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EIGENVALUES OF THE UNITARY PART OF A MATRIX

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1. Introduction. It is well known that every matrix A (square and with complex entries) has a polar decomposition $A = P_1U_1 = U_2P_2$, where U_i are unitary and P_i are unique positive semi-definite Hermitian matrices. If A is non-singular then $U_1 = U_2 = U$, where U is also unique. In this case we call U the unitary part of A. The eigenvalues of P_1 are the same as those of P_2.

In [2] the following problem was solved. Given the eigenvalues of P_1, what is the exact range of variation of the eigenvalues of A ? The answer shows that a knowledge of the eigenvalues of P_1 puts restrictions only on the moduli of the eigenvalues of A. In this paper we are going to consider the corresponding question for the unitay part U of A. In turns out that a knowledge of the eigenvalues of U restricts only the arguments of the eigenvalues of A.

Before stating the result, we need some definitions. An ordered pair of n-tuples (λ_i), (α_i) of complex numbers is said to be *realizable* if there exists a non-singular matrix A of order n with eigenvalues λ_i such that the unitary part of A has eigenvalues α_i. If (γ_j) is an n-tuple of complex numbers of modulus 1, and if two of the γ_j are of the form e^{ib}, e^{ic} with $0 < b - c < \pi$ and $0 \leqq d \leqq (b-c)/2$, then the operation of replacing e^{ib}, e^{ic} by $e^{i(b-d)}$, $e^{i(c+d)}$ is called a *pinch* of (γ_j). In other words, a pinch of (γ_j) consists in choosing two of the γ_j which do not lie on the same line through 0 and turning them toward each other through equal angles.

If (a_i), (b_i) are n-tuples of real numbers, and if (a'_i), (b'_i) are their rearrangements in non-decreasing order, then we write $(a_i) \prec (b_i)$ when $\sum_r^n a'_i \leqq \sum_r^n b'_i$, $r = 2, \cdots, n$ and $\sum_1^n a'_i = \sum_1^n b'_i$. It is easily seen that the conditions are equivalent to the conditions $\sum_1^r a'_i \geqq \sum_1^r b'_i$, $r = 1, \cdots, n-1$, and $\sum_1^n a'_i = \sum_1^n b'_i$.

Our main theorem is the following.

THEOREM 1. *Let (λ_i), (α_i) be n-tuples of complex numbers such that $\lambda_i \neq 0$ and $|\alpha_i| = 1$. Then the following statements are equivalent:*

(1) *the pair (λ_i), (α_i) is realizable;*

(2) *(α_i) can be reduced to $(\lambda_i/|\lambda_i|)$ by a finite sequence of pinches;*

(3) *$\prod_1^n \alpha_i = \prod_1^n (\lambda_i/|\lambda_i|)$, and exactly one of the following hold:*

(a) *there is a line through 0 containing all the α_i and $(\lambda_i/|\lambda_i|)$ is a rearrangement of (α_i);*

(b) *there is no line through 0 containing all α_i but there is*

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a closed half plane H with 0 on its boundary containing all α_i, and, if we choose a branch of the argument function which is continuous in $H - \{0\}$, then $(\arg \lambda_i) \prec (\arg \alpha_i)$;

(c) *there is no closed half plane with* 0 *on its boundary which contains all α_i.*

The proof of Theorem 1 will be given at the end of the paper.

2. Definitions and preliminary results. Two matrices A and B are said to be *congruent* if there exists a non-singular matrix X such that $B = X^*AX$. A *triangular* matrix is a matrix such that all entries below the main diagonal are 0. If P is a positive definite matrix, then $P^{1/2}$ denotes the unique positive definite matrix whose square is P. We will use the symbol diag $(a_1, \cdots, a_n)$ to denote the diagonal matrix with diagonal elements $a_1, \cdots, a_n$.

LEMMA 1. *If $\lambda_i \neq 0$ and $|\alpha_i| = 1$, then the pair (λ_i), (α_i) is realizable if and only if there exists a matrix A with eigenvalues λ_i which is congruent to* $D = \operatorname{diag}(\alpha_1, \cdots, \alpha_n)$.

Proof. We use the fact that for any two matrices B and C, BC and CB have the same eigenvalues. If (λ_i), (α_i) is realizable, there exists a unitary matrix U with eigenvalues α_i and a positive definite matrix P such that PU has eigenvalues λ_i. Let V be a unitary matrix such that $U = V^*DV$. Then PU has the same eigenvalues as $P^{1/2}V^*DVP^{1/2}$, which is congruent to D. Conversely, if X^*DX has eigenvalues λ_i, then so does $A = XX^*D$, and D is the unitary part of A since XX^* is positive definite.

LEMMA 2. *If (λ_i), (α_i) is realizable and $\rho_i > 0$ for each i, then $(\rho_i\lambda_i)$, (α_i) is realizable.*

Proof. Suppose $D = \operatorname{diag}(\alpha_1, \cdots, \alpha_n)$ is congruent to a matrix A with eigenvalues λ_i. Then A is congruent to a triangular matrix B with diagonal elements λ_i. If $X = \operatorname{diag}(\rho_1^{1/2}, \cdots, \rho_n^{1/2})$, then X^*BX obviously has eigenvalues $\rho_i\lambda_i$ and is congruent to D.

LEMMA 3. *If (λ_i), (α_i) is realizable and z is any complex number of modulus* 1, *then $(z\lambda_i)$, $(z\alpha)$ is realizable.*

LEMMA 4. *If (μ_1, μ_2) results from (λ_1, λ_2) by a pinch and T is a triangular matrix with diagonals elements λ_1, λ_2, then T is congruent to a matrix with eigenvalues μ_1, μ_2.*

Proof. By multiplication by a suitable constant, we may suppose that $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$, and $\mu_1 = e^{i\phi}$, $\mu_2 = e^{-i\phi}$, where $0 \leqq \phi \leqq \theta < \pi/2$. It suffices to find a positive matrix P such that PT has eigenvalues $e^{\pm i\phi}$. Suppose

$$T = \begin{pmatrix} e^{i\theta} & a \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Let

$$P = \begin{pmatrix} x & \bar{y} \\ y & x \end{pmatrix},$$

where $x \geqq 1$, $|y|^2 = x^2 - 1$ and $ya = |a|(x^2 - 1)^{1/2}$. Since P has determinant 1, we need only choose x so that the trace of PT is $2\cos\phi$. The trace of PT is $f(x) = xe^{i\theta} + xe^{-i\theta} + ya = 2x\cos\theta + |a|(x^2 - 1)^{1/2}$. When $x = 1$, this is $2\cos\theta$, and for $x \geqq 1$, $f(x)$ increases to infinity.

LEMMA 5. *If (α_i) can be reduced to $(\lambda_i/|\lambda_i|)$ by a finite number of pinches, then (λ_i), (α_i) is realizable.*

Proof. By Lemma 2 we may assume $|\lambda_i| = 1$. We need only prove the following: if (λ_i), (α) is realizable, if $|\lambda_i| = 1$ and if (μ_i) is a pinch of (λ_i), then (μ_i), (α_i) is realizable. We may suppose that the pinch consists in replacing λ_1, λ_2 by μ_1, μ_2. By hypothesis there exists a triangular matrix A with eigenvalues λ_i which is congruent to diag $(\alpha_1, \cdots, \alpha_n)$. By Lemma 4 there exists a two rowed non-singular matrix Z such that

$$B = Z^*\begin{pmatrix} \lambda_1 & a_{12} \\ 0 & \lambda_2 \end{pmatrix}Z$$

has eigenvalues μ_1, μ_2. Here a_{12} is the (1, 2) entry of A. If we set

$$Y = \begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity matrix of order $n - 2$, then

$$Y^*AY = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where D is triangular with diagonal elements $\lambda_3, \cdots, \lambda_n$. But this last matrix obviously has eigenvalues $(\mu_1, \mu_2, \lambda_3, \cdots, \lambda_n) = (\mu_1, \cdots, \mu_n)$.

LEMMA 6. *If $(a_1, \cdots, a_k) \prec (b_1, \cdots, b_k)$ and $(c_1, \cdots, c_p) \prec (d_1, \cdots, d_p)$ then $(a_1, \cdots, a_k, c_1, \cdots, c_p) \prec (b_1, \cdots, b_k, d_1, \cdots, d_p)$.*

Proof. A proof is given in [1; 63].

LEMMA 7. *If A is a matrix such that $(Ax, x) \neq 0$ and $0 < \arg(Ax, x) < \pi$ for all $x \neq 0$, then A is congruent to a unitary matrix.*

Proof. Let $H = (A + A^*)/2$, $K = (A - A^*)/2i$. Then $A = H + iK$, and H, K are Hermitian. Since $(Ax, x) = (Hx, x) + i(Kx, x)$, the hypothesis implies that $(Kx, x) > 0$ for all $x \neq 0$, so that K is positive definite. Therefore by [3; 261] H and K are simultaneously congruent to real diagonal matrices. Hence $A = H + iK$ is congruent to a diagonal unitary matrix.

LEMMA 8. *If A is congruent to a unitary matrix U with eigenvalues α_i, and if $0 < \arg \alpha_1 < \cdots < \arg \alpha_n < \pi$, then $(Ax, x) \neq 0$ for all $x \neq 0$ and*

$$\arg \alpha_j = \inf_{\substack{\dim S \\ = j}} \sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) = \sup_{\substack{\dim S \\ = n-j+1}} \inf_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x)$$

where S ranges over subspaces of n-dimensional complex Euclidean space.

Proof. Let (u_i) be an ortho-normal sequence of eigenvectors of U corresponding to (α_i). If $A = X^*UX$, then $(Ax, x) = \sum_1^n \alpha_i |(Xx, u_i)|^2$. If S is the space spanned by $X^{-1}u_1, \cdots, X^{-1}u_j$, then

$$\sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) = \arg \alpha_j.$$

Now let S be any subspace of dimension j. Let M be the space spanned by $X^{-1}u_j, \cdots, X^{-1}u_n$. Then there exists a non-zero vector x in $M \cap S$. But

$$\arg(Ax, x) \geqq \inf_{y \neq 0} \arg \sum_j^n \alpha_i |(y, u_i)|^2 = \arg \alpha_j .$$

Therefore

$$\sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) \geqq \arg \alpha_j .$$

The proof of the second statement is analogous.

Lemma 8 is of course the analogue of the minimax principle for Hermitian matrices. The generalization due to Wielandt [4] also has an analogue for unitary matrices, which we mention without proof since it will not be used.

If A and U satisfy the hypotheses of Lemma 8 and $1 \leqq i_1 < \cdots < i_k \leqq n$, then

$$\arg \alpha_{i_1} + \cdots + \arg \alpha_{i_k} = \inf_{\substack{M_1 \subset \cdots \subset M_k \\ \dim M_p = i_p}} \sup_{x_p \in M_p} (\arg \beta_1 + \cdots + \arg \beta_k)$$

where $(x_1, \cdots, x_k)$ *ranges over linearly independent sequences of vectors, and the* β_j *are the eigenvalues of the matrix of order* k *whose* (i, j) *entry is* (Ux_i, x_j). *The number* $\arg \beta_1 + \cdots + \arg \beta_k$ *depends only on the subspace generated by* $x_1, \cdots, x_k$.

LEMMA 9. *If* $(\lambda_i), (\alpha_i)$ *is realizable and* $0 \leqq \arg \alpha_1 \leqq \cdots \leqq \arg \alpha_n \leqq \pi$, *then* $(\arg \lambda_i) \prec (\arg \alpha_i)$.

Proof. By Lemma 1, λ_i are the eigenvalues of X^*DX, where X is non-singular and $D = \operatorname{diag}(\alpha_1, \cdots, \alpha_n)$. Since the eigenvalues of X^*DX vary continuously with the α_i, we need only prove the theorem for the case where $0 < \arg \alpha_1$, $\arg \alpha_n < \pi$. We proceed by induction on n. The statement being obvious when $n = 1$, suppose $n > 1$ and the theorem holds for matrices of order $n - 1$. Let A be a triangular matrix with eigenvalues λ_i which is congruent to D. Suppose the λ_i are arranged so that $\arg \lambda_1 \leqq \cdots \leqq \arg \lambda_n$. Let B be the principal minor of A formed from the first $n - 1$ rows and columns of A. If $x = (x_1, \cdots, x_{n-1})$ is a vector with $n - 1$ components and $y = (x_1, \cdots, x_{n-1}, 0)$ then $(Bx, x) = (Ay, y)$. Therefore for any such $x \neq 0$, $(Ax, x) \neq 0$ and

$$0 < \arg \alpha_1 \leqq \arg (Ay, y) = \arg (Bx, x) \leqq \arg \alpha_n < \pi ,$$

by Lemma 8, since A is congruent to D.

By Lemma 7, B is congruent to a unitary matrix V. Let the eigenvalues of V be β_i, where $\arg \beta_1 \leqq \cdots \leqq \arg \beta_{n-1}$. Since the quadratic form (Bx, x) associated with B is a restriction of the quadratic form associated with A, it follows from Lemma 8 that $\arg \alpha_{j+1} \geqq \arg \beta_j \geqq \arg \alpha_j$, $j = 1, \cdots, n - 1$. Also by the induction hypothesis $(\arg \lambda_1, \cdots, \arg \lambda_{n-1}) \prec (\arg \beta_1, \cdots, \arg \beta_{n-1})$. Therefore

$\arg \lambda_1 + \cdots + \arg \lambda_r \geqq \arg \beta_1 + \cdots + \arg \beta_r \geqq \arg \alpha_1 + \cdots + \arg \alpha_r$, $r = 1, \cdots, n - 1$

and

$$\arg \alpha_2 + \cdots + \arg \alpha_n \geqq \arg \lambda_1 + \cdots + \arg \lambda_{n-1} \\ \geqq \arg \alpha_1 + \cdots + \arg \alpha_{n-1} .$$

Hence

$$-\pi < \arg \lambda_n - \arg \alpha_n \leqq \sum_1^n (\arg \lambda_i - \arg \alpha_i) \leqq \arg \lambda_n - \arg \alpha_1 < \pi .$$

But

$$\prod_1^n \lambda_i = |\det X|^2 \cdot \prod_1^n \alpha_i .$$

Therefore

$$\sum_1^n \arg \lambda_i = \sum_1^n \arg \alpha_i .$$

The proof is complete.

LEMMA 10. *If* (β_i), (α_i) *are n-tuples of complex numbers of modulus* 1 *which lie on a line through* 0, *and if* $(\beta\ )$, (α_i) *is realizable, then* (β_i) *must be a rearrangement of* (α_i).

Proof. By Lemma 3 we may suppose that the α_i and β_i are all real. Let A be a matrix with eigenvalues β_i which is congruent to $\operatorname{diag}(\alpha_1, \cdots, \alpha_n)$. Then A is Hermitian and therefore A is also congruent to $\operatorname{diag}(\beta_1, \cdots, \beta_n)$. But by Lemma 1 it follows that (α_i), (β_i) is realizable. Therefore by Lemma 9 we have $(\arg \beta_i) \prec (\arg \alpha_i) \prec (\arg \beta_i)$, from which the present theorem follows immediately.

LEMMA 11. *Suppose* (β_i), (α_i) *are n-tuples of complex numbers of modulus* 1 *such that* $\prod_1^n \beta_i = \prod_1^n \alpha_i$. *Then there exist determinations of* $\arg \alpha_i$, $\arg \beta_i$ *such that*

$$\max \arg \alpha_i - \min \arg \alpha_i \leqq 2\pi$$

and

$$(\arg \beta_i) \prec (\arg \alpha_i) .$$

Proof. The statement is obvious for $n = 1$. Suppose $n > 1$ and it holds for n-1-tuples. If any of the β_i is equal to any of the α_i, say $\beta_1 = \alpha_1$, then by the induction hypothesis, we can find determinations of the remaining $\arg \alpha_i$, $\arg \beta_i$ as stated. If we now choose a value of $\arg \alpha_1$ which lies between μ and $\mu + 2\pi$, where $\mu = \min_{i>1} \arg \alpha_i$, and set $\arg \beta_1 = \arg \alpha_1$, then the conditions of our theorem will be satisfied, by Lemma 6. So henceforth we may assume that $\beta_i \neq \alpha_j$ for all i, j.

As another special case, suppose the α_i are all equal, say to 1. If we assign arguments to the β_i such that $0 < \arg \beta_i < 2\pi$, then $\sum_1^n \arg \beta_i = 2\pi k$, where k is some positive integer $< n$. We need only assign arguments to the α_i such that exactly k of them have argument 2π and the remaining ones have argument 0.

Now assume the previous two cases do not occur. The α_i divide the unit circle into arcs. At least one of them must contain more than one of the β_i, for if not the α_i would be all distinct and each of the n arcs determined by them would contain exactly one of the β_i. We could then assign arguments to arrangements of the α_i, β_i so that

$$\arg \alpha_1 < \arg \beta_1 < \arg \alpha_2 < \cdots < \arg \alpha_n < \arg \beta_n < \arg \alpha_1 + 2\pi .$$

But then $0 < \sum_1^n \arg \beta_i - \sum_1^n \arg \alpha_i < 2\pi$, contradicting the hypothesis $\prod_1^n \alpha_i = \prod_1^n \beta_i$.

Let C be an arc containing more than one of the β_i. By changing subscripts, we may assume that the endpoints of C when described counterclockwise are α_1 and α_2. Let β_1 be one of the β_i in C which is nearest to α_1 and β_2 be one of the β_i with subscript $\neq 1$ which is nearest to α_2. Note that β_1 may equal β_2, but $\alpha_1 \neq \alpha_2$. As will be seen from the following argument, we may assume the subarc $\alpha_1\beta_1$ of $C \leqq$ the subarc $\beta_2\alpha_2$ of C, (all arcs are described counterclockwise). Let $\beta_1' = \alpha_1$ and let β_2' be the point in $\beta_2\alpha_2$ such that $\beta_2\beta_2' = \alpha_1\beta_1 = \delta$. By the first case of the proof, we may assign arguments to β_1', β_2', $\beta_3, \cdots, \beta_n$ and $\alpha_1, \cdots, \alpha_n$ so that

(1) $\max \arg \alpha_i - \min \arg \alpha_i \leqq 2\pi$

and

(2) $(\arg \beta_1', \arg \beta_2', \arg \beta_3, \cdots, \arg \beta_n) \prec (\arg \alpha_1, \cdots, \arg \alpha_n)$.

If $\arg \alpha_1$ happens to be the largest of $\arg \alpha_i$, and therefore $\arg \alpha_2$ is the smallest of $\arg \alpha_i$, then none of β_1', β_2', $\beta_3, \cdots, \beta_n$ can lie in the interior of C. Therefore $\beta_2' = \alpha_2$, and if we decrease $\arg \alpha_1$ and $\arg \beta_1$ by 2π, then (1) and (2) will still hold. Thus we may assume $\arg \alpha_1 < \arg \alpha_2$, and therefore $\arg \beta_1' < \arg \beta_2'$. Now assign to β_1 the argument $\beta_1' + \delta$ and to β_2 the argument $\arg \beta_2' - \delta$. Since

$$(\arg \beta_1' + \delta,\ \arg \beta_2' - \delta) \prec (\arg \beta_1',\ \arg \beta_2')\ ,$$

we have by Lemma 6,

$$\begin{aligned}(\arg \beta_1, \cdots, \arg \beta_n) &\prec (\arg \beta_1', \arg \beta_2', \arg \beta_3, \cdots, \arg \beta_n) \\ &\prec (\arg \alpha_1, \cdots, \arg \alpha_n)\ .\end{aligned}$$

This completes the proof.

LEMMA 12. *If (β_i), (α_i) are n-tuples of complex numbers of modulus* 1 *which can be assigned arguments such that*

$$\arg \alpha_1 \leqq \cdots \leqq \arg \alpha_n \leqq \arg \alpha_1 + 2\pi\ ,$$
$$\arg \beta_1 \leqq \cdots \leqq \arg \beta_n\ ,$$
$$(\arg \beta_i) \prec (\arg \alpha_i)\ ,$$

and

$$\arg \alpha_{i+1} - \arg \alpha_i < \pi,\ i = 1, \cdots, n-1\ ,$$

then a finite number of pinches will reduce (α_i) to (β_i).

Proof. We proceed by induction on n. When $n = 2$, we have $\arg \alpha_1 \leqq \arg \beta_1 \leqq \arg \beta_2 \leqq \arg \alpha_2$, $\arg \alpha_1 + \arg \alpha_2 = \arg \beta_1 + \arg \beta_2$ and $\arg \alpha_2 - \arg \alpha_1 < \pi$. Therefore $\arg \beta_1 - \arg \alpha_1 = \arg \alpha_2 - \arg \beta_2$ and so

(β_1, β_2) is a pinch of (α_1, α_2).

Suppose $n > 2$ and the theorem holds for all m-tuples, $m < n$. Let

$$\delta = \min_{1 \leqq p \leqq n-1} \sum_1^p (\arg \beta_i - \arg \alpha_i) .$$

There exists k such that $\sum_1^k \arg \beta_i - \sum_1^k \arg \alpha_i = \delta$. It is easy to verify that

$$(\arg \beta_1, \cdots, \arg \beta_k) \prec (\arg \alpha_1 + \delta, \arg \alpha_2, \cdots, \arg \alpha_k)$$

and

$$(\arg \beta_{k+1}, \cdots, \arg \beta_n) \prec (\arg \alpha_{k+1}, \cdots, \arg \alpha_{n-1}, \arg \alpha_n - \delta) .$$

Also

$$\arg \alpha_1 + \delta \leqq \arg \beta_1 \leqq \arg \beta_n \leqq \arg \alpha_n - \delta .$$

By the induction hypothesis, we can reduce $(\alpha_1 e^{i\delta}, \alpha_2, \cdots, \alpha_k)$ to $(\beta_1, \cdots, \beta_k)$ and $(\alpha_{k+1}, \cdots, \alpha_{n-1}, \alpha_n e^{-i\delta})$ to $(\beta_{k+1}, \cdots, \beta_n)$ by a finite number of pinches. We need only show that $(\alpha_1, \cdots, \alpha_n)$ can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n e^{-i\delta})$ by a finite number of pinches. This will follow from the next lemma if we consider only the distinct α_i.

If the α_i all coincide, then so do the β_i and the statement of our theorem is trivial.

LEMMA 13. *If (α_i) is an m-tuple of numbers of modulus* 1 *with assigned arguments such that*

$$\arg \alpha_1 < \cdots < \arg \alpha_m \leqq \arg \alpha_1 + 2\pi$$

and

$$\arg \alpha_{i+1} - \arg \alpha_i < \pi, \ i = 1, \cdots, m-1 ,$$

and if δ is a positive number such that $\arg \alpha_1 + \delta \leqq \arg \alpha_m - \delta$, *then (α_i) can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \cdots, \alpha_{m-1}, \alpha_m e^{-i\delta})$ by a finite number of pinches.*

Proof. This is obvious for $m = 2$. Assume $m > 2$ and the lemma holds for $m - 1 -$ tuples. If

$$\eta = \min(\arg \alpha_2 - \arg \alpha_1, \pi - (\arg \alpha_3 - \arg \alpha_2), \cdots, \\ \pi - (\arg \alpha_m - \arg \alpha_{m-1})) ,$$

and $0 < \varepsilon < \eta$, then each sequence in the following list is a pinch of the preceeding sequence:

$$\alpha_1, \cdots, \alpha_m$$

$$\alpha_1 e^{i\varepsilon},\ \alpha_2 e^{-i\varepsilon},\ \alpha_3,\ \cdots,\ \alpha_m$$
$$\alpha_1 e^{i\varepsilon},\ \alpha_2,\ \alpha_3 e^{-i\varepsilon},\ \cdots,\ \alpha_m$$
$$\cdots$$
$$\alpha_1 e^{i\varepsilon},\ \alpha_2,\ \cdots,\ \alpha_{m-2},\ \alpha_{m-1} e^{-i\varepsilon},\ \alpha_m$$
$$\alpha_1 e^{i\varepsilon},\ \alpha_2,\ \cdots,\ \alpha_{m-1},\ \alpha_m e^{-i\varepsilon}\ .$$

Note that $\arg \alpha_1 + \varepsilon$ need not be $\leqq \arg \alpha_2 - \varepsilon$, and $\arg \alpha_2$ need not be $\leqq \arg \alpha_3 - \varepsilon$, etc.

We may repeat this cycle of m pinches $k-1$ more times to pass from

$$\alpha_1 e^{i\varepsilon},\ \alpha_2,\ \cdots,\ \alpha_{m-1},\ \alpha_m e^{-i\varepsilon} \text{ to } \alpha_1 e^{ki\varepsilon},\ \alpha_2,\ \cdots,\ \alpha_{m-1},\ \alpha_m e^{-ki\varepsilon}$$

as long as $\arg \alpha_1 + k\varepsilon \leqq \arg \alpha_2$, since

$$\arg \alpha_2 + p\varepsilon - \arg \alpha_1 > \arg \alpha_2 - \arg \alpha_1$$

and

$$\pi - (\arg \alpha_n - p\varepsilon - \arg \alpha_{m-1}) > \pi - (\arg \alpha_n - \arg \alpha_{m-1})$$

for $p < k$. Therefore if $\delta \leqq \arg \alpha_2 - \arg \alpha_1$, we need only choose $\varepsilon = \delta/k$, where k is an integer so large that $\delta/k < \eta$. If $\delta > \arg \alpha_2 - \arg \alpha_1$, choose $\varepsilon = (\arg \alpha_2 - \arg \alpha_1)/k$, where k is so large that $\varepsilon < \eta$. Then $(\alpha_1, \cdots, \alpha_m)$ is reduced to $(\alpha_2, \alpha_2, \cdots, \alpha_{m-1}, \alpha_m e^{-ik\varepsilon})$ by the above sequence of pinches. By the induction hypothesis, $(\alpha_2, \alpha_3, \cdots, \alpha_{m-1}, \alpha_m e^{-ik\varepsilon})$ can by a finite number of pinches be reduced to $(\alpha_1 e^{i\delta}, \alpha_3, \cdots, \alpha_{m-1}, \alpha_m e^{-i\delta})$. (The fact that $\alpha_m e^{-ik\varepsilon}$ might be equal to one of the α_j is clearly unimportant.) Therefore $(\alpha_1, \cdots, \alpha_m)$ can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \cdots, \alpha_{m-1}, \alpha_m e^{-i\delta})$, and the proof is complete.

3. Proof of Theorem 1.

(2) → (1): This is the statement of Lemma 5.

(1) → (3): If (λ_i), (α_i) is realizable, then by Lemma 1 there exists a matrix A and a non-singular matrix X such that $A = X^* \operatorname{diag}(\alpha_1, \cdots, \alpha_n) X$ and A has eigenvalues λ_i. Therefore $\prod \lambda_i = \prod \alpha_i \cdot |\det X|^2$ and hence $\prod \lambda_i / |\lambda_i| = \prod \alpha_i$. If the α_i lie on a line through 0, then $(\lambda_i / |\lambda_i|)$ is a rearrangement of (α_i) by Lemmas 2 and 10. If the α_i lie in a closed half plane through 0, then by Lemma 3 we may assume they lie in the upper half plane. By Lemma 9 it follows that $(\arg \lambda_i) \prec (\arg \alpha_i)$.

(3) → (2): In case (a), the statement is obvious. In case (c), Lemma 11 and the fact that the α_i do not lie in any closed half plane with 0 on its boundary show that the hypotheses of Lemma 12 are satisfied by arrangements of $(\lambda_i / |\lambda_i|)$, (α_i). In case (b), the hypotheses of

Lemma 12 also are satisfied by arrangements of $(\lambda_i/|\lambda_i|)$, (α_i). Thus an application of Lemma 12 completes the proof.

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