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EIGENVALUES OF THE UNITARY PART OF A MATRIX

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1. Introduction. It is well known that every matrix A (square and with complex entries) has a polar decomposition $A = P_1 U_1 = U_2 P_2$, where U_i are unitary and P_i are unique positive semi-definite Hermitian matrices. If A is non-singular then $U_1 = U_2 = U$, where U is also unique. In this case we call U the unitary part of A . The eigenvalues of P_1 are the same as those of P_2 .

In [2] the following problem was solved. Given the eigenvalues of P_1 , what is the exact range of variation of the eigenvalues of A ? The answer shows that a knowledge of the eigenvalues of P_1 puts restrictions only on the moduli of the eigenvalues of A . In this paper we are going to consider the corresponding question for the unitary part U of A . It turns out that a knowledge of the eigenvalues of U restricts only the arguments of the eigenvalues of A .

Before stating the result, we need some definitions. An ordered pair of n -tuples $(\lambda_i), (\alpha_i)$ of complex numbers is said to be *realizable* if there exists a non-singular matrix A of order n with eigenvalues λ_i such that the unitary part of A has eigenvalues α_i . If (γ_j) is an n -tuple of complex numbers of modulus 1, and if two of the γ_j are of the form e^{ib}, e^{ic} with $0 < b - c < \pi$ and $0 \leq d \leq (b - c)/2$, then the operation of replacing e^{ib}, e^{ic} by $e^{i(b-d)}, e^{i(c+d)}$ is called a *pinch* of (γ_j) . In other words, a pinch of (γ_j) consists in choosing two of the γ_j which do not lie on the same line through 0 and turning them toward each other through equal angles.

If $(a_i), (b_i)$ are n -tuples of real numbers, and if $(a'_i), (b'_i)$ are their rearrangements in non-decreasing order, then we write $(a_i) < (b_i)$ when $\sum_r^n a'_i \leq \sum_r^n b'_i, r = 2, \dots, n$ and $\sum_1^n a'_i = \sum_1^n b'_i$. It is easily seen that the conditions are equivalent to the conditions $\sum_1^r a'_i \geq \sum_1^r b'_i, r = 1, \dots, n - 1$, and $\sum_1^n a'_i = \sum_1^n b'_i$.

Our main theorem is the following.

THEOREM 1. *Let $(\lambda_i), (\alpha_i)$ be n -tuples of complex numbers such that $\lambda_i \neq 0$ and $|\alpha_i| = 1$. Then the following statements are equivalent:*

- (1) *the pair $(\lambda_i), (\alpha_i)$ is realizable;*
- (2) *(α_i) can be reduced to $(\lambda_i/|\lambda_i|)$ by a finite sequence of pinches;*
- (3) *$\prod_1^n \alpha_i = \prod_1^n (\lambda_i/|\lambda_i|)$, and exactly one of the following hold:*
 - (a) *there is a line through 0 containing all the α_i and $(\lambda_i/|\lambda_i|)$ is a rearrangement of (α_i) ;*
 - (b) *there is no line through 0 containing all α_i but there is*

a closed half plane H with 0 on its boundary containing all α_i , and, if we choose a branch of the argument function which is continuous in $H - \{0\}$, then $(\arg \lambda_i) < (\arg \alpha_i)$;

(c) there is no closed half plane with 0 on its boundary which contains all α_i .

The proof of Theorem 1 will be given at the end of the paper.

2. Definitions and preliminary results. Two matrices A and B are said to be *congruent* if there exists a non-singular matrix X such that $B = X^*AX$. A *triangular* matrix is a matrix such that all entries below the main diagonal are 0. If P is a positive definite matrix, then $P^{1/2}$ denotes the unique positive definite matrix whose square is P . We will use the symbol $\text{diag}(a_1, \dots, a_n)$ to denote the diagonal matrix with diagonal elements a_1, \dots, a_n .

LEMMA 1. *If $\lambda_i \neq 0$ and $|\alpha_i| = 1$, then the pair $(\lambda_i), (\alpha_i)$ is realizable if and only if there exists a matrix A with eigenvalues λ_i which is congruent to $D = \text{diag}(\alpha_1, \dots, \alpha_n)$.*

Proof. We use the fact that for any two matrices B and C , BC and CB have the same eigenvalues. If $(\lambda_i), (\alpha_i)$ is realizable, there exists a unitary matrix U with eigenvalues α_i and a positive definite matrix P such that PU has eigenvalues λ_i . Let V be a unitary matrix such that $U = V^*DV$. Then PU has the same eigenvalues as $P^{1/2}V^*DVP^{1/2}$, which is congruent to D . Conversely, if X^*DX has eigenvalues λ_i , then so does $A = XX^*D$, and D is the unitary part of A since XX^* is positive definite.

LEMMA 2. *If $(\lambda_i), (\alpha_i)$ is realizable and $\rho_i > 0$ for each i , then $(\rho_i\lambda_i), (\alpha_i)$ is realizable.*

Proof. Suppose $D = \text{diag}(\alpha_1, \dots, \alpha_n)$ is congruent to a matrix A with eigenvalues λ_i . Then A is congruent to a triangular matrix B with diagonal elements λ_i . If $X = \text{diag}(\rho_1^{1/2}, \dots, \rho_n^{1/2})$, then X^*BX obviously has eigenvalues $\rho_i\lambda_i$ and is congruent to D .

LEMMA 3. *If $(\lambda_i), (\alpha_i)$ is realizable and z is any complex number of modulus 1, then $(z\lambda_i), (z\alpha_i)$ is realizable.*

LEMMA 4. *If (μ_1, μ_2) results from (λ_1, λ_2) by a pinch and T is a triangular matrix with diagonal elements λ_1, λ_2 , then T is congruent to a matrix with eigenvalues μ_1, μ_2 .*

Proof. By multiplication by a suitable constant, we may suppose

that $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$, and $\mu_1 = e^{i\phi}$, $\mu_2 = e^{-i\phi}$, where $0 \leq \phi \leq \theta < \pi/2$. It suffices to find a positive matrix P such that PT has eigenvalues $e^{\pm i\phi}$. Suppose

$$T = \begin{pmatrix} e^{i\theta} & a \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Let

$$P = \begin{pmatrix} x & \bar{y} \\ y & x \end{pmatrix},$$

where $x \geq 1$, $|y|^2 = x^2 - 1$ and $ya = |a|(x^2 - 1)^{1/2}$. Since P has determinant 1, we need only choose x so that the trace of PT is $2 \cos \phi$. The trace of PT is $f(x) = xe^{i\theta} + xe^{-i\theta} + ya = 2x \cos \theta + |a|(x^2 - 1)^{1/2}$. When $x = 1$, this is $2 \cos \theta$, and for $x \geq 1$, $f(x)$ increases to infinity.

LEMMA 5. *If (α_i) can be reduced to $(\lambda_i/|\lambda_i|)$ by a finite number of pinches, then (λ_i) , (α_i) is realizable.*

Proof. By Lemma 2 we may assume $|\lambda_i| = 1$. We need only prove the following: if (λ_i) , (α) is realizable, if $|\lambda_i| = 1$ and if (μ_i) is a pinch of (λ_i) , then (μ_i) , (α_i) is realizable. We may suppose that the pinch consists in replacing λ_1, λ_2 by μ_1, μ_2 . By hypothesis there exists a triangular matrix A with eigenvalues λ_i which is congruent to $\text{diag}(\alpha_1, \dots, \alpha_n)$. By Lemma 4 there exists a two rowed non-singular matrix Z such that

$$B = Z^* \begin{pmatrix} \lambda_1 & a_{12} \\ 0 & \lambda_2 \end{pmatrix} Z$$

has eigenvalues μ_1, μ_2 . Here a_{12} is the (1, 2) entry of A . If we set

$$Y = \begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity matrix of order $n - 2$, then

$$Y^*AY = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where D is triangular with diagonal elements $\lambda_3, \dots, \lambda_n$. But this last matrix obviously has eigenvalues $(\mu_1, \mu_2, \lambda_3, \dots, \lambda_n) = (\mu_1, \dots, \mu_n)$.

LEMMA 6. *If $(a_1, \dots, a_k) < (b_1, \dots, b_k)$ and $(c_1, \dots, c_p) < (d_1, \dots, d_p)$ then $(a_1, \dots, a_k, c_1, \dots, c_p) < (b_1, \dots, b_k, d_1, \dots, d_p)$.*

Proof. A proof is given in [1; 63].

LEMMA 7. *If A is a matrix such that $(Ax, x) \neq 0$ and $0 < \arg(Ax, x) < \pi$ for all $x \neq 0$, then A is congruent to a unitary matrix.*

Proof. Let $H = (A + A^*)/2$, $K = (A - A^*)/2i$. Then $A = H + iK$, and H, K are Hermitian. Since $(Ax, x) = (Hx, x) + i(Kx, x)$, the hypothesis implies that $(Kx, x) > 0$ for all $x \neq 0$, so that K is positive definite. Therefore by [3; 261] H and K are simultaneously congruent to real diagonal matrices. Hence $A = H + iK$ is congruent to a diagonal unitary matrix.

LEMMA 8. *If A is congruent to a unitary matrix U with eigenvalues α_i , and if $0 < \arg \alpha_1 < \dots < \arg \alpha_n < \pi$, then $(Ax, x) \neq 0$ for all $x \neq 0$ and*

$$\arg \alpha_j = \inf_{\substack{\dim S \\ =j}} \sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) = \sup_{\substack{\dim S \\ =n-j+1}} \inf_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x)$$

where S ranges over subspaces of n -dimensional complex Euclidean space.

Proof. Let (u_i) be an ortho-normal sequence of eigenvectors of U corresponding to (α_i) . If $A = X^*UX$, then $(Ax, x) = \sum_1^n \alpha_i |(Xx, u_i)|^2$. If S is the space spanned by $X^{-1}u_1, \dots, X^{-1}u_j$, then

$$\sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) = \arg \alpha_j.$$

Now let S be any subspace of dimension j . Let M be the space spanned by $X^{-1}u_j, \dots, X^{-1}u_n$. Then there exists a non-zero vector x in $M \cap S$. But

$$\arg(Ax, x) \geq \inf_{y \neq 0} \arg \sum_j^n \alpha_i |(y, u_i)|^2 = \arg \alpha_j.$$

Therefore

$$\sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) \geq \arg \alpha_j.$$

The proof of the second statement is analogous.

Lemma 8 is of course the analogue of the minimax principle for Hermitian matrices. The generalization due to Wielandt [4] also has an analogue for unitary matrices, which we mention without proof since it will not be used.

If A and U satisfy the hypotheses of Lemma 8 and $1 \leq i_1 < \dots < i_k \leq n$, then

$$\arg \alpha_{i_1} + \dots + \arg \alpha_{i_k} = \inf_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \sup_{x_p \in M_p} (\arg \beta_1 + \dots + \arg \beta_k)$$

where (x_1, \dots, x_k) ranges over linearly independent sequences of vectors, and the β_j are the eigenvalues of the matrix of order k whose (i, j) entry is (Ux_i, x_j) . The number $\arg \beta_1 + \dots + \arg \beta_k$ depends only on the subspace generated by x_1, \dots, x_k .

LEMMA 9. *If $(\lambda_i), (\alpha_i)$ is realizable and $0 \leq \arg \alpha_1 \leq \dots \leq \arg \alpha_n \leq \pi$, then $(\arg \lambda_i) < (\arg \alpha_i)$.*

Proof. By Lemma 1, λ_i are the eigenvalues of X^*DX , where X is non-singular and $D = \text{diag}(\alpha_1, \dots, \alpha_n)$. Since the eigenvalues of X^*DX vary continuously with the α_i , we need only prove the theorem for the case where $0 < \arg \alpha_1, \arg \alpha_n < \pi$. We proceed by induction on n . The statement being obvious when $n = 1$, suppose $n > 1$ and the theorem holds for matrices of order $n - 1$. Let A be a triangular matrix with eigenvalues λ_i which is congruent to D . Suppose the λ_i are arranged so that $\arg \lambda_1 \leq \dots \leq \arg \lambda_n$. Let B be the principal minor of A formed from the first $n - 1$ rows and columns of A . If $x = (x_1, \dots, x_{n-1})$ is a vector with $n - 1$ components and $y = (x_1, \dots, x_{n-1}, 0)$ then $(Bx, x) = (Ay, y)$. Therefore for any such $x \neq 0$, $(Ax, x) \neq 0$ and

$$0 < \arg \alpha_1 \leq \arg (Ay, y) = \arg (Bx, x) \leq \arg \alpha_n < \pi,$$

by Lemma 8, since A is congruent to D .

By Lemma 7, B is congruent to a unitary matrix V . Let the eigenvalues of V be β_i , where $\arg \beta_1 \leq \dots \leq \arg \beta_{n-1}$. Since the quadratic form (Bx, x) associated with B is a restriction of the quadratic form associated with A , it follows from Lemma 8 that $\arg \alpha_{j+1} \geq \arg \beta_j \geq \arg \alpha_j$, $j = 1, \dots, n - 1$. Also by the induction hypothesis $(\arg \lambda_1, \dots, \arg \lambda_{n-1}) < (\arg \beta_1, \dots, \arg \beta_{n-1})$. Therefore

$$\arg \lambda_1 + \dots + \arg \lambda_r \geq \arg \beta_1 + \dots + \arg \beta_r \geq \arg \alpha_1 + \dots + \arg \alpha_r, \\ r = 1, \dots, n - 1$$

and

$$\arg \alpha_2 + \dots + \arg \alpha_n \geq \arg \lambda_1 + \dots + \arg \lambda_{n-1} \\ \geq \arg \alpha_1 + \dots + \arg \alpha_{n-1}.$$

Hence

$$-\pi < \arg \lambda_n - \arg \alpha_n \leq \sum_1^n (\arg \lambda_i - \arg \alpha_i) \leq \arg \lambda_n - \arg \alpha_1 < \pi.$$

But

$$\prod_1^n \lambda_i = |\det X|^2 \cdot \prod_1^n \alpha_i.$$

Therefore

$$\sum_1^n \arg \lambda_i = \sum_1^n \arg \alpha_i .$$

The proof is complete.

LEMMA 10. *If (β_i) , (α_i) are n -tuples of complex numbers of modulus 1 which lie on a line through 0, and if (β) , (α_i) is realizable, then (β_i) must be a rearrangement of (α_i) .*

Proof. By Lemma 3 we may suppose that the α_i and β_i are all real. Let A be a matrix with eigenvalues β_i which is congruent to $\text{diag}(\alpha_1, \dots, \alpha_n)$. Then A is Hermitian and therefore A is also congruent to $\text{diag}(\beta_1, \dots, \beta_n)$. But by Lemma 1 it follows that (α_i) , (β_i) is realizable. Therefore by Lemma 9 we have $(\arg \beta_i) < (\arg \alpha_i) < (\arg \beta_i)$, from which the present theorem follows immediately.

LEMMA 11. *Suppose (β_i) , (α_i) are n -tuples of complex numbers of modulus 1 such that $\prod_1^n \beta_i = \prod_1^n \alpha_i$. Then there exist determinations of $\arg \alpha_i$, $\arg \beta_i$ such that*

$$\max \arg \alpha_i - \min \arg \alpha_i \leq 2\pi$$

and

$$(\arg \beta_i) < (\arg \alpha_i) .$$

Proof. The statement is obvious for $n = 1$. Suppose $n > 1$ and it holds for $n-1$ -tuples. If any of the β_i is equal to any of the α_i , say $\beta_1 = \alpha_1$, then by the induction hypothesis, we can find determinations of the remaining $\arg \alpha_i$, $\arg \beta_i$ as stated. If we now choose a value of $\arg \alpha_1$ which lies between μ and $\mu + 2\pi$, where $\mu = \min_{i>1} \arg \alpha_i$, and set $\arg \beta_1 = \arg \alpha_1$, then the conditions of our theorem will be satisfied, by Lemma 6. So henceforth we may assume that $\beta_i \neq \alpha_j$ for all i, j .

As another special case, suppose the α_i are all equal, say to 1. If we assign arguments to the β_i such that $0 < \arg \beta_i < 2\pi$, then $\sum_1^n \arg \beta_i = 2\pi k$, where k is some positive integer $< n$. We need only assign arguments to the α_i such that exactly k of them have argument 2π and the remaining ones have argument 0.

Now assume the previous two cases do not occur. The α_i divide the unit circle into arcs. At least one of them must contain more than one of the β_i , for if not the α_i would be all distinct and each of the n arcs determined by them would contain exactly one of the β_i . We could then assign arguments to arrangements of the α_i , β_i so that

$$\arg \alpha_1 < \arg \beta_1 < \arg \alpha_2 < \dots < \arg \alpha_n < \arg \beta_n < \arg \alpha_1 + 2\pi .$$

But then $0 < \sum_1^n \arg \beta_i - \sum_1^n \arg \alpha_i < 2\pi$, contradicting the hypothesis $\prod_1^n \alpha_i = \prod_1^n \beta_i$.

Let C be an arc containing more than one of the β_i . By changing subscripts, we may assume that the endpoints of C when described counterclockwise are α_1 and α_2 . Let β_1 be one of the β_i in C which is nearest to α_1 and β_2 be one of the β_i with subscript $\neq 1$ which is nearest to α_2 . Note that β_1 may equal β_2 , but $\alpha_1 \neq \alpha_2$. As will be seen from the following argument, we may assume the subarc $\alpha_1\beta_1$ of $C \leq$ the subarc $\beta_2\alpha_2$ of C , (all arcs are described counterclockwise). Let $\beta'_1 = \alpha_1$ and let β'_2 be the point in $\beta_2\alpha_2$ such that $\beta_2\beta'_2 = \alpha_1\beta_1 = \delta$. By the first case of the proof, we may assign arguments to $\beta'_1, \beta'_2, \beta_3, \dots, \beta_n$ and $\alpha_1, \dots, \alpha_n$ so that

$$(1) \quad \max \arg \alpha_i - \min \arg \alpha_i \leq 2\pi$$

and

$$(2) \quad (\arg \beta'_1, \arg \beta'_2, \arg \beta_3, \dots, \arg \beta_n) < (\arg \alpha_1, \dots, \arg \alpha_n).$$

If $\arg \alpha_1$ happens to be the largest of $\arg \alpha_i$, and therefore $\arg \alpha_2$ is the smallest of $\arg \alpha_i$, then none of $\beta'_1, \beta'_2, \beta_3, \dots, \beta_n$ can lie in the interior of C . Therefore $\beta'_2 = \alpha_2$, and if we decrease $\arg \alpha_1$ and $\arg \beta_1$ by 2π , then (1) and (2) will still hold. Thus we may assume $\arg \alpha_1 < \arg \alpha_2$, and therefore $\arg \beta'_1 < \arg \beta'_2$. Now assign to β_1 the argument $\beta'_1 + \delta$ and to β_2 the argument $\arg \beta'_2 - \delta$. Since

$$(\arg \beta'_1 + \delta, \arg \beta'_2 - \delta) < (\arg \beta'_1, \arg \beta'_2),$$

we have by Lemma 6,

$$\begin{aligned} (\arg \beta_1, \dots, \arg \beta_n) &< (\arg \beta'_1, \arg \beta'_2, \arg \beta_3, \dots, \arg \beta_n) \\ &< (\arg \alpha_1, \dots, \arg \alpha_n). \end{aligned}$$

This completes the proof.

LEMMA 12. *If $(\beta_i), (\alpha_i)$ are n -tuples of complex numbers of modulus 1 which can be assigned arguments such that*

$$\begin{aligned} \arg \alpha_1 &\leq \dots \leq \arg \alpha_n \leq \arg \alpha_1 + 2\pi, \\ \arg \beta_1 &\leq \dots \leq \arg \beta_n, \\ (\arg \beta_i) &< (\arg \alpha_i), \end{aligned}$$

and

$$\arg \alpha_{i+1} - \arg \alpha_i < \pi, \quad i = 1, \dots, n - 1,$$

then a finite number of pinches will reduce (α_i) to (β_i) .

Proof. We proceed by induction on n . When $n = 2$, we have $\arg \alpha_1 \leq \arg \beta_1 \leq \arg \beta_2 \leq \arg \alpha_2$, $\arg \alpha_1 + \arg \alpha_2 = \arg \beta_1 + \arg \beta_2$ and $\arg \alpha_2 - \arg \alpha_1 < \pi$. Therefore $\arg \beta_1 - \arg \alpha_1 = \arg \alpha_2 - \arg \beta_2$ and so

(β_1, β_2) is a pinch of (α_1, α_2) .

Suppose $n > 2$ and the theorem holds for all m -tuples, $m < n$. Let

$$\delta = \min_{1 \leq p \leq n-1} \sum_1^p (\arg \beta_i - \arg \alpha_i) .$$

There exists k such that $\sum_1^k \arg \beta_i - \sum_1^k \arg \alpha_i = \delta$. It is easy to verify that

$$(\arg \beta_1, \dots, \arg \beta_k) < (\arg \alpha_1 + \delta, \arg \alpha_2, \dots, \arg \alpha_k)$$

and

$$(\arg \beta_{k+1}, \dots, \arg \beta_n) < (\arg \alpha_{k+1}, \dots, \arg \alpha_{n-1}, \arg \alpha_n - \delta) .$$

Also

$$\arg \alpha_1 + \delta \leq \arg \beta_1 \leq \arg \beta_n \leq \arg \alpha_n - \delta .$$

By the induction hypothesis, we can reduce $(\alpha_1 e^{i\delta}, \alpha_2, \dots, \alpha_k)$ to $(\beta_1, \dots, \beta_k)$ and $(\alpha_{k+1}, \dots, \alpha_{n-1}, \alpha_n e^{-i\delta})$ to $(\beta_{k+1}, \dots, \beta_n)$ by a finite number of pinches. We need only show that $(\alpha_1, \dots, \alpha_n)$ can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \dots, \alpha_{n-1}, \alpha_n e^{-i\delta})$ by a finite number of pinches. This will follow from the next lemma if we consider only the distinct α_i .

If the α_i all coincide, then so do the β_i and the statement of our theorem is trivial.

LEMMA 13. *If (α_i) is an m -tuple of numbers of modulus 1 with assigned arguments such that*

$$\arg \alpha_1 < \dots < \arg \alpha_m \leq \arg \alpha_1 + 2\pi$$

and

$$\arg \alpha_{i+1} - \arg \alpha_i < \pi, \quad i = 1, \dots, m-1,$$

and if δ is a positive number such that $\arg \alpha_1 + \delta \leq \arg \alpha_m - \delta$, then (α_i) can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \dots, \alpha_{m-1}, \alpha_m e^{-i\delta})$ by a finite number of pinches.

Proof. This is obvious for $m = 2$. Assume $m > 2$ and the lemma holds for $m-1$ -tuples. If

$$\eta = \min(\arg \alpha_2 - \arg \alpha_1, \pi - (\arg \alpha_3 - \arg \alpha_2), \dots, \pi - (\arg \alpha_m - \arg \alpha_{m-1})) ,$$

and $0 < \varepsilon < \eta$, then each sequence in the following list is a pinch of the preceding sequence:

$$\alpha_1, \dots, \alpha_m$$

$$\begin{aligned} &\alpha_1 e^{i\varepsilon}, \alpha_2 e^{-i\varepsilon}, \alpha_3, \dots, \alpha_m \\ &\alpha_1 e^{i\varepsilon}, \alpha_2, \alpha_3 e^{-i\varepsilon}, \dots, \alpha_m \\ &\quad \cdot \quad \cdot \quad \cdot \\ &\alpha_1 e^{i\varepsilon}, \alpha_2, \dots, \alpha_{m-2}, \alpha_{m-1} e^{-i\varepsilon}, \alpha_m \\ &\alpha_1 e^{i\varepsilon}, \alpha_2, \dots, \alpha_{m-1}, \alpha_m e^{-i\varepsilon}. \end{aligned}$$

Note that $\arg \alpha_1 + \varepsilon$ need not be $\leq \arg \alpha_2 - \varepsilon$, and $\arg \alpha_2$ need not be $\leq \arg \alpha_3 - \varepsilon$, etc.

We may repeat this cycle of m pinches $k - 1$ more times to pass from

$$\alpha_1 e^{i\varepsilon}, \alpha_2, \dots, \alpha_{m-1}, \alpha_m e^{-i\varepsilon} \text{ to } \alpha_1 e^{ki\varepsilon}, \alpha_2, \dots, \alpha_{m-1}, \alpha_m e^{-ki\varepsilon}$$

as long as $\arg \alpha_1 + k\varepsilon \leq \arg \alpha_2$, since

$$\arg \alpha_2 + p\varepsilon - \arg \alpha_1 > \arg \alpha_2 - \arg \alpha_1$$

and

$$\pi - (\arg \alpha_n - p\varepsilon - \arg \alpha_{m-1}) > \pi - (\arg \alpha_n - \arg \alpha_{m-1})$$

for $p < k$. Therefore if $\delta \leq \arg \alpha_2 - \arg \alpha_1$, we need only choose $\varepsilon = \delta/k$, where k is an integer so large that $\delta/k < \eta$. If $\delta > \arg \alpha_2 - \arg \alpha_1$, choose $\varepsilon = (\arg \alpha_2 - \arg \alpha_1)/k$, where k is so large that $\varepsilon < \eta$. Then $(\alpha_1, \dots, \alpha_m)$ is reduced to $(\alpha_2, \alpha_2, \dots, \alpha_{m-1}, \alpha_m e^{-ik\varepsilon})$ by the above sequence of pinches. By the induction hypothesis, $(\alpha_2, \alpha_3, \dots, \alpha_{m-1}, \alpha_m e^{-ik\varepsilon})$ can by a finite number of pinches be reduced to $(\alpha_1 e^{i\delta}, \alpha_3, \dots, \alpha_{m-1}, \alpha_m e^{-i\delta})$. (The fact that $\alpha_m e^{-ik\varepsilon}$ might be equal to one of the α_j is clearly unimportant.) Therefore $(\alpha_1, \dots, \alpha_m)$ can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \dots, \alpha_{m-1}, \alpha_m e^{-i\delta})$, and the proof is complete.

3. Proof of Theorem 1.

(2) \rightarrow (1): This is the statement of Lemma 5.

(1) \rightarrow (3): If $(\lambda_i), (\alpha_i)$ is realizable, then by Lemma 1 there exists a matrix A and a non-singular matrix X such that $A = X^* \text{diag} (\alpha_1, \dots, \alpha_n)$. X and A has eigenvalues λ_i . Therefore $\prod \lambda_i = \prod \alpha_i \cdot |\det X|^2$ and hence $\prod |\lambda_i| / |\lambda_i| = \prod \alpha_i$. If the α_i lie on a line through 0, then $(|\lambda_i| / |\lambda_i|)$ is a rearrangement of (α_i) by Lemmas 2 and 10. If the α_i lie in a closed half plane through 0, then by Lemma 3 we may assume they lie in the upper half plane. By Lemma 9 it follows that $(\arg \lambda_i) < (\arg \alpha_i)$.

(3) \rightarrow (2): In case (a), the statement is obvious. In case (c), Lemma 11 and the fact that the α_i do not lie in any closed half plane with 0 on its boundary show that the hypotheses of Lemma 12 are satisfied by arrangements of $(|\lambda_i| / |\lambda_i|), (\alpha_i)$. In case (b), the hypotheses of

Lemma 12 also are satisfied by arrangements of $(\lambda_i / |\lambda_i|), (\alpha_i)$. Thus an application of Lemma 12 completes the proof.

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