ON STRICTLY SEMI-SIMPLE BANACH ALGEBRAS

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I. Introduction. Define the *strict radical* of an algebra to be the intersection of just those of its two-sided ideals which are regular maximal right ideals. Call the algebra *strictly semi-simple* (sss) if its strict radical is the zero ideal. This note proves that the strict radical of a real Banach algebra $B$ contains the set of topologically nilpotent elements of $B$. Also, it gives a condition which is both necessary and sufficient for $B$ to be sss.

II. Preliminaries. For any ring or algebra $A$ let $T(A)$ denote the set of all those two-sided ideals in $A$ which are regular maximal right ideals. The intersection of the elements of $T(A)$ is the *strict radical* of $A$. $A$ is strictly semi-simple (sss) if its strict radical is the zero ideal.

**Lemma 1.** Let $I$ be a two-sided ideal in the algebra (ring) $A$. Then the following are equivalent:

(a) $I \in T(A)$, that is, $I$ is a regular maximal right ideal.
(b) $I$ is a regular maximal left ideal.
(c) $A/I$ is a division algebra (division ring).

*Proof.* Use is made of the theorem [4, Theorem 24.6.1] that a division algebra has no proper right or left ideals and that an algebra with no proper right ideals either is trivial or is a division algebra.

If (a) holds, then $A/I$ has no proper right ideals. Now $A/I$ is not trivial since if $j$ is a left unit element of $A$ modulo $I$, $j' \cdot j' = j' \neq 0$ (where $x'$ denotes the image of $x \in A$ under the canonical homomorphism of $A$ onto $A/I$). The cited theorem shows $A/I$ is a division algebra. Thus (a) implies (c) and, similarly, (b) implies (c). Moreover, if (a) holds, then $j'$ is a left identity for $A/I$ and hence an identity for it, so that $I$ is regular with $j$ as its associated unit element. If $I \subset L$, $L$ a left ideal in $A$, then $L/I$ is a left ideal in $A/I$, and an improper ideal by the cited theorem, so that $L = I$ or $A$ and $I$ is a regular maximal left ideal. Thus (a) implies (b).

Suppose (c) holds and $e'$ is a unit of $A/I$. Then $I$ is regular with

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e as its associated unit element. If $I \subseteq J$, $J$ a right ideal in $A$, $J/I$ is a right improper ideal in $A/I$ so that $J = I$ or $A$ and $I$ is a regular maximal right ideal. Thus (c) implies (a).

Theorem 1 relates the strict radical of $A$ to the Jacobson radical [5] and to the Segal [9] or Brown-McCoy radical [2], which is the intersection of the regular maximal two-sided ideals in $A$. $A$ is called strongly semi-simple (semi-simple) if the Segal (Jacobson) radical is the zero ideal.

$A$ satisfies Property $M$ if each of its regular maximal right ideals is a two-sided ideal.

**Theorem 1.** The strict radical contains the Segal and Jacobson radicals so that if $A$ is sss then it is necessarily strongly semi-simple and semi-simple. These radicals coincide if Property $M$ is satisfied.

**Proof.** Let $W$ be the set of all regular maximal two-sided ideals and $W_r$ the set of all regular maximal right ideals in $A$. If $I \in T(A)$ then $A/I$ is a division algebra by Lemma 1 so that $I \in W$. Therefore the strict radical contains the Segal radical which contains the Jacobson radical. Now, if Property $M$ holds, $I \in W_r$ implies $I \in T(A)$ which shows the Jacobson radical contains the strict radical and hence all these radicals coincide.

**Examples.** 1. An example of an algebra which is semi-simple and strongly semi-simple but not sss is furnished by the algebra of all 2 by 2 matrices, which is radical in the sense of the strict radical.

2. Arens' $BQ^*$-algebra [1] are examples of Banach algebras which are sss and satisfy Property $M$. Indeed, Arens establishes that such algebras are semi-simple and have the property that every closed ideal—and, a fortiori, every regular maximal right ideal—is two-sided.

3. Let $C(X, D)$ be the ring of all continuous functions on $X$ with values in $D$, where $X$ is a compact $T_\infty$-space and $D$ a division ring that admits a continuous function $f(x)$ such that $xf(x) + yf(y) = 0$ implies that $x = y = 0$. Kaplansky [6, p. 179] notes that such a function $f(x)$ cannot exist in a ring of characteristic 2 (take $x = y$) but exists in every ring of characteristic different from 2 that he has examined. The maximal right (or left) ideals in $C(X, D)$ are two-sided [6, p. 180], so that $C(X, D)$ satisfies Property $M$ and, since it is semi-simple, it is necessarily sss.

4. If a ring $A$ is strongly regular (that is, if for every $a \in A$ there exists $x \in A$ such that $ax = a$) then $A$ is semi-simple and every ideal in it is two-sided [2, pp. 462-4]. Hence a strongly regular ring satisfies Property $M$ and is sss.
III. Necessary and sufficient condition for a Banach algebra to be sss. Henceforth the algebras considered are over the real field and the homomorphisms considered are algebraic (real-linear). Let $Q$ denote the quaternions, $H(A, Q)$ the set of nonzero homomorphisms of the algebra $A$ into $Q$, $|q|$ the absolute value of the quaternion $q$, and $C(X, Q)$ the algebra of quaternion-valued functions, continuous on and vanishing at the infinite point of a locally compact Hausdorff space $X$.

**Lemma 2.** An algebra $A$ is mapped onto the reals, onto a field isomorphic to the complexes, or onto the quaternions by any $h \in H(A, Q)$ and the kernel of $h$ belongs to $T(A)$. If $A$ is a Banach algebra each member of $T(A)$ is the kernel of some member of $H(A, Q)$.

**Proof.** Let $h(A)$ denote the image of $A$ under $h$. For any $u, v \in h(A)$

$$|u \cdot v| = |u| \cdot |v|.$$  

Under the norm $|u|$, $h(A)$ is a normed algebra. A normed algebra in which the norm satisfies property (1) is isomorphic to either the reals, complexes, or quaternions [7, Theorem II]. Hence $A/h^{-1}(0)$ is a division algebra and $h^{-1}(0) \in T(A)$ by Lemma 1.

Let $A$ be a Banach algebra and $I \in T(A)$. Then $A/I$ is a division algebra by Lemma 1 and a Banach algebra since $I$ is closed. A normed division algebra is isomorphic to the reals, complexes, or quaternions [7, Theorem I]. Hence $I$ is the kernel of some $h \in H(A, Q)$.

**Theorem 2.** Any subalgebra $A$ of $C(X, Q)$ is sss.

**Proof.** Let $f \in A$, $f \neq 0$. Then there is an $x \in X$ such that $f(x) \neq 0$. Let $I = \{g \in A : g(x) = 0\}$. Then $A/I$ is naturally isomorphic to a subalgebra of $Q$. Hence $I \in T(A)$ by Lemma 2. But $f \notin I$. Therefore $A$ is sss.

**Theorem 3.** If a Banach algebra $B$ is sss, then $B$ is isomorphic with a subalgebra of some $C(X, Q)$.

**Proof.** Let $X = H(B, Q)$. There is a natural homomorphism of $B$ into $C(X, Q)$: $f \rightarrow \varphi$ where $\varphi(x) = x(f)$. It remains only to show that the homomorphism is $1 - 1$. Let $f \in B$, $f \neq 0$. Since $B$ is sss there is an $I \in T(B)$ such that $f \notin I$. By Lemma 2, $I = x^{-1}(0)$ for some $x \in X$. Hence $\varphi(x) \neq 0$.

**Corollary 1.** An algebra isomorphic to a subalgebra of a sss Banach algebra is itself sss. Hence any subalgebra, whether closed or not, of a sss Banach algebra is itself sss.
IV. The strict radical of a Banach algebra contains the set of topologically nilpotent elements. An element \( x \) of a normed algebra is called topologically nilpotent if \( r(x) = 0 \) where \( r(x) = \lim_{n \to \infty} \| x^n \|^{1/n} = \sup |\beta|: \beta \in \text{ spectrum of } x \) [8, pp. 617–618].

**Theorem 4.** Let \( N \) be the set of topologically nilpotent elements of a Banach algebra \( B \) and \( S \) the strict radical of \( B \). Let \( J' \) be the Jacobson radical of any subalgebra of \( B \). Then \( J' \subset N \subset S \).

*Proof.* That \( J' \subset N \) is known [8, Lemma 1.2]. If it is shown that every \( h \in H(B, \mathbb{Q}) \) maps \( x \in N \) into the zero element, then it follows from Lemma 2 that \( x \) belongs to every member of \( T(B) \) and therefore to \( S \). The spectrum of \( h(x) \) contains the spectrum of \( x \); hence \( r[h(x)] = 0 \) since \( r(x) = 0 \). Since a topologically nilpotent element is singular [4, p. 121], \( h(x) = 0 \). Hence \( N \subset S \).

**Corollary 2.** If a Banach algebra is sss then zero is its only topologically nilpotent element.

**Corollary 3.** Let \( N \) and \( S \) be defined as in Theorem 4 and let \( J \) be the Jacobson radical of \( B \). Then \( J = S \) if and only if \( N = S \). If \( B \) satisfies Property \( M \), then \( J = N = S \).

*Proof.* Theorem 4 yields Corollary 2 as an immediate consequence and also shows that if \( J = S \), then \( J = N = S \). If \( N = S \), \( N \) is an ideal composed of topologically nilpotent elements and therefore \( N \subset J \) since \( J \) is the union of such ideals [8, p. 617]; hence \( J = N \). If Property \( M \) is satisfied then \( J = S \) by Theorem 1 so that \( J = N = S \).

**References**


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