

# Pacific Journal of Mathematics

**AVERAGES OF FOURIER COEFFICIENTS**

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We shall say the sequence  $a_n(n = 1, 2, \dots)$  is a  $p$ -sequence ( $1 \leq p < \infty$ ) if there is a function  $f \in L^p(0, \pi)$  such that

$$a_n = \int_0^\pi f(t) \cos nt \, dt \quad n = 1, 2, \dots ;$$

(i.e. the  $a_n$  are Fourier cosine coefficients of an  $L^p$  function).

A famous theorem of Hardy [1] states that if  $a_n$  is a  $p$ -sequence ( $1 \leq p < \infty$ ) and  $b_n = \frac{1}{n}(a_1 + a_2 + \dots + a_n)$ , then  $b_n$  is also a  $p$ -sequence.

In this paper we shall prove the following generalization of Hardy's theorem:

**THEOREM 1.** *Let  $\psi(x)$  be of bounded variation on  $0 \leq x \leq 1$ , and let  $1 \leq p < \infty$ . Then, if  $a_n$  is a  $p$ -sequence and*

$$b_n = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{m}{n}\right) a_m ,$$

*$b_n$  is also a  $p$ -sequence.*

Hardy's theorem is the special case  $\psi(x) = 1$  for  $0 \leq x \leq 1$ .

If the conclusion of Theorem 1 holds for each of two functions  $\psi$  it clearly holds for their difference. Hence it is sufficient to prove Theorem 1 in the case where  $\psi(x)$  is non-decreasing for  $0 \leq x \leq 1$ . Further, since any non-decreasing function may be written as the difference of two non-negative non-decreasing functions (the second of which is constant) to prove Theorem 1 it is sufficient to prove

**THEOREM 1A.** *Let  $\psi(x)$  be non-negative and non-decreasing on  $0 \leq x \leq 1$  and let  $1 \leq p < \infty$ . Then, if  $a_n$  is a  $p$ -sequence and*

$$b_n = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{m}{n}\right) a_m ,$$

*$b_n$  is also a  $p$ -sequence.*

The proof of Theorem 1A will follow a sequence of lemmas.

**LEMMA 1.** *Let  $B_i(x) = \int_0^x \cos yt \, d(y - [y])$ . Then there is an  $M > 0$*

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such that

$$|B_t(x)| \leq M \quad 0 \leq t \leq \pi; 0 \leq x < \infty .$$

The symbol  $[y]$  denotes the greatest integer not exceeding  $y$ .

*Proof.* Let  $n$  be any non-negative integer. Then for  $t > 0$

$$\int_0^n \cos yt \, dy = \frac{\sin nt}{t}$$

and

$$\int_0^n \cos yt \, d[y] = \sum_{m=1}^n \cos mt = \frac{\sin (n + 1/2)t}{2 \sin t/2} - \frac{1}{2} .$$

Hence

$$\begin{aligned} B_t(n) &= \frac{\sin nt}{t} - \frac{\sin (n + 1/2)t}{2 \sin t/2} + \frac{1}{2} \\ &= \sin nt \left( \frac{1}{t} - \frac{1}{2} \cot \frac{t}{2} \right) - \frac{\cos nt}{2} + \frac{1}{2} \end{aligned}$$

and so

$$(1) \quad |B_t(n)| \leq \left| \frac{1}{t} - \frac{1}{2} \cot \frac{t}{2} \right| + 1 \quad n = 0, 1, 2, \dots$$

The right side of (1) is bounded for  $0 < t \leq \pi$ . Thus for some  $M \geq 1$

$$(2) \quad |B_t(n)| \leq M - 1 \quad n = 0, 1, 2, \dots; 0 < t \leq \pi .$$

Now take any  $x \geq 0$  and let  $n = [x]$ . Then

$$B_t(x) = B_t(n) + \int_n^x \cos ytd(y - [y])$$

so that from (2) we have for any  $x \geq 0$

$$|B_t(x)| \leq M - 1 + \int_n^x |d(y - [y])| \leq M - 1 + x - n \leq M, 0 < t \leq \pi$$

and the proof is complete since  $B_0(x) : x - [x] \leq 1 \leq M$ .

(Henceforth we assume  $\psi(x) \geq 0$  and  $\psi(x)$  non-decreasing for  $0 \leq x \leq 1$ .)

LEMMA 2. *There is an  $M > 0$  such that*

$$\left| \int_0^n \psi \left( \frac{x}{n} \right) \cos xt \, d(x - [x]) \right| \leq M \quad 0 \leq t \leq \pi; n = 1, 2, \dots$$

*Proof.* With  $B_t(x)$  as in Lemma 1 we have

$$\int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, d(x - [x]) = \int_0^n \psi\left(\frac{x}{n}\right) dB_i(x) \\ = \psi(1)B_i(n) - \int_0^n B_i(x) d\psi\left(\frac{x}{n}\right).$$

Thus with  $M$  as in Lemma 1

$$\left| \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, d(x - [x]) \right| \leq M\psi(1) + M \int_0^n d\psi\left(\frac{x}{n}\right) \leq 2M\psi(1),$$

and the lemma is proved (with  $2M\psi(1)$  instead of  $M$ ).

LEMMA 3. *Let  $f \in L'(0, \pi)$  and let*

$$d_n = \frac{1}{n} \int_0^\pi f(t) dt \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, d(x - [x]) \quad n = 1, 2, \dots$$

Then

$$(3) \quad d_n = O\left(\frac{1}{n}\right) \quad n \rightarrow \infty$$

and hence  $d_n$  is a  $p$ -sequence for every  $p \geq 1$ .

*Proof.* By Lemma 2 there is an  $M > 0$  such that  $|d_n| \leq \frac{M}{n} \int_0^\pi |f(t)| dt$  from which (3) follows. From (3) it follows that  $\sum_{n=1}^\infty |d_n|^q < \infty$ , for every  $q > 1$ . By the Hausdorff-Young theorem and the fact that  $L^p \subseteq L^{p'}$  if  $1 \leq p' \leq p$ , this implies that  $d_n$  is a  $p$ -sequence for every  $p \geq 1$ . (See [2].)

From now on we shall write  $f \sim a_n$  as an abbreviation for  $a_n = \int_0^\pi f(t) \cos nt \, dt, n = 1, 2, \dots$

LEMMA 4. *Let  $1 \leq p < \infty, f \in L^p(0, \pi)$  and  $a(x) = \int_0^\pi f(t) \cos xt \, dt$  so that*

$$f \sim a_n = a(n).$$

Let

$$g(x) = \int_x^\pi \frac{1}{t} \psi\left(\frac{x}{t}\right) f(t) dt \quad c_n = \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) a(x) dx.$$

Then  $g \in L^p(0, \pi)$  and

$$g \sim c_n.$$

*Proof.* Since  $|g(x)| \leq \psi(1) \int_x^\pi \frac{|f(t)|}{t} dt$  it follows from the proof in [1] that  $g \in L^p$ . Also

$$\begin{aligned}
\int_0^\pi g(x) \cos nx \, dx &= \int_0^\pi \cos nx \, dx \int_x^\pi \frac{1}{t} \psi\left(\frac{x}{t}\right) f(t) \, dt \\
&= \int_0^\pi \frac{1}{t} f(t) \, dt \int_0^t \psi\left(\frac{x}{t}\right) \cos nx \, dx = \int_0^\pi f(t) \, dt \int_0^1 \psi(x) \cos nxt \, dt \\
&= \frac{1}{n} \int_0^\pi f(t) \, dt \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, dt = \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) \int_0^\pi f(t) \cos xt \, dt = c_n .
\end{aligned}$$

The changes in order of integration are valid since

$$\int_0^\pi |f(t)| \, dt \int_0^1 |\psi(x) \cos nxt| \, dx \leq \psi(1) \int_0^\pi |f(t)| \, dt < \infty .$$

(Note  $f \in L^p(0, \pi)$  since  $f \in L^p(0, \pi)$ .) Thus  $g \sim c_n$ , which is what we wished to show.

We can now establish our principal result.

*Proof of Theorem 1A.* Let  $f \in L^p(0, \pi)$  be such that  $f \sim a_n$  and let  $a(x), g(x), c_n$  be as in Lemma 4. Then

$$b_n = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{m}{n}\right) a_m = \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) a(x) \, d[x]$$

so that

$$\begin{aligned}
c_n - b_n &= \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) a(x) \, d(x - [x]) = \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) d(x - [x]) \int_0^\pi f(t) \cos xt \, dt \\
&= \frac{1}{n} \int_0^\pi f(t) \, dt \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, d(x - [x]) .
\end{aligned}$$

The last iterated integral clearly converges absolutely, justifying the change in order of integration. By Lemma 3  $c_n - b_n$  is a  $p$ -sequence. Also  $c_n$  is a  $p$ -sequence since, by Lemma 4,  $g \in L^p(0, \pi)$  and  $g \sim c_n$ . Hence  $b_n = c_n - (c_n - b_n)$  is a  $p$ -sequence and the theorem is proved.

REMARK. Note that except for the result of Lemma 1 the only properties of the cosine function used were its boundedness and the fact that  $O\left(\frac{1}{n}\right)$  is a  $p$ -sequence for all  $p \geq 1$ .

LEMMA 5. Let  $C_t(x) = \int_0^x \sin yt \, d(y - [y])$ . Then there is an  $M > 0$  such that

$$|C_t(x)| \leq M \quad 0 \leq t \leq \pi; 0 \leq x < \infty .$$

*Proof.* Let  $n$  be any non-negative integer. Then for  $t > 0$

$$\int_0^n \sin yt \, dy = \frac{1}{t} - \frac{\cos nt}{t}$$

and

$$\int_0^n \sin yt \, d[y] = \sum_{k=1}^n \sin kt = \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2}.$$

Hence

$$\begin{aligned} C_i(n) &= \frac{1}{t} - \frac{\cos nt}{t} - \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2} \\ &= (1 - \cos nt) \left( \frac{1}{t} - \frac{1}{2} \cot \frac{t}{2} \right) - \frac{\sin nt}{2}. \end{aligned}$$

The remainder of the proof follows as in Lemma 1.

In view of Lemma 5 and the remark preceding it the exact analogue of Theorem 1 for sine coefficients must hold. This we now state:

**THEOREM 2.** *Fix  $p \geq 1$ . If, for some  $f \in L^p$ ,*

$$a_n = \int_0^\pi f(t) \sin ntdt \qquad n = 1, 2, \dots,$$

and if  $b_n = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{m}{n}\right) a_m$  where  $\psi(x)$  is of bounded variation on  $0 \leq x \leq 1$  then there exists  $g \in L^p$  such that

$$b_n = \int_0^\pi g(t) \sin ntdt \qquad n = 1, 2, \dots$$

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2. A. Zygmund, *Trigonometrical Series*, Warsaw 1935 p. 190.

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