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CONNECTIVITY OF TOPOLOGICAL LATTICES

ELDON DYER AND ALLEN LOWELL SHIELDS

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In this note we show that compact connected topological lattices have strong acyclicity properties, both globally and locally. This yields a proof of a conjecture of A. D. Wallace [6] in the finite dimensional case.

A *topological lattice* is a topological space (Hausdorff) upon which is imposed a lattice structure compatible with the topology. More explicitly, the topological space M is a topological lattice if there are maps

$$(1) \quad \mathbf{\Lambda} : M \times M \rightarrow M \text{ and } \mathbf{V} : M \times M \rightarrow M$$

which define a lattice structure on M . This means that for $x, y, z \in M$

$$(2) \quad \mathbf{\Lambda}(x, x) = x \text{ and } \mathbf{V}(x, x) = x ,$$

$$(3) \quad \mathbf{\Lambda}(x, y) = \mathbf{\Lambda}(y, x) \text{ and } \mathbf{V}(x, y) = \mathbf{V}(y, x) ,$$

$$(4) \quad \mathbf{\Lambda}(x, \mathbf{\Lambda}(y, z)) = \mathbf{\Lambda}(\mathbf{\Lambda}(x, y), z) \text{ and} \\ \mathbf{V}(x, \mathbf{V}(y, z)) = \mathbf{V}(\mathbf{V}(x, y), z), \text{ and}$$

$$(5) \quad \mathbf{\Lambda}(x, \mathbf{V}(x, y)) = x \text{ and } \mathbf{V}(x, \mathbf{\Lambda}(x, y)) = x .$$

It is customary to write $x \wedge y$ in place of $\mathbf{\Lambda}(x, y)$ and $x \vee y$ in place of $\mathbf{V}(x, y)$. Relation (5) implies that $x \wedge y = x$ if and only if $x \vee y = y$. We shall say that $x \leq y$ if and only if $x \wedge y = x$. It is easily seen that the relation $x \leq y$ induces a *partial ordering* on M . The element $1 \in M$ is a *unit* in M provided $m \leq 1$ for all $m \in M$. Similarly, an element $0 \in M$ is a *zero* in M if $0 \leq m$ for all $m \in M$. Clearly, if such elements exist they are unique.

We shall need several elementary lemmas on topological lattices. Lemmas 2 and 4 were proved in [1], however for completeness we prove them here. Lemma 1 was proven by A. D. Wallace [7].

LEMMA 1. *If M is a compact topological lattice, then it has a unit and a zero.*

LEMMA 2. *If M is a topological lattice, then*

(a) *if U is a neighborhood of $x \in M$, there is a neighborhood V of x such that if $y, z \in V$, then $y \vee z \in U$ and $y \wedge z \in U$, and*

(b) *if $y \leq x$ and U_x is a neighborhood of x , there are neighbor-*

hoods V_x of x and V_y of y such that if $x' \in V_x$ and $y' \in V_y$, then $x' \vee y' \in U_x$.

To prove (a) we observe that $\bigwedge^{-1}(U) \cap \bigvee^{-1}(U)$ is a neighborhood of $(x, x) \in M \times M$ and that there there is a neighborhood V of x such that $V \times V \subset \bigwedge^{-1}(U) \cap \bigvee^{-1}(U)$. Then if y and z lie in V , (y, z) lies in this intersection, and so $y \vee z \in U$, $y \wedge z \in U$.

In (b) $\bigvee^{-1}(U_x)$ is a neighborhood of $(x, y) \in M \times M$ and there are neighborhoods $V_x \times V_y \subset \bigvee^{-1}(U_x)$. Clearly these neighborhoods have the asserted properties.

LEMMA 3. *If M is a compact topological lattice and U is a neighborhood of $x \in M$, there is a neighborhood W of x such that if $y, z \in W$ and $m \in M$, then $(m \wedge y) \vee z \in U$.*

For $m \in M$ let $V(m)$ and $N(m)$ be neighborhoods of x and $x \wedge m$ as in Lemma 2(b). Let $P(m)$ and $Q(m)$ be neighborhoods of m and x such that $P(m) \times Q(m) \subset \bigwedge^{-1}(N(m))$ and let $R(m) = V(m) \cap Q(m)$. Then if $m' \in P(m)$ and $y, z \in R(m)$, $m' \wedge y \in N(m)$ and $(m' \wedge y) \vee z \in U$. Since M is compact, there is a finite set $[m_i]_1^n$ of points of M such that $\bigcup_1^n P(m_i) = M$. Let $W = \bigcap_1^n R(m_i)$. Then W is the required neighborhood of x .

For $x, y \in M, x \leq y$, we define

$$(6) \quad C_{x,y} = \{z \in M \mid x \leq z \leq y\} .$$

Such sets $C_{x,y}$ will be called *convex sets*. It is clear that if M is a compact topological lattice, then its convex subsets are also compact topological lattices in their natural lattice and topological structures.

LEMMA 4. *If M is a compact topological lattice and U is a neighborhood of x , then there is a neighborhood V of x such that if $y \in V$, then $C_{y,y \vee x} \cup C_{x,y \vee x} \subset U$.*

Let W be a neighborhood of x as in Lemma 3 and V be a neighborhood of x for W as in Lemma 2(a). Then if $y \in V, y \vee x \in W$. If $y \leq z \leq y \vee x$, then $z = (z \wedge \{y \vee x\}) \vee y \in U$ and if $x \leq z \leq y \vee x$, then $z = (z \wedge \{y \vee x\}) \vee x \in U$.

A space X is *acyclic* if $H^*(X) = 0$, where $H^*()$ denotes the reduced cohomology ring; X is *clc* if for each $x \in X$ and closed neighborhood U of x , there is a closed neighborhood V of $x, V \subset U$, such that the homomorphism of reduced groups $H^*(U) \rightarrow H^*(V)$ induced by inclusion is trivial.

Before proceeding to the proof of Theorem 1 we recall a well-known generalization of the fact that homotopic maps induce the same homomorphisms of cohomology.

THEOREM. *If X and Y are compact, N is compact and connected, n_1, n_2 are two points of N , $f, g: X \rightarrow Y$, and $F: X \times N \rightarrow Y$ are given such that $F|X \times n_1 = f$ and $F|X \times n_2 = g$, then $f^* = g^*: H^*(Y) \rightarrow H^*(X)$.*

THEOREM 1. *If the compact topological lattice M is connected, then it is acyclic and clc.*

The fact that M is acyclic was first proven by A. D. Wallace [8]. We give here a slightly different proof.

Let $\hat{M} = (M \times 1) \cup (1 \times M) \subset M \times M$. Since M is connected and $1 \times 1 \in (M \times 1) \cap (1 \times M)$, \hat{M} is connected. For $x, y \in M$ define

$$f_{x,y}: \hat{M} \rightarrow C_{y,y \vee x} \cup C_{x,y \vee x}$$

by

$$f_{x,y}(m, 1) = (m \wedge \{y \vee x\}) \vee y \text{ and } f_{x,y}(1, m) = (m \wedge \{y \vee x\}) \vee x .$$

Note that

$$(1 \wedge \{y \vee x\}) \vee y = y \vee x = (1 \wedge \{y \vee x\}) \vee x .$$

Define $G: M \times \hat{M} \rightarrow M$ by $G(m, \hat{m}) = f_{0,m}(\hat{m})$. Then

$$G(m, (0, 1)) = f_{0,m}(0, 1) = (0 \wedge \{m \vee 0\}) \vee m = m$$

and

$$G(m, (1, 0)) = f_{0,m}(1, 0) = (0 \vee \{m \vee 0\}) \wedge 0 = 0 .$$

If $i: M \rightarrow M$ is the identity and $j: M \rightarrow 0 \xrightarrow{\cong} M$, then $i = G|M \times (0, 1)$ and $j = G|M \times (1, 0)$. Hence $i^* = j^*$. But i^* is the identity isomorphism of $H^*(M)$, and j^* is trivial. Hence, M is acyclic.

For a closed neighborhood U of $x \in M$, let V be a closed neighborhood of x as in Lemma 4. Define $F: V \times \hat{M} \rightarrow U$ by

$$F(v, \hat{m}) = f_{x,v}(\hat{m}) \subset C_{v,v \vee x} \cup C_{x,v \vee x} \subset U .$$

Note that $F|V \times (0, 1)$ is the inclusion map of V into U and that $F|V \times (1, 0)$ is the trivial map of V onto x . It follows as before that the inclusion map induces the trivial cohomology homomorphism, and hence, that M is clc.

In this connection we remark that Lee Anderson [1] has shown that a locally compact connected lattice is locally connected.

An immediate consequence of Theorem 1 and results of E. G. Begle [2] is the following.

COROLLARY 1. *If M is a finite dimensional, compact connected topological lattice, then M has the fixed point property.*

A slightly stronger statement is also true; namely, if M is such a lattice and f is an upper semi-continuous mapping of M into the set of its convex subsets, then some element of M lies in its image.

THEOREM 2. *If the compact metric topological lattice M is connected, then it is contractible and locally contractible.*

Since M is clc° , it is locally connected. Thus, \hat{M} is a compact, connected, locally connected metric space. It follows that there is a mapping $h: I \rightarrow \hat{M}$ such that $h(0) = (0, 1) \in \hat{M}$ and $h(1) = (1, 0) \in \hat{M}$. Here I denotes the unit interval.

Define $H: M \times I \rightarrow M$ by $H(m, t) = G(m, h)$. Then H is the contracting homotopy sought. For $V \subset U$ as in the proof of Theorem 1, define $J: V \times I \rightarrow U$ by $J(v, t) = F(v, h(t))$. Then J is a contraction of V to x within U .

A consequence of this theorem and standard results on absolute neighborhood retracts (see, for example, [5] Propositions 12.2b, 16.4, 19.2) is the following.

COROLLARY 2. *If M is a finite dimensional, compact metric, connected topological lattice, then it is an absolute retract.*

Any convex subset of a compact connected topological lattice has these same properties itself, and is thus acyclic and clc . Furthermore, the intersection of finitely many convex subsets is a convex subset. We shall show that if M satisfies certain additional conditions, it has a neighborhood basis of convex subsets.

A lattice M is said to be *distributive* if for $x, y, z \in M$

$$(7) \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z).$$

A lattice M is said to be of *breadth* b if for each finite set $[x]$ of more than b elements of M , there is a subset $[y_i]$ of b elements such that $x_1 \wedge x_2 \wedge \cdots = y_1 \wedge y_2 \wedge \cdots \wedge y_b$, and b is the least number for which this holds. Similarly, one can define the breadth, b_1 , using joins instead of meets. It is then a simple fact that $b = b_1$ (see [3] p. 20).

THEOREM 3. *If M is a compact distributive topological lattice of finite breadth and U is a neighborhood of a point $x \in M$, then there is a convex set $C_{y,z}$ that is a neighborhood of x and lies in U .*

Let b denote the breadth of M , let W_1 denote a closed neighborhood of x as in Lemma 3 for the neighborhood U , and let W_2, \dots, W_{2b} denote neighborhoods of x such that for $2 \leq i \leq 2b$, if $y, z \in W_i$ then $y \vee z \in W_{i-1}$ and $y \wedge z \in W_{i-1}$. Let R denote the union of the ranges of all lattice polynomials over the domain W_{2b} . By Theorem 12, p. 145 of [3], any such polynomial can be written in the form

$$\bigwedge_{h=1}^r \left[\bigvee_{k=1}^{n(h)} x_{i(h,k)} \right].$$

Since each $\bigvee x_{i(h,k)}$ is the join of not more than b elements of W_{2b} , every such element lies in W_b . Hence, any element in the range of a lattice polynomial over W_{2b} is the meet of not more than b elements of W_b and so lies in W_1 . Thus, $R \subset W_1$. Since R is a sublattice of M , the closure \bar{R} of R is a sublattice of M , and $\bar{R} \subset W_1$. \bar{R} is a compact topological lattice and by Lemma 1 has a unit a and a zero b . By Lemma 3, $C_{b,a} \subset U$. $C_{b,a}$ is a neighborhood of x since $W_{2b} \subset R \subset \bar{R} \subset C_{b,a}$.

In closing we would like to note the following conjectures.

Suppose M is a compact, metric, connected, distributive topological lattice. Then

- (i) M admits sufficiently many lattice homomorphisms onto the unit interval to separate points;
- (ii) M is an absolute retract;
- (iii) $\dim M = \text{breadth of } M$; and
- (iv) if $\dim M = n$, M is homeomorphic to a subset of an n -cell.

D. E. Edmondson has announced [4] an example of a compact, metric connected two dimensional lattice that is modular but not distributive, and that cannot be imbedded in the plane. Lee Anderson has a proof (unpublished) that $\text{breadth } M \leq \dim M$. Therefore in Theorem 3 the hypothesis that M has finite breadth may be replaced by the hypothesis that M is finite dimensional.

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