

# Pacific Journal of Mathematics

**ON STRICTLY SEMI-SIMPLE BANACH ALGEBRAS**

EDITH HIRSCH LUCHINS

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**I. Introduction.** Define the *strict radical* of an algebra to be the intersection of just those of its two-sided ideals which are regular maximal right ideals. Call the algebra *strictly semi-simple* (sss) if its strict radical is the zero ideal. This note proves that the strict radical of a real Banach algebra  $B$  contains the set of topologically nilpotent elements of  $B$ . Also, it gives a condition which is both necessary and sufficient for  $B$  to be sss.

**II. Preliminaries.** For any ring or algebra  $A$  let  $T(A)$  denote the set of all those two-sided ideals in  $A$  which are regular maximal right ideals. The intersection of the elements of  $T(A)$  is the *strict radical* of  $A$ .  $A$  is *strictly semi-simple* (sss) if its strict radical is the zero ideal.

**LEMMA 1.** *Let  $I$  be a two-sided ideal in the algebra (ring)  $A$ . Then the following are equivalent:*

- (a)  $I \in T(A)$ , that is,  $I$  is a regular maximal right ideal.
- (b)  $I$  is a regular maximal left ideal.
- (c)  $A/I$  is a division algebra (division ring).

*Proof.* Use is made of the theorem [4, Theorem 24.6.1] that a division algebra has no proper right or left ideals and that an algebra with no proper right ideals either is trivial or is a division algebra.

If (a) holds, then  $A/I$  has no proper right ideals. Now  $A/I$  is not trivial since if  $j$  is a left unit element of  $A$  modulo  $I$ ,  $j' \cdot j' = j' \neq 0$  (where  $x'$  denotes the image of  $x \in A$  under the canonical homomorphism of  $A$  onto  $A/I$ ). The cited theorem shows  $A/I$  is a division algebra. Thus (a) implies (c) and, similarly, (b) implies (c). Moreover, if (a) holds, then  $j'$  is a left identity for  $A/I$  and hence an identity for it, so that  $I$  is regular with  $j$  as its associated unit element. If  $I \subset L$ ,  $L$  a left ideal in  $A$ , then  $L/I$  is a left ideal in  $A/I$ , and an improper ideal by the cited theorem, so that  $L = I$  or  $A$  and  $I$  is a regular maximal left ideal. Thus (a) implies (b).

Suppose (c) holds and  $e'$  is a unit of  $A/I$ . Then  $I$  is regular with

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$e$  as its associated unit element. If  $I \subset J$ ,  $J$  a right ideal in  $A$ ,  $J/I$  is a right improper ideal in  $A/I$  so that  $J = I$  or  $A$  and  $I$  is a regular maximal right ideal. Thus (c) implies (a).

Theorem 1 relates the strict radical of  $A$  to the Jacobson radical [5] and to the Segal [9] or Brown-McCoy radical [2], which is the intersection of the regular maximal two-sided ideals in  $A$ .  $A$  is called strongly semi-simple (semi-simple) if the Segal (Jacobson) radical is the zero ideal.

$A$  satisfies *Property M* if each of its regular maximal right ideals is a two-sided ideal.

**THEOREM 1.** *The strict radical contains the Segal and Jacobson radicals so that if  $A$  is sss then it is necessarily strongly semi-simple and semi-simple. These radicals coincide if Property M is satisfied.*

*Proof.* Let  $W$  be the set of all regular maximal two-sided ideals and  $W_r$  the set of all regular maximal right ideals in  $A$ . If  $I \in T(A)$  then  $A/I$  is a division algebra by Lemma 1 so that  $I \in W$ . Therefore the strict radical contains the Segal radical which contains the Jacobson radical. Now, if Property *M* holds,  $I \in W_r$  implies  $I \in T(A)$  which shows the Jacobson radical contains the strict radical and hence all these radicals coincide.

**EXAMPLES.** 1. An example of an algebra which is semi-simple and strongly semi-simple but not sss is furnished by the algebra of all  $2 \times 2$  matrices, which is radical in the sense of the strict radical.

2. Arens'  $BQ^*$ -algebra [1] are examples of Banach algebras which are sss and satisfy Property *M*. Indeed, Arens establishes that such algebras are semi-simple and have the property that every closed ideal—and, *a fortiori*, every regular maximal right ideal—is two-sided.

3. Let  $C(X, D)$  be the ring of all continuous functions on  $X$  with values in  $D$ , where  $X$  is a compact  $T_0$ -space and  $D$  a division ring that admits a continuous function  $f(x)$  such that  $xf(x) + yf(y) = 0$  implies that  $x = y = 0$ . Kaplansky [6, p. 179] notes that such a function  $f(x)$  cannot exist in a ring of characteristic 2 (take  $x = y$ ) but exists in every ring of characteristic different from 2 that he has examined. The maximal right (or left) ideals in  $C(X, D)$  are two-sided [6, p. 180], so that  $C(X, D)$  satisfies Property *M* and, since it is semi-simple, it is necessarily sss.

4. If a ring  $A$  is strongly regular (that is, if for every  $a \in A$  there exists  $x \in A$  such that  $a^2x = a$ ) then  $A$  is semi-simple and every ideal in it is two-sided [2, pp. 462-4]. Hence a strongly regular ring satisfies Property *M* and is sss.

**III. Necessary and sufficient condition for a Banach algebra to be sss.** Henceforth the algebras considered are over the real field and the homomorphisms considered are algebraic (real-linear). Let  $Q$  denote the quaternions,  $H(A, Q)$  the set of nonzero homomorphisms of the algebra  $A$  into  $Q$ ,  $|q|$  the absolute value of the quaternion  $q$ , and  $C(X, Q)$  the algebra of quaternion-valued functions, continuous on and vanishing at the infinite point of a locally compact Hausdorff space  $X$ .

**LEMMA 2.** *An algebra  $A$  is mapped onto the reals, onto a field isomorphic to the complexes, or onto the quaternions by any  $h \in H(A, Q)$  and the kernel of  $h$  belongs to  $T(A)$ . If  $A$  is a Banach algebra each member of  $T(A)$  is the kernel of some member of  $H(A, Q)$ .*

*Proof.* Let  $h(A)$  denote the image of  $A$  under  $h$ . For any  $u, v \in h(A)$

$$(1) \quad |u \cdot v| = |u| \cdot |v|.$$

Under the norm  $|u|$ ,  $h(A)$  is a normed algebra. A normed algebra in which the norm satisfies property (1) is isomorphic to either the reals, complexes, or quaternions [7, Theorem II]. Hence  $A/h^{-1}(0)$  is a division algebra and  $h^{-1}(0) \in T(A)$  by Lemma 1.

Let  $A$  be a Banach algebra and  $I \in T(A)$ . Then  $A/I$  is a division algebra by Lemma 1 and a Banach algebra since  $I$  is closed. A normed division algebra is isomorphic to the reals, complexes, or quaternions [7, Theorem I]. Hence  $I$  is the kernel of some  $h \in H(A, Q)$ .

**THEOREM 2.** *Any subalgebra  $A$  of  $C(X, Q)$  is sss.*

*Proof.* Let  $f \in A$ ,  $f \neq 0$ . Then there is an  $x \in X$  such that  $f(x) \neq 0$ . Let  $I = \{g \in A: g(x) = 0\}$ . Then  $A/I$  is naturally isomorphic to a subalgebra of  $Q$ . Hence  $I \in T(A)$  by Lemma 2. But  $f \notin I$ . Therefore  $A$  is sss.

**THEOREM 3.** *If a Banach algebra  $B$  is sss, then  $B$  is isomorphic with a subalgebra of some  $C(X, Q)$ .*

*Proof.* Let  $X = H(B, Q)$ . There is a natural homomorphism of  $B$  into  $C(X, Q)$ :  $f \rightarrow \varphi$  where  $\varphi(x) = x(f)$ . It remains only to show that the homomorphism is 1-1. Let  $f \in B$ ,  $f \neq 0$ . Since  $B$  is sss there is an  $I \in T(B)$  such that  $f \notin I$ . By Lemma 2,  $I = x^{-1}(0)$  for some  $x \in X$ . Hence  $\varphi(x) \neq 0$ .

**COROLLARY 1.** *An algebra isomorphic to a subalgebra of a sss Banach algebra is itself sss. Hence any subalgebra, whether closed or not, of a sss Banach algebra is itself sss.*

**IV. The strict radical of a Banach algebra contains the set of topologically nilpotent elements.** An element  $x$  of a normed algebra is called topologically nilpotent if  $r(x) = 0$  where  $r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n} = \sup |\beta|$ ;  $\beta \in$  spectrum of  $x$  [8, pp. 617-618].

**THEOREM 4.** *Let  $N$  be the set of topologically nilpotent elements of a Banach algebra  $B$  and  $S$  the strict radical of  $B$ . Let  $J'$  be the Jacobson radical of any subalgebra of  $B$ . Then  $J' \subset N \subset S$ .*

*Proof.* That  $J' \subset N$  is known [8, Lemma 1.2]. If it is shown that every  $h \in H(B, \mathbb{Q})$  maps  $x \in N$  into the zero element, then it follows from Lemma 2 that  $x$  belongs to every member of  $T(B)$  and therefore to  $S$ . The spectrum of  $h(x)$  contains the spectrum of  $x$ ; hence  $r[h(x)] = 0$  since  $r(x) = 0$ . Since a topologically nilpotent element is singular [4, p. 121],  $h(x) = 0$ . Hence  $N \subset S$ .

**COROLLARY 2.** *If a Banach algebra is sss then zero is its only topologically nilpotent element.*

**COROLLARY 3.** *Let  $N$  and  $S$  be defined as in Theorem 4 and let  $J$  be the Jacobson radical of  $B$ . Then  $J = S$  if and only if  $N = S$ . If  $B$  satisfies Property  $M$ , then  $J = N = S$ .*

*Proof.* Theorem 4 yields Corollary 2 as an immediate consequence and also shows that if  $J = S$ , then  $J = N = S$ . If  $N = S$ ,  $N$  is an ideal composed of topologically nilpotent elements and therefore  $N \subset J$  since  $J$  is the union of such ideals [8, p. 617]; hence  $J = N$ . If Property  $M$  is satisfied then  $J = S$  by Theorem 1 so that  $J = N = S$ .

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Lee William Anderson, <i>On the breadth and co-dimension of a topological lattice</i> .....	327
Frank W. Anderson and Robert L. Blair, <i>Characterizations of certain lattices of functions</i> .....	335
Donald Charles Benson, <i>Extensions of a theorem of Loewner on integral operators</i> .....	365
Errett Albert Bishop, <i>A duality theorem for an arbitrary operator</i> .....	379
Robert McCallum Blumenthal and Ronald Kay Getoor, <i>The asymptotic distribution of the eigenvalues for a class of Markov operators</i> .....	399
Delmar L. Boyer and Elbert A. Walker, <i>Almost locally pure Abelian groups</i> .....	409
Paul Civin and Bertram Yood, <i>Involutions on Banach algebras</i> .....	415
Lincoln Kearney Durst, <i>Exceptional real Lehmer sequences</i> .....	437
Eldon Dyer and Allen Lowell Shields, <i>Connectivity of topological lattices</i> .....	443
Ronald Kay Getoor, <i>Markov operators and their associated semi-groups</i> .....	449
Bernard Greenspan, <i>A bound for the orders of the components of a system of algebraic difference equations</i> .....	473
Branko Grünbaum, <i>On some covering and intersection properties in Minkowski spaces</i> .....	487
Bruno Harris, <i>Derivations of Jordan algebras</i> .....	495
Henry Berge Helson, <i>Conjugate series in several variables</i> .....	513
Isidore Isaac Hirschman, Jr., <i>A maximal problem in harmonic analysis. II</i> .....	525
Alfred Horn and Robert Steinberg, <i>Eigenvalues of the unitary part of a matrix</i> .....	541
Edith Hirsch Luchins, <i>On strictly semi-simple Banach algebras</i> .....	551
William D. Munro, <i>Some iterative methods for determining zeros of functions of a complex variable</i> .....	555
John Rainwater, <i>Spaces whose finest uniformity is metric</i> .....	567
William T. Reid, <i>Variational aspects of generalized convex functions</i> .....	571
A. Sade, <i>Isomorphisme d'hypergroupoï des isotopes</i> .....	583
Isadore Manual Singer, <i>The geometric interpretation of a special connection</i> .....	585
Charles Andrew Swanson, <i>Asymptotic perturbation series for characteristic value problems</i> .....	591
Jack Phillip Tull, <i>Dirichlet multiplication in lattice point problems. II</i> .....	609
Richard Steven Varga, <i>p-cyclic matrices: A generalization of the Young-Frankel successive overrelaxation scheme</i> .....	617