, \dots)\beta_a &= (0, 0, \dots, 0, h_{a_1}, 0, 0, \dots)\beta_a \\ &= (0, 0, \dots, 0, h_{a_1}, 0, 0, \dots)\beta_{a_1} = (0, 0, \dots, 0, h_{a_1}, 0, 0, \dots) \\ &\neq (0, 0, \dots, 0, \dots) . \end{aligned}$$

Assume $n > 1$. If $(h_1, h_2, \dots, h_a, \dots)\beta_a = (0, 0, \dots, 0, \dots)$, then $k_c = 0$ for $c \leq a_n$, and

$$(h_1, h_2, \dots, h_a, \dots)\beta_{a_{n-1}} = (0, 0, \dots, 0, h_{a_n}, 0, 0, \dots) .$$

Therefore $H_{a_n} \cap (G'\beta_{a_{n-1}}) \neq 0$, and so $H_{a_n} \cap (\bigcup_{c < a} G'\beta_c) \neq 0$. Hence $k_{a_n} = h_{a_n} \neq 0$, and this contradicts $k_c = 0$ for $c \leq a_n$. Therefore

$$(h_1, h_2, \dots, h_a, \dots)\beta_a \neq (0, 0, \dots, 0, \dots) ,$$

and the kernel of β_a is 0. Hence each β_a is an isomorphism. Now let $(h_1, h_2, \dots, h_a, \dots)\beta = (k_1, k_2, \dots, k_a, \dots)$. Clearly β is a homomorphism of G' into $\sum_{a \in I} \bigoplus H_a$. But the kernel of β is 0 because every element in G' has only a finite number of non-zero coordinates. Let I' be the set of indices such that $a \notin I'$ implies that the image of the projection of $G'\beta$ into H_a is 0. $G'\beta$ is isomorphic to a subdirect sum of the groups $\{H_a\}_{a \in I'}$. If $G'\beta \cap H_1 = 0$, then for $(h_1, h_2, \dots, h_a, \dots) \in G'$ we have $(h_1, h_2, \dots, h_a, \dots)\beta_1 = (0, h_2, \dots, h_a, \dots)$, so that

$$(h_1, h_2, \dots, h_a, \dots)\beta = (0, k_2, k_3, \dots, k_a, \dots) .$$

Hence the image of the projection of $G'\beta$ into H_1 is 0. Therefore $1 \notin I'$. Let $a > 1$. Suppose $G'\beta \cap H_a = 0$ and $H_a \cap (\bigcup_{c < a} G'\beta_c) \neq 0$. Then there exists $b < a$ such that $H_a \cap G'\beta_b \neq 0$. Let $(0, 0, \dots, 0, k_a, 0, 0, \dots) \in H_a \cap G'\beta_b$, where $k_a \neq 0$. Let $(h_1, h_2, \dots, h_a, \dots)\beta_b = (0, 0, \dots, 0, k_a, 0, 0, \dots)$. Then $(h_1, h_2, \dots, h_a, \dots)\beta = (0, 0, \dots, 0, k_a, 0, 0, \dots)$, and so $G'\beta \cap H_a \neq 0$. Therefore if $G'\beta \cap H_a = 0$, then $H_a \cap (\bigcup_{c < a} G'\beta_c) = 0$. This implies for every $(h_1, h_2, \dots, h_a, \dots) \in G'$ that

$$(h_1, h_2, \dots, h_a, \dots)\beta_a = (k_1, k_2, \dots, k_a, h_{a+1}, h_{a+2}, \dots) ,$$

where $k_a = 0$, and hence that

$$(h_1, h_2, \dots, h_a, \dots)\beta = (k_1, k_2, \dots, 0, k_{a+1}, k_{a+2}, \dots) .$$

Thus the image of the projection of $G'\beta$ into H_a is 0 so that $a \notin I'$. Hence for $a \in I'$, $G'\beta \cap H_a \neq 0$. Since G is isomorphic to $G'\beta$, the theorem follows.

3. REMARKS. Theorem 9 in [1] is an immediate corollary of the preceding theorem, as are some other known theorems in Abelian group theory. In [2], Scott proves that every uncountable Abelian group G has, for every possible infinite index α , $2^{o(G)}$ subgroups of order equal to $o(G)$ and of index α , and that for each given infinite index, their intersection is 0. The following theorem shows that if G is torsion free, one can say more.

4. THEOREM. *Every torsion free Abelian group G of infinite rank has, for every possible infinite index α , $2^{o(G)}$ pure subgroups of order equal to $o(G)$ and of index α . Furthermore, the intersection of these pure subgroups of index α is 0.*

Proof. Represent G as a subdirect sum G' of rational groups $\{H_a\}_{a \in I}$ such that for each $a \in I$, $G' \cap H_a \neq 0$. Let α be an infinite cardinal such that $\alpha \leq o(G)$. $o(I) = o(G)$ since G has infinite rank. Let $I = S_1 \cup S_2$ where $o(S_1) = \alpha$, $o(S_2) = o(G)$, and $S_1 \cap S_2 = \emptyset$. Let T be a subset of S_2 such that $o(S_2 - T) = o(G)$. There are $2^{o(G)}$ such T 's. Let $(h_1, h_2, \dots, h_a, \dots)$ be in G' , and let

$$(h_1, h_2, \dots, h_a, \dots)t = \left(\sum_{j \in T} h_j, k_1, k_2, \dots, k_a, \dots \right),$$

where $k_i = h_i$ if $i \in S_1$ and $k_i = 0$ otherwise. The mapping t is a homomorphism and the order of its image is equal to $o(S_1)$. That is, the index of the kernel of t is α . The order of the kernel of t is equal to $o(G)$ since $o(S_2 - T) = o(G)$, and $G' \cap H_a \neq 0$ for all $a \in I$. Let $T, T' \subseteq S_2$, $T \neq T'$. Then there is a $j \in T$ such that $j \notin T'$, say. Let $h_j \in G'$, $h_j \neq 0$. Then

$$(0, 0, \dots, h_j, 0, 0, \dots)t = (h_j, 0, \dots).$$

However, $(0, 0, \dots, h_j, 0, 0, \dots)t' = (0, 0, 0, \dots)$. Hence the kernel of t is not the same as the kernel of t' . These kernels are pure in G' since the quotient groups are torsion free. Thus G has $2^{o(G)}$ pure subgroups of index α , and of order equal to $o(G)$. Suppose $(h_1, h_2, \dots, h_a, \dots)$ is in the intersection of all these pure subgroups of index α . Then if $b \in S_1$, $h_b = 0$. Hence if $h_c \neq 0$, letting $T = \{c\}$, we have

$$(h_1, h_2, \dots, h_c, \dots, h_a, \dots)t = (h_c, 0, 0, \dots) \neq 0,$$

which is impossible. Therefore for each $a \in I$, $h_a = 0$, and this shows that the intersection of these subgroups is 0.

5. REMARKS. Every torsion free divisible group D of rank α is a direct sum of α copies of the additive group of rational numbers, and D contains an isomorphic copy of every torsion free Abelian group of rank α . The following theorem says that if α is infinite, every torsion free Abelian group of rank α is represented in a special way in D .

6. THEOREM. *Every torsion free Abelian group G of infinite rank can be represented as a subdirect sum G' of copies of the additive group of rational numbers, and in such a way that G' intersects each subdirect summand non-trivially.*

Proof. Represent G as a subdirect sum G' of the rational groups

$\{H_a\}_{a \in I}$ such that for each $a \in I$, $G' \cap H_a \neq 0$. Suppose first that G has countably infinite rank. That is, suppose I is the set of positive integers. Each H_a is a subgroup of the additive group of rational numbers, since G is torsion free. Let k_1, k_2, k_3, \dots be a sequence of non-zero rational numbers such that $k_i \in G' \cap H_i$. Let r_1, r_2, r_3, \dots be the non-zero rational numbers arranged in a sequence. Let $s_i = r_i/k_i$. Let $(h_1, h_2, \dots, h_n, \dots)$ be an element of G' . Let

$$(h_1, h_2, \dots, h_n, \dots)\beta = \left(\sum_{i=1}^{\infty} s_i h_i, \sum_{i=2}^{\infty} s_i h_i, \dots, \sum_{i=n}^{\infty} s_i h_i, \dots \right).$$

Since only a finite number of the h_i 's are non-zero, for each k , $\sum_{i=k}^{\infty} s_i h_i$ is a rational number, and for only a finite number of k 's is $\sum_{i=k}^{\infty} s_i h_i$ non-zero.

$$\begin{aligned} & ((h_1, h_2, \dots, h_n, \dots) + (g_1, g_2, \dots, g_n, \dots))\beta \\ &= (h_1 + g_1, h_2 + g_2, \dots, h_n + g_n, \dots)\beta \\ &= \left(\sum_{i=1}^{\infty} s_i (h_i + g_i), \dots, \sum_{i=n}^{\infty} s_i (h_i + g_i), \dots \right) \\ &= \left(\sum_{i=1}^{\infty} s_i h_i + \sum_{i=1}^{\infty} s_i g_i, \dots, \sum_{i=n}^{\infty} s_i h_i + \sum_{i=n}^{\infty} s_i g_i, \dots \right) \\ &= (h_1, h_2, \dots, h_n, \dots)\beta + (g_1, g_2, \dots, g_n, \dots)\beta. \end{aligned}$$

Hence β is a homomorphism of G' into a direct sum of copies of the additive group R of rationals. Let R_n be the set of n th coordinates of elements of $G'\beta$. R_n is a subgroup of R since it is the image of the projection of $G'\beta$ onto its n th coordinates. Let $m \geq n$.

$$(0, 0, \dots, 0, k_m, 0, 0, \dots) \in G'$$

and

$$(0, 0, \dots, 0, k_m, 0, 0, \dots)\beta = (r_m, r_m, \dots, r_m, 0, 0, \dots),$$

so that $r_m \in R_n$. Thus R_n contains all but at most a finite number of elements of R , and being a subgroup of R , must then be R . Therefore $G'\beta$ is a subdirect sum of copies of R . Let $x \in G'$, $x \neq 0$, and let h_r be the last non-zero coordinate of x . Then the r th coordinate of $x\beta$ is $s_r h_r \neq 0$. Hence the kernel of β is 0 and β is an isomorphism of G onto a subdirect sum of copies of R . Now consider the case where I is not countable. Let I be the union of the set of mutually disjoint countably infinite sets $\{I_j\}_{j \in J}$. Denote by S_j the image of the projection of G' into $\sum_{a \in I_j} \oplus H_a$. Then G' is a subdirect sum of the set of groups $\{S_j\}_{j \in J}$, and each S_j is of countably infinite rank. Hence each S_j may be represented as a subdirect sum of copies of the additive group of rational numbers, and it follows that G may be so represented. In light of the proof of 2, this representation may be assumed to intersect each subdirect summand non-trivially.

REFERENCES

1. W. R. Scott, *Groups, and cardinal numbers*, Amer. J. Math. **74** (1952), 187-197.
2. W. R. Scott, *The number of subgroups of given index in nondenumerable Abelian groups*, Proc. Amer. Math. Soc. **5** (1954), 19-22.

UNIVERSITY OF KANSAS AND
NEW MEXICO STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG

Stanford University
Stanford, California

R. A. BEAUMONT

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

L. J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

C. E. BURGESS

E. HEWITT

A. HORN

V. GANAPATHY IYER

R. D. JAMES

M. S. KNEBELMAN

L. NACHBIN

I. NIVEN

T. G. OSTROM

H. L. ROYDEN

M. M. SCHIFFER

E. G. STRAUS

G. SZEKERES

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA

OREGON STATE COLLEGE

UNIVERSITY OF OREGON

OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE COLLEGE

UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

CALIFORNIA RESEARCH CORPORATION

HUGHES AIRCRAFT COMPANY

SPACE TECHNOLOGY LABORATORIES

Printed in Japan by Kokusai Bunken Insatsusha
(International Academic Printing Co., Ltd.), Tokyo, Japan

Pacific Journal of Mathematics

Vol. 9, No. 1

May, 1959

Julius Rubin Blum and Murray Rosenblatt, <i>On the structure of infinitely divisible distributions</i>	1
Robert Geroge Buschman, <i>Asymptotic expressions for</i> $\sum n^a f(n) \log^r n$	9
Eckford Cohen, <i>A class of residue systems (mod r) and related arithmetical functions. I. A generalization of Möbius inversion</i>	13
Paul F. Conrad, <i>Non-abelian ordered groups</i>	25
Richard Henry Crowell, <i>On the van Kampen theorem</i>	43
Irving Leonard Glicksberg, <i>Convolution semigroups of measures</i>	51
Seymour Goldberg, <i>Linear operators and their conjugates</i>	69
Olof Hanner, <i>Mean play of sums of positional games</i>	81
Erhard Heinz, <i>On one-to-one harmonic mappings</i>	101
John Rolfe Isbell, <i>On finite-dimensional uniform spaces</i>	107
Erwin Kreyszig and John Todd, <i>On the radius of univalence of the function</i> $\exp z^2 \int_0^z \exp(-t^2) dt$	123
Roger Conant Lyndon, <i>An interpolation theorem in the predicate calculus</i>	129
Roger Conant Lyndon, <i>Properties preserved under homomorphism</i>	143
Roger Conant Lyndon, <i>Properties preserved in subdirect products</i>	155
Robert Osserman, <i>A lemma on analytic curves</i>	165
R. S. Phillips, <i>On a theorem due to Sz.-Nagy</i>	169
Richard Scott Pierce, <i>A generalization of atomic Boolean algebras</i>	175
J. B. Roberts, <i>Analytic continuation of meromorphic functions in valued fields</i>	183
Walter Rudin, <i>Idempotent measures on Abelian groups</i>	195
M. Schiffer, <i>Fredholm eigen values of multiply-connected domains</i>	211
V. N. Singh, <i>A note on the computation of Alder's polynomials</i>	271
Maurice Sion, <i>On integration of 1-forms</i>	277
Elbert A. Walker, <i>Subdirect sums and infinite Abelian groups</i>	287
John W. Woll, <i>Homogeneous stochastic processes</i>	293