

# Pacific Journal of Mathematics

**ALMOST LOCALLY PURE ABELIAN GROUPS**

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# ALMOST LOCALLY PURE ABELIAN GROUPS

D. L. BOYER AND E. A. WALKER

0. **Introduction.** It is the purpose of this paper to introduce and to give a preliminary investigation of almost locally pure Abelian groups [see definition 1]. For primary groups the concept of almost locally pure Abelian group coincides with that of no elements of infinite height [Theorem 9].

1. **DEFINITION.** A group (= Abelian group),  $G$ , is *almost locally pure* (hereafter abbreviated a.l.p.) if for every finite set of elements  $g_1, \dots, g_n$  of  $G$  there exists a finitely generated pure subgroup,  $P$ , of  $G$  which contains  $g_1, \dots, g_n$ .

2. **EXAMPLES.** Direct sums of cyclic groups are clearly a.l.p. The complete direct sum of copies of the integers is a.l.p. since by [1] every finite subset is contained in a completely decomposable direct summand and each such summand is free of finite rank.

3. **REMARK.** If one defines a group  $G$  to be locally pure if every finite subset generates a pure subgroup, then it is easy to see that  $G$  is a direct sum of cyclic groups of prime order, for various primes.

4. **THEOREM.** *A direct sum of a. l. p. groups is a.l.p.*

*Proof.* Let  $G = \sum_{\alpha} \oplus H_{\alpha}$ , where  $\oplus$  denotes the weak direct sum, and where  $H_{\alpha}$  is a.l.p. for all  $\alpha$ . Let  $g_1, \dots, g_n$  be in  $G$ . Now let  $H_{\beta}$  be a summand in which some  $g_i$  has a non-zero component, and consider the components  $g_{\beta_1}, \dots, g_{\beta_n}$  of  $g_1, \dots, g_n$  in  $H_{\beta}$ . In each such  $H_{\beta}$  (there are only a finite number) there exists a finitely generated pure subgroup  $P_{\beta}$  containing  $g_{\beta_1}, \dots, g_{\beta_n}$ . Then  $\sum_{\beta} \oplus P_{\beta}$  is a finitely generated pure subgroup containing  $g_1, \dots, g_n$ .

5. **THEOREM.** *If  $G$  is a.l.p., if  $K$  is a subgroup of  $G$ , and if for every finite set of elements  $g_1, \dots, g_n$  of  $G$ , there exists a pure subgroup,  $P$ , of  $G$  such that the group generated by  $K$  and  $g_1, \dots, g_n$  is a subgroup of  $P$  and  $P/K$  is finitely generated, then  $G/K$  is a.l.p.*

*If  $G$  and  $G/K$  are a.l.p., where  $K$  is pure in  $G$ , then for every finite set of elements  $g_1, \dots, g_n$  of  $G$ , there exists a pure subgroup,  $P$ , of  $G$  such that the group generated by  $K$  and  $g_1, \dots, g_n$  is a subgroup of  $P$ , and  $P/K$  is finitely generated.*

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*Proof.* For the proof of the first statement, assume there exists such a  $P$  in  $G$  for each finite set of elements of  $G$ . Then if  $g_1 + K, \dots, g_n + K$  are elements of  $G/K$ , there exists a pure subgroup,  $P$ , of  $G$  such that the group generated by  $K$  and  $g_1, \dots, g_n$  is a subgroup of  $P$  and  $P/K$  is finitely generated. Now  $P/K$  is pure in  $G/K$  and  $G/K$  is a.i.p.

If  $G/K$  is a.i.p. and  $g_1, \dots, g_n$  are elements of  $G$ , then there exists a finitely generated pure subgroup,  $P/K$ , of  $G/K$  which contains  $g_1 + K, \dots, g_n + K$ . The inverse image,  $P$ , of  $P/K$  has the desired properties.

6. COROLLARY. *If  $G$  is a.i.p. and  $H$  is a finitely generated subgroup, then  $G/H$  is a.i.p.*

7. COROLLARY. *If  $T$  is the torsion subgroup of an a.i.p. group, then  $G/T$  is a.i.p.*

*Proof.* For  $g_1, \dots, g_n$  in  $G$ , let  $H$  be a finitely generated pure subgroup containing  $g_1, \dots, g_n$ . Then by [2],  $P$ , the subgroup generated by  $H$  and  $T$  is pure. Clearly the subgroup generated by  $T$  and  $g_1, \dots, g_n$  is a subgroup of  $P$  and  $P/T$  is finitely generated. Hence by the theorem  $G/T$  is a.i.p.

8. EXAMPLE. A strong direct sum of a.i.p. groups is not necessarily a.i.p. Consider  $G = \sum_n \odot C(p^n)$ , where  $\odot$  denotes the strong direct sum and  $C(p^n)$  is cyclic of order  $p^n$ . Then if  $G$  were a.i.p.,  $G/T$  would be a.i.p., where  $T$  is the torsion subgroup. But a torsion-free a.i.p. group,  $F$ , is only finitely divisible (i.e. for each  $x \neq 0$  in  $F$  there exists a maximum positive integer,  $m_x$ , such that  $m_x y = x$  has a solution in  $F$ ), whereas the element  $(0, 1/p, 0, 0, 1/p^2, 0, 0, 0, 1/p^3, 0, 0, 0, 0, 1/p^4, \dots) + T$  is not zero and is divisible by all powers of  $p$ .

9. THEOREM. *A torsion group is a.i.p. if and only if its  $p$ -components have no elements of infinite height.*

*Proof.* This follows from the footnote on page 79 of [1] and from 4.

10. LEMMA. *Every subgroup of a torsion-free a.i.p. group is a.i.p.*

*Proof.* Let  $H$  be a subgroup of the torsion-free group,  $G$ , and let  $h_1, \dots, h_n$  be elements of  $H$ . Then there exists a finitely generated pure subgroup,  $P$ , of  $G$  which contains  $h_1, \dots, h_n$ . Since  $P \cap H$  is a finitely generated pure subgroup of  $H$ ,  $H$  is a.i.p.

11. LEMMA. *The torsion subgroup,  $T$ , of an a.i.p. group,  $G$ , is a.i.p.*

*Proof.* The proof is similar to the proof of Lemma 10.

12. THEOREM. *Every subgroup of an a.l.p. group is a.l.p.*

*Proof.* Let  $G$  be a.l.p., let  $S$  be an arbitrary subgroup of  $G$  and let  $T$  be the torsion subgroup of  $G$ . By 7  $G/T$  is a.l.p. and by 10  $(S \cup T)/T$  is a.l.p. Thus  $S/(S \cap T)$  is a.l.p. Now let  $s_1, \dots, s_n$  be elements of  $S$ . Since  $S/(S \cap T)$  is a.l.p. there exists a finitely generated pure subgroup,  $P/(S \cap T)$ , of  $S/(S \cap T)$  such that  $s_1 + (S \cap T), \dots, s_n + (S \cap T)$  are elements of  $P/(S \cap T)$ . Since  $P/(S \cap T)$  is finitely generated and torsion-free,  $P = (S \cap T) \oplus K$ , where  $K$  is finitely generated and torsion-free. Since  $K$  is finitely generated, it is clearly a.l.p., and it follows from 11 and 9 that  $S \cap T$  is a.l.p. Hence by 4,  $P$  is a.l.p. and  $s_1, \dots, s_n$  are elements of  $P$ . Thus there exists a finitely generated pure subgroup,  $P_1$ , of  $P$  containing  $s_1, \dots, s_n$ . Since  $S \cap T$  is a pure subgroup of  $S$ ,  $P$  is a pure subgroup of  $S$ . Hence  $P_1$  is pure in  $S$ .

13. LEMMA. *A countable torsion-free a.l.p. group,  $G$ , is free.*

*Proof.* Let  $H_1 \subset H_2 \subset \dots \subset H_n \subset \dots$  be an ascending chain of subgroups of  $G$ , each having finite rank  $r$ . Let  $h_1, \dots, h_r$  be a maximal linearly independent subset of  $H_1$ , and hence of all the  $H_i$ 's. Since  $G$  is a.l.p., there exists a finitely generated pure subgroup  $P$  containing  $h_1, \dots, h_r$ . Hence each  $H_i$  is contained in  $P$ . Since  $P$  is free of finite rank, it satisfies the ascending chain condition, so that by Theorem *E*, page 168, of [3],  $G$  is free.

14. THEOREM. *If the torsion subgroup  $T$ , of an a.l.p. group,  $G$ , has countable index, then  $T$  is a direct summand (and the complementary summand is free).*

*Proof.* By 7  $G/T$  is a.l.p., countable and torsion-free. Thus by 13  $G/T$  is free. Hence  $G = T \oplus K$ .

15. LEMMA. *A countable a.l.p.  $p$ -group,  $G$ , is a direct of sum cyclic groups.*

*Proof.* By Theorem 9 this is a restatement of a theorem of Prüfer [4].

Now we prove a generalization of Prüfer's theorem.

16. THEOREM. *A countable a.l.p. group,  $G$ , is a direct sum of cyclic groups.*

*Proof.* Let  $T$  be the torsion subgroup of  $G$ . Then by 14,  $G = T \oplus K = T_{p_1} \oplus \cdots \oplus K$ , where  $T_{p_i}$  is the  $p_i$ -component of  $T$ . Since  $G$  is countable it follows from 4 and 15 that  $G$  is a direct sum of cyclic groups.

17. REMARKS. From 12 and 16 it follows that every countable subgroup of an a.l.p. group is a direct sum of cyclic groups.

If one represents the group of rational numbers as a quotient group of a free group, one obtains a pure subgroup (the kernel of the mapping) of an a.l.p. group which is not a direct summand.

From 16 it follows that if  $H$  is a pure subgroup of  $G$  and  $G/H$  is both a.l.p. and countable, then  $H$  is a direct summand of  $G$  and the complementary summand is a direct sum of cyclic groups.

It follows from Corollary 6 that if  $T$  is the torsion subgroup of an a.l.p. group,  $G$ , and if  $H/T$  is finitely generated then  $G/H \cong (G/T)/(H/T)$  is also a.l.p.

18. THEOREM. *If  $H$  is pure in  $G$  and if  $H$  and  $G/H$  are a.l.p., the  $G$  is a.l.p.*

*Proof.* Let  $g_1, \dots, g_n$  be elements of  $G$ . Since  $G/H$  is a.l.p. there exists a finitely generated pure subgroup,  $L/H$ , of  $G/H$  which contains  $g_1 + H, \dots, g_n + H$ . Since  $H$  is pure and  $L/H$  is finitely generated,  $L = H \oplus K$ ,  $K$  finitely generated. Since  $g_i$  is in  $L$  for  $i = 1, \dots, n$ , let  $g_i = h_i + k_i$ . Since  $H$  is a.l.p. let  $P$  be a finitely generated pure subgroup of  $H$  which contains the  $h_i$ . Now  $g_i$  is in  $P \oplus K$  and  $P \oplus K$  is pure in  $L$ , which is pure in  $G$ . Hence  $P \oplus K$  is a finitely generated pure subgroup of  $G$  which contains the  $g_i$ . Hence  $G$  is a.l.p.

19. THEOREM. *Every group,  $G$ , has a maximal pure a.l.p. subgroup,  $M$ , (which may be 0) and 0 is the only pure a. l. p. subgroup of  $G/M$ .*

*Proof.* The existence of  $M$  is easily proved by applying Zorn's lemma. If  $P/M$  were a non-zero pure a.l.p. subgroup of  $G/M$ , then  $P$  would be a pure subgroup of  $G$  and by Theorem 18  $P$  would be a.l.p., contradicting the maximality of  $M$ .

20. COROLLARY. *If  $G$  is a  $p$ -group and  $M$  is a maximal pure a.l.p. subgroup of  $G$ , then  $G/M$  is divisible.*

*Proof.* Otherwise  $G/M = D \oplus R$ , with  $D$  divisible and  $R$  reduced and  $R$  has a finite cyclic direct summand,  $P$ , which is a pure a.l.p. subgroup of  $G/M$ .

21. COROLLARY. *If  $G$  is a  $p$ -group and  $M$  is a countable maximal pure a.l.p. subgroup, then  $M$  is a basic subgroup of  $G$ .*

*Proof.* By Theorem 16  $M$  is a direct sum of cyclic groups and by 20  $G/M$  is divisible. Hence  $M$  is a basic subgroup of  $G$ .

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