

Pacific Journal of Mathematics

**MULTIPLICATION FORMULAS FOR PRODUCTS OF
BERNOULLI AND EULER POLYNOMIALS**

L. CARLITZ

MULTIPLICATION FORMULAS FOR PRODUCTS OF BERNOULLI AND EULER POLYNOMIALS

L. CARLITZ

1. Put

$$(1.1) \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

The following multiplication formulas are familiar [5, pp. 18, 24]:

$$(1.2) \quad B_m(kx) = k^{m-1} \sum_{r=0}^{k-1} B_m\left(x + \frac{r}{k}\right),$$

$$(1.3) \quad E_m(kx) = k^m \sum_{r=0}^{k-1} (-1)^r E_m\left(x + \frac{r}{k}\right) \quad (k \text{ odd}).$$

Let $\overline{B}_m(x)$, $\overline{E}_m(x)$ denote, respectively, the Bernoulli and Euler functions defined by

$$\begin{aligned} \overline{B}_m(x) &= B_m(x) (0 \leq x < 1), \quad \overline{B}_m(x+1) = \overline{B}_m(x), \\ \overline{E}_m(x) &= E_m(x) (0 \leq x < 1), \quad \overline{E}_m(x+1) = -\overline{E}_m(x), \quad (m \geq 1). \end{aligned}$$

Then $\overline{B}_m(x)$ and $\overline{E}_m(x)$ also satisfy the multiplication formulas (1.2), (1.3).

In this note we obtain some generalizations of (1.2) and (1.3) suggested by a recent result of Mordell [4]. In extending some results of Mikolás [3], Mordell proves the following theorem. Let $f_1(x), \dots, f_n(x)$ denote functions of x of period 1 that satisfy the relations

$$(1.4) \quad \sum_{r=0}^{k-1} f_i\left(r + \frac{r}{k}\right) = C_i^{(k)} f_i(kx) \quad (i = 1, \dots, n),$$

where $C_i^{(k)}$ is independent of x . Let a_1, \dots, a_n be positive integers that are relatively prime in pairs. Then if the integrals exist and $A = a_1 a_2 \cdots a_n$,

$$\begin{aligned} (1.5) \quad & \int_0^A f_1\left(\frac{x}{a_1}\right) f_2\left(\frac{x}{a_2}\right) \cdots f_n\left(\frac{x}{a_n}\right) dx \\ &= A \int_0^1 f_1\left(\frac{Ax}{a_1}\right) f_2\left(\frac{Ax}{a_2}\right) \cdots f_n\left(\frac{Ax}{a_n}\right) dx \\ &= C_1^{(a_1)} C_2^{(a_2)} \cdots C_n^{(a_n)} \int_0^1 f_1(x) f_2(x) \cdots f_n(x) dx. \end{aligned}$$

2. We first prove

THEOREM 1. Let $n \geq 1$; $m_1, \dots, m_n \geq 1$; a_1, a_2, \dots, a_n positive integers that are relative prime in pairs; $A = a_1 a_2 \dots a_n$. Then

$$(2.1) \quad \sum_{r=0}^{kA-1} \bar{B}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right) \bar{B}_{m_2}\left(x_2 + \frac{r}{a_2 k}\right) \cdots \bar{B}_{m_n}\left(x_n + \frac{r}{a_n k}\right) \\ = C \sum_{r=0}^{k-1} \bar{B}_{m_1}\left(a_1 x_1 + \frac{r}{k}\right) \bar{B}_{m_2}\left(a_2 x_2 + \frac{r}{k}\right) \cdots \bar{B}_{m_n}\left(a_n x_n + \frac{r}{k}\right),$$

where

$$(2.2) \quad C = a_1^{1-m_1} a_2^{1-m_2} \cdots a_n^{1-m_n}.$$

In the first place for $n = 1$ it follows from (1.2) for arbitrary $a \geq 1$ that

$$\sum_{r=0}^{ka-1} \bar{B}_m\left(x + \frac{r}{ak}\right) = \sum_{r=0}^{k-1} \sum_{s=0}^{a-1} \bar{B}_m\left(r + \frac{s}{a} + \frac{r}{ak}\right) \\ = \sum_{r=0}^{k-1} \bar{B}_m\left(ax + \frac{r}{k}\right),$$

which agrees with (2.1).

For the general case, let S denote the left member of (2.1). Put

$$A_s = a_1 a_2 \cdots a_s \quad (1 \leq s \leq n)$$

and replace r by $skA_{n-1} + r$. Then

$$S = \sum_{r=0}^{kA_{n-1}-1} \bar{B}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right) \cdots \bar{B}_{m_{n-1}}\left(x_{n-1} + \frac{r}{a_{n-1} k}\right) \\ \cdot \sum_{s=0}^{a_n-1} \bar{B}_{m_n}\left(x_n + \frac{A_{n-1}s}{a_n} + \frac{r}{a_n k}\right) \\ = \sum_{r=0}^{kA_{n-1}-1} \bar{B}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right) \cdots \bar{B}_{m_{n-1}}\left(x_{n-1} + \frac{r}{a_{n-1} k}\right) \\ \cdot \sum_{s=0}^{a_n-1} \bar{B}_{m_n}\left(x_n + \frac{s}{a_n} + \frac{r}{a_n k}\right) \\ = a_n^{1-m_n} \sum_{r=0}^{kA_{n-1}-1} \bar{B}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right) \cdots \bar{B}_{m_{n-1}}\left(x_{n-1} + \frac{r}{a_{n-1} k}\right) \\ \cdot \bar{B}_{m_n}\left(a_n x_n + \frac{r}{k}\right).$$

Continuing in this way we get

$$\begin{aligned}
 S &= a_{n-1}^{1-m} a_n^{1-m} \sum_{r=0}^{kA_{n-2}-1} \bar{B}_{m_1} \left(x_1 + \frac{r}{a_1 k} \right) \cdots \bar{B}_{m_{n-2}} \left(x_{n-2} + \frac{r}{a_{n-2} k} \right) \\
 &\quad \cdot \bar{B}_{m_{n-1}} \left(a_{n-1} x_{n-1} + \frac{r}{k} \right) \bar{B}_{m_n} \left(a_n x_n + \frac{r}{k} \right) \\
 &= a_1^{1-m_1} \cdots a_n^{1-m_n} \sum_{r=0}^{k-1} \bar{B}_{m_1} \left(a_1 x_1 + \frac{r}{k} \right) \bar{B}_2 \left(a_2 x_2 + \frac{r}{k} \right) \\
 &\quad \cdots \bar{B}_n \left(a_n x_n + \frac{r}{k} \right).
 \end{aligned}$$

For $k = 1$, (2.1) reduces to

$$\begin{aligned}
 (2.3) \quad &\sum_{r=0}^{A-1} \bar{B}_{m_1} \left(x_1 + \frac{r}{a_1} \right) \bar{B}_2 \left(x_2 + \frac{r}{a_2} \right) \cdots \bar{B}_n \left(x_n + \frac{r}{a_n} \right) \\
 &= C \cdot \bar{B}_{m_1}(a_1 x_1) \bar{B}_{m_2}(a_2 x_2) \cdots \bar{B}_{m_n}(a_n x_n),
 \end{aligned}$$

where C is defined by (2.2); (2.3) may be considered a direct generalization of (1.2).

We remark that a formula like (2.1) holds for any set of functions satisfying (1.4).

We note also that the formula (2.2) can be proved by means of the Chinese remainder theorem. This remarks applies also to formulas (3.4) and (4.8) below.

3. In the next place we have

THEOREM 2. *Let n be odd and ≥ 1 ; $m_1, \dots, m_n \geq 1$; a_1, a_2, \dots, a_n positive odd integers that are relatively prime in pairs; $A = a_1 a_2 \cdots a_n$; k odd ≥ 1 . Then*

$$\begin{aligned}
 (3.1) \quad &\sum_{r=0}^{kA-1} (-1)^r \bar{E}_{m_1} \left(x_1 + \frac{r}{a_1 k} \right) \cdots \bar{E}_{m_n} \left(x_n + \frac{r}{a_n k} \right) \\
 &= C' \sum_{r=0}^{k-1} (-1)^r \bar{E}_{m_1} \left(a_1 x_1 + \frac{r}{k} \right) \cdots \bar{E}_{m_n} \left(a_n x_n + \frac{r}{k} \right),
 \end{aligned}$$

where

$$(3.2) \quad C' = a_1^{-m_1} a_2^{-m_2} \cdots a_n^{-m_n}.$$

The proof is similar to that of Theorem 1, but makes use of (1.3) in place of (1.2); also the formula

$$(3.3) \quad \bar{E}_m(x+r) = (-1)^r \bar{E}_m(x) \quad (m \geq 1)$$

is needed.

For $n = 1$ and a odd, we have

$$\begin{aligned} \sum_{r=0}^{ka-1} (-1)^r \bar{E}_{m_1} \left(x + \frac{r}{ak} \right) &= \sum_{r=0}^{k-1} (-1)^{sk} \bar{E}_m \left(x + \frac{s}{a} + \frac{r}{ak} \right) \\ &= a^{-m} \sum_{r=0}^{k-1} (-1)^r \bar{E}_m \left(ax + \frac{r}{k} \right), \end{aligned}$$

which agrees with (3.1). For the general case let S' denote the left member of (3.1). Then

$$\begin{aligned} S' &= \sum_{r=0}^{kA_{n-1}-1} \sum_{s=0}^{a_n-1} (-1)^{r+s} \bar{E}_{m_1} \left(x_1 + \frac{sA_{n-1}}{a_1} + \frac{r}{a_1k} \right) \cdots \\ &\quad \cdot \bar{E}_{m_{n-1}} \left(x_{n-1} + \frac{sA_{n-1}}{a_{n-1}} + \frac{r}{a_{n-1}k} \right) \\ &\quad \cdot \bar{E}_{m_n} \left(x_n + \frac{sA_{n-1}}{a_n} + \frac{r}{a_nk} \right). \end{aligned}$$

If we put

$$sA_{n-1} = qa_n + t \quad (0 \leq t < a_n),$$

then $s \equiv q + t \pmod{2}$, so that

$$\bar{E}_{m_n} \left(x_n + \frac{sA_{n-1}}{a_n} + \frac{r}{a_nk} \right) = (-1)^q \bar{E}_{m_n} \left(x_n + \frac{t}{a_n} + \frac{r}{a_nk} \right).$$

Since n is odd we therefore get

$$\begin{aligned} S' &= \sum_{r=0}^{kA_{n-1}-1} (-1)^r \bar{E}_{m_1} \left(x_1 + \frac{r}{a_1k} \right) \cdots \bar{E}_{m_{n-1}} \left(x_{n-1} + \frac{r}{a_{n-1}k} \right) \\ &\quad \cdot \sum_{t=0}^{a_n-1} (-1)^t \bar{E}_{m_n} \left(x_n + \frac{t}{a_n} + \frac{r}{a_nk} \right) \\ &= \sum_{r=0}^{kA_{n-1}-1} (-1)^r \bar{E}_{m_1} \left(x_1 + \frac{r}{a_1k} \right) \cdots \bar{E}_{m_{n-1}} \left(x_{n-1} + \frac{r}{a_{n-1}k} \right) \\ &\quad \cdot a_n^{-m_n} \bar{E}_{m_n} \left(a_n x_n + \frac{r}{k} \right). \end{aligned}$$

Continuing in this way we ultimately reach (3.1).

For $k = 1$, (3.1) becomes

$$\begin{aligned} (3.4) \quad &\sum_{r=0}^{A-1} (-1)^r \bar{E}_{m_1} \left(x_1 + \frac{r}{a_1} \right) \cdots \bar{E}_{m_n} \left(x_n + \frac{r}{a_n} \right) \\ &= C' E_{m_1}(a_1 x_1) \cdots E_{m_n}(a_n x_n), \end{aligned}$$

subject to the conditions of the theorem.

4. **Theorem 2 can be extended further** by introducing the “ Eulerian ” polynomial [2] $\phi_m(x, \rho)$ defined by

$$(4.1) \quad \frac{1 - \rho}{1 - \rho e^t} e^{xt} = \sum_{m=0}^{\infty} \phi_m(x, \rho) \frac{t^m}{m!} \quad (\rho \neq 1).$$

In particular $\phi_m(x, -1) = E_m(x)$.

We shall assume that the parameter ρ is an f th root of unity. It follows easily from (4.1) that

$$(4.2) \quad \phi_{m-1}(kx, \rho) = \frac{(\rho - 1)f^{m-1}}{m} \sum_{r=0}^{f-1} \rho^r B_m\left(x + \frac{r}{f}\right).$$

We accordingly define the function $\bar{\phi}_n(x, \rho)$ by means of

$$(4.3) \quad \bar{\phi}_{m-1}(kx, \rho) = \frac{(\rho - 1)e^{m-1}}{m} \sum_{r=0}^{f-1} \rho^r \bar{B}_m\left(x + \frac{r}{f}\right).$$

It follows from (4.3) that

$$(4.4) \quad \bar{\phi}_n(x + 1, \rho) = \rho^{-1} \bar{\phi}_n(x, \rho),$$

so that if ρ is a primitive f th root of unity, $\bar{\phi}_n(x, \rho)$ has period f . Also by means of (4.1) we readily obtain the multiplication theorem [1] valid for $k \equiv 1 \pmod{f}$

$$(4.5) \quad \sum_{r=0}^{k-1} \rho^r \phi_m\left(x + \frac{r}{k}, \rho\right) = k^{-m} \phi_m(kx, \rho)$$

and consequently

$$(4.6) \quad \sum_{r=0}^{k-1} \rho^r \bar{\phi}_m\left(x + \frac{r}{k}, \rho\right) = k^{-m} \bar{\phi}_m(kx, \rho).$$

We may now state

THEOREM 3. *Let $f > 1, n \equiv 1 \pmod{f}; m_1, \dots, m_n \geq 1, a_1, a_2, \dots, a_n$ positive integers that are relatively prime in pairs and such that $a_i \equiv 1 \pmod{f}$ for $i=1, \dots, n$; also let $k \equiv 1 \pmod{f}$. Then if $A = a_1 a_2 \dots a_n$, we have*

$$(4.7) \quad \sum_{r=0}^{kA-1} \rho^r \bar{\phi}_{m_1}\left(x_1 + \frac{r}{a_1 k}, \rho\right) \cdots \bar{\phi}_{m_n}\left(x_n + \frac{r}{a_n k}, \rho\right)$$

$$= C' \sum_{r=0}^{k-1} \rho^r \bar{\phi}_{m_1} \left(a_1 x_1 + \frac{r}{k}, \rho \right) \cdots \bar{\phi}_{m_n} \left(a_n x_n + \frac{r}{k}, \rho \right),$$

where C' is defined by (3.2).

The proof is very much like that of Theorem 2 and will be omitted. We remark that for $k = 1$, (4.7) becomes

$$(4.8) \quad \sum_{r=0}^{A-1} \rho^r \bar{\phi}_{m_1} \left(x_1 + \frac{r}{a_1}, \rho \right) \cdots \bar{\phi}_{m_n} \left(x_n + \frac{r}{a_n}, \rho \right) \\ = C' \bar{\phi}_{m_1} (a_1 x_1, \rho) \cdots \bar{\phi}_{m_n} (a_n x_n, \rho).$$

REFERENCES

1. L. Carlitz, *The multiplication formulas for the Bernoulli and Euler polynomials*, Mathematics Magazine, **27** (1953), 59-64.
2. G. Frobenius, *Über die Bernoulli'schen Zahlen und die Euler'schen Polynome*, Sitzungsberichte der Preussischen Akademie der Wissenschaften (1910), 809-847.
3. M. Mikolás, *Integral formulas of arithmetical characteristics relating to the zeta-function of Hurwitz*, Publicationes Mathematicae, **5** (1957), 44-53.
4. L. J. Mordell, *Integral formulas of arithmetical character*, Journal of the London Mathematical Society, **33** (1957), 371-375.
5. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, 1924.

DUKE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG

Stanford University
Stanford, California

R. A. BEAUMONT

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

L. J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

C. E. BURGESS

E. HEWITT

A. HORN

V. GANAPATHY IYER

R. D. JAMES

M. S. KNEBELMAN

L. NACHBIN

I. NIVEN

T. G. OSTROM

H. L. ROYDEN

M. M. SCHIFFER

E. G. STRAUS

G. SZEKERES

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA

OREGON STATE COLLEGE

UNIVERSITY OF OREGON

OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE COLLEGE

UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

CALIFORNIA RESEARCH CORPORATION

HUGHES AIRCRAFT COMPANY

SPACE TECHNOLOGY LABORATORIES

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chivoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Errett Albert Bishop, <i>A minimal boundary for function algebras</i>	629
John W. Brace, <i>The topology of almost uniform convergence</i>	643
Cecil Edmund Burgess, <i>Chainable continua and indecomposability</i>	653
L. Carlitz, <i>Multiplication formulas for products of Bernoulli and Euler polynomials</i>	661
Eckford Cohen, <i>A class of residue systems (mod r) and related arithmetical functions. II. Higher dimensional analogues</i>	667
Shaul Foguel, <i>Boolean algebras of projections of finite multiplicity</i>	681
Richard Robinson Goldberg, <i>Averages of Fourier coefficients</i>	695
Seymour Goldberg, <i>Ranges and inverses of perturbed linear operators</i>	701
Philip Hartman, <i>On functions representable as a difference of convex functions</i> ...	707
Milton Vernon Johns, Jr. and Ronald Pyke, <i>On conditional expectation and quasi-rings</i>	715
Robert Jacob Koch, <i>Arcs in partially ordered spaces</i>	723
Gregers Louis Krabbe, <i>A space of multipliers of type $L^p(-\infty, \infty)$</i>	729
John W. Lamperti and Patrick Colonel Suppes, <i>Chains of infinite order and their application to learning theory</i>	739
Edith Hirsch Luchins, <i>On radicals and continuity of homomorphisms into Banach algebras</i>	755
T. M. MacRobert, <i>Multiplication formulae for the E-functions regarded as functions of their parameters</i>	759
Michael Bahir Maschler, <i>Classes of minimal and representative domains and their kernel functions</i>	763
William Schumacher Massey, <i>On the imbeddability of the real projective spaces in Euclidean space</i>	783
Thomas Wilson Mullikin, <i>Semi-groups of class (C_0) in L_p determined by parabolic differential equations</i>	791
Steven Orey, <i>Recurrent Markov chains</i>	805
Ernest Tilden Parker, <i>On quadruply transitive groups</i>	829
Calvin R. Putnam, <i>On Toeplitz matrices, absolute continuity, and unitary equivalence</i>	837
Helmut Heinrich Schaefer, <i>On nonlinear positive operators</i>	847
Robert Seall and Marion Wetzel, <i>Some connections between continued fractions and convex sets</i>	861
Robert Steinberg, <i>Variations on a theme of Chevalley</i>	875
Olga Taussky and Hans Zassenhaus, <i>On the similarity transformation between a matrix and its transpose</i>	893
Emery Thomas, <i>The suspension of the generalized Pontrjagin cohomology operations</i>	897
Joseph L. Ullman, <i>On Tchebycheff polynomials</i>	913
Richard Steven Varga, <i>Orderings of the successive overrelaxation scheme</i>	925
Orlando Eugenio Villamayor, Sr., <i>On weak dimension of algebras</i>	941