

Pacific Journal of Mathematics

A SPACE OF MULTIPLIERS OF TYPE $L^p(-\infty, \infty)$

GREGERS LOUIS KRABBE

A SPACE OF MULTIPLIERS OF TYPE $L^p(-\infty, \infty)$

GREGERS L. KRABBE

1. Introduction. Let $V(G)$ denote the set of all functions having finite variation on G . Set $G = (-\infty, \infty) = \hat{G}$, and let $V_\infty(G)$ be the Banach space of all functions in $V(G)$ which vanish at infinity. If $f \in V_\infty(G)$, then there exists a bounded linear operator $(t_p f)$ on $L^p(\hat{G})$ such that

$$(i_0) \quad (\text{Fourier transform of } (t_p f)x) = (\text{Fourier transform of } x) \cdot f$$

for all x in $L^p(\hat{G})$. This will be shown in 7.2. In the terminology of Hille [3, p. 566], functions f having property (i_0) are called "factor functions for Fourier transforms of type (L_p, L_p) ".

Suppose $1 < p < \infty$. When $f \in L^1(G) \cap V(G) \subset V_\infty(G)$, then $(t_p f)$ is a singular integral operator: for all x in $L^p(\hat{G})$ it is found that $(t_p f)x$ has the form

$$[(t_p f)x]_\lambda = \frac{1}{2\pi i} \int_{-\infty}^{\infty} x(\theta) \frac{F(\theta - \lambda)}{\theta - \lambda} d\theta \quad (\lambda \in \hat{G}),$$

where the integral is taken in the Cauchy principal value sense.

In 6.2 will be defined a set $\blacktriangle(L^p(\hat{G}))$ which contains all factor functions for Fourier transforms of type (L_p, L_p) ; the set $\blacktriangle(L^p(\hat{G}))$ is a slight extension of what Mihlin [6] calls "multipliers of Fourier integrals". We will find a number N_p such that

$$(i) \quad \text{if } f \in V_\infty(G) \text{ then } f \in \blacktriangle(L^p(\hat{G})) \text{ and } \|(t_p f)\| \leq N_p \cdot \|f\|_v,$$

where $\|f\|_v$ denotes the total variation on G of the function f . Let F_* be the mapping $\{x \rightarrow x * F\}$, where $x * F$ is the convolution of the functions x and F ;

$$[x * F]_\lambda = \int_{-\infty}^{\infty} x(\theta) \cdot F(\theta - \lambda) d\theta \quad (\lambda \in \hat{G}).$$

Let (Yf) denote the Fourier transform of the function f :

(ii) *if $f \in L^1(G) \cap V(G)$, then the transformation $(Yf)_*$ is a densely defined bounded operator, and $(t_p f)$ is its continuous linear extension to the whole space $L^p(\hat{G})$.*

Let us for a moment call $G = \{0, \pm 1, \pm 2, \dots\}$ and $\hat{G} = [0, 1]$. In

Received December 8, 1958, and in revised form February 11, 1959. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under contract No. AF 49(638)-505.

a sense, the following relations are duals of (i) and (ii), respectively:

(i') if $F \in V(\hat{G})$ then $(YF) \in \blacktriangle(L^p(\hat{G}))$ and $\|t_p(YF)\| \leq k_p \cdot \|F\|_v$

(ii') if $F \in V(\hat{G})$ then $F_* = t_p(YF)$ is a bounded operator on $L^p(\hat{G})$.

When $\hat{G} = [0, 1]$ these properties are easily verified (see 8.1). We will not¹ prove (i')-(ii') for other choices of G .

When $G = [0, 1]$, then (ii) is seen to be a theorem due to Stečkin [10]; by means of appropriate definitions, it could be shown that (i) also holds for this particular choice of G .

2. Applications. If f belongs to the class S of members of $L^1(G) \cap V(G)$ such that $(Yf) \in L^1(\hat{G})$, then $(Yf)_* = (t_p f)$ is a bounded operator defined on all of $L^p(\hat{G})$; it is interesting to compare this result with the conclusion $F_* = t_p(YF)$ of (ii'). All the classical convolution operators (Poisson, Picard, Weierstrass, Stieltjes, Dirichlet, Fejér,...etc. [7]) are of the form $(t_p f)$, where $f \in S$. See § 8.

3. Preliminaries. We assume $1 < p < \infty$ throughout, and write $G = (-\infty, \infty)$. Denote by L^0 the set of step functions with compact support. Let V be the set of all functions a defined on G and such that $\|a\|_v \neq \infty$, where $\|a\|_v$ denotes the total variation on G .

3.1 DEFINITIONS. Let V_∞ be the set of all functions a in V such that $\lim a(\theta) = 0$ whenever $|\theta| \rightarrow \infty$. We will write L^p instead of $L^p(G)$. If $\iota = 0, 1$ and $f \in L^1$, then the Fourier transforms $[_\iota Yf]$ are the functions g_ι defined by

$$(1) \quad [_\iota Yf]_\lambda = g_\iota(\lambda) = \int_{-\infty}^{\infty} \exp(2\pi i \lambda (-1)^\iota \theta) \cdot f(\theta) d\theta \quad (\lambda \in G).$$

To lighten the notation, we will write Yf for $[_1 Yf]$ and ψf for $[_0 Yf]$.

3.2 LEMMA. If $a \in L^1 \cap V$, then $a \in V_\infty$ and

$$(2) \quad \int_{-\infty}^{\infty} e^{-2\pi i \theta t} da(t) = 2\pi i \theta \cdot [Ya]_0 \quad (\theta \in G).$$

Proof. Since $a \in V$, the limits $a(\pm\infty) = \lim a(\theta)$ (when $\theta \rightarrow \pm\infty$) exist. Since $\|a\|_1 < \infty$ we have

$$(3) \quad \lim_{\theta \rightarrow \pm\infty} \int_{\theta}^{\theta+1} |a| = 0.$$

The eventuality $a(\pm\infty) \neq 0$ implies a contradiction of (3). Therefore

¹ It would be of interest to determine the validity of (i)-(ii) and (i')-(ii') in the general case where G is a connected locally compact abelian group with dual group \hat{G} . It is mainly in order to suggest such an investigation that (i')-(ii') are mentioned here.

$a(\pm\infty) = 0$, which permits the integration of (1) by parts to obtain (2).

3.3 DEFINITIONS. Let $\delta_* = (-\infty, -\delta] \cup [\delta, \infty)$ and let $(T_\delta a)x$ be the function defined by

$$(4) \quad [(T_\delta a)x]_\lambda = \int_{\delta_*} d\theta \frac{x(\lambda - \theta)}{2\pi i \theta} \int_{-\infty}^{\infty} e^{-2\pi i \theta t} da(t)$$

for all λ in G . We denote by V_1 the set of all members a of V such that, for all x in L^0 , the limit

$$[(Ta)x]_\lambda = \lim_{\delta \rightarrow 0+} [(T_\delta a)x]_\lambda$$

exists almost-everywhere on G . Let Ta be the operator $\{x \rightarrow (Ta)x\}$ defined on L^0 .

3.4 LEMMA. *If $h(\theta) = i\theta/|\theta|$, then $h \in V_1$ and Th is the restriction to L^0 of the Hilbert transformation. Moreover $\|(T_\delta h)x\|_p \leq c_p \cdot \|x\|_p$, where c_p is the norm of Th .*

Proof. This follows from the statement in [8, p. 241] that $\|(T_\delta h)x\|_p \leq \|(Th)x\|_p$. Theorem G in [1, p. 251] yields a less precise result.

3.5. LEMMA. *If $a \in L^1 \cap V$ then $a \in V_1$ and $x * [Ya] = (Ta)x$ whenever $x \in L^0$.*

Proof. Suppose $\delta > 0$. By definition

$$(x * [Ya])_\lambda = \int_{-\infty}^{\infty} d\theta \cdot x(\lambda - \theta) \cdot [Ya]_\theta = E^\delta(\lambda) + G^\delta(\lambda),$$

where

$$G^\delta(\lambda) = \int_{\delta_*} d\theta \cdot x(\lambda - \theta) \cdot [Ya]_\theta \quad (\lambda \in G),$$

while $E^\delta(\lambda)$ is the same integral over the interval $(-\delta, \delta)$. It is clear that $\lim E^\delta(\lambda) = 0$ when $\delta \rightarrow 0+$. On the other hand, $G^\delta = (T_\delta a)x$ follows immediately from (2) and (4). This concludes the proof.

3.6 LEMMA. *Suppose $a \in V_1$ and $x \in L^0$. If there exists a number k_p such that $\|(T_\delta a)x\|_p \leq k_p$ for all $\delta > 0$, then $\|(Ta)x\|_p \leq k_p$.*

Proof. Set $q = p/(p - 1)$. Observe first that

$$(5) \quad \|g\|_p = \sup \left\{ \left| \int g \cdot \varphi \right| : \varphi \in L^q \text{ and } \|\varphi\|_q \leq 1 \right\}.$$

Next, we infer from a theorem of F. Riesz ([8], p. 227 footnote 10) that the uniform boundedness of $\|(T_\delta a)x\|_p$ implies that, for all φ in L^q with $\|\varphi\|_q \leq 1$:

$$(6) \quad \int [(Ta)x] \cdot \varphi = \lim_{\delta \rightarrow 0+} \int [T_\delta a]x \cdot \varphi .$$

By (5) we have $\left| \int [(T_\delta a)x] \cdot \varphi \right| \leq k_p$; this enables us to use (6) to deduce $\left| \int [(Ta)x] \cdot \varphi \right| \leq k_p$. The conclusion is reached by another application of (5).

3.7 LEMMA. *If $a \in L^1 \cap V$ and $x \in L^p$, then*

$$\|(Ta)x\|_p \leq 2^{-1}c_p \|a\|_v \|x\|_p .$$

Proof. Suppose $\delta > 0$. Apply Fubini's theorem to (4):

$$[(T_\delta a)x]_\lambda = \int_{-\infty}^{\infty} da(t) e^{-2\pi i \lambda t} \int_{\delta_*} d\theta \frac{x(\lambda - \theta)}{2\pi i \theta} e^{2\pi i t(\lambda - \theta)} .$$

Set $x^t(\beta) = x(\beta) \exp(2\pi i t\beta)$. Keeping both (4) and 3.4 in mind, we can therefore write

$$(7) \quad [(T_\delta a)x]_\lambda = (2i)^{-1} \int_{-\infty}^{\infty} da(t) \{ e^{-2\pi i \lambda t} [(T_\delta h)x^t]_\lambda \} .$$

This implies

$$(8) \quad \|(T_\delta a)x\|_p \leq 2^{-1} \|a\|_v \sup_{t \in G} \|(T_\delta h)x^t\|_p .$$

The derivation of (8) from (7) is obtained by a standard procedure (e.g. as in [3, Lemma 21.2.1]); it rests upon (5) and requires a single application of the Fubini theorem. On the other hand, 3.4 implies that

$$\|(T_\delta h)x^t\|_p \leq c_p \cdot \|x^t\|_p \leq c_p \cdot \|x\|_p .$$

In view of (8) therefore: $\|(T_\delta a)x\|_p \leq 2^{-1}c_p \|a\|_v \|x\|_p$. Use now 3.6 to reach the conclusion.

4. The Banach space V_∞ . Let V_s denote the set of all functions in V which have compact support. The norm $\{a \rightarrow \|a\|_v\}$ makes the set $\{a \in V; a(-\infty) = 0\}$ into a Banach space V_0 . Note that $V_s \subset V_\infty \subset V_0$. Henceforth V_∞ will be given the topology of V_0 . We will write $\|a\|_\infty = \sup\{|a(\theta)|; \theta \in G\}$; it is easily checked that

$$(9) \quad \|a\|_\infty \leq \|a\|_v \quad (\text{when } a \in V_0) .$$

Let χ_n denote the characteristic function of the interval $(-n, n)$, and set $a_n = \chi_n \cdot a$.

4.1 LEMMA. If $a \in V_\infty$, then $\lim_{n \rightarrow \infty} \|a - a_n\|_v = 0$.

Proof. Suppose $f \in V$. Using the notation δ_* of 3.3, we have

$$(iii) \quad \|f\|_v = v(f; [-\delta, \delta]) + v(f; \delta_*),$$

where $v(f; I)$ denotes the total variation over I . Set $\delta = n$ and $h_n = a - a_n$; therefore $v(h_n; [-\delta, \delta]) = |a(-\delta)| + |a(\delta)|$ and $v(h_n; \delta_*) = v(a; \delta_*)$. From (iii) therefore $\|h_n\|_v = |a(-\delta)| + |a(\delta)| + v(a; \delta_*)$, and the conclusion follows by letting $\delta \rightarrow \infty$.

4.2 REMARK. The set V_s is dense in V_∞ (since 4.1 and the fact that $a_n \in V_s$).

4.3 THEOREM. The set V_∞ is a Banach space.

Proof. Since V_∞ is a metric subspace of the Banach space V_0 , it will suffice to show that V_∞ is complete. To that effect, consider a Cauchy sequence $\{g_k\}$ in V_∞ ; since $\{g_k\}$ is also in V_0 , it will converge to some function f in V_0 ; therefore $f(-\infty) = 0$ and we need only establish that $f(\infty) = 0$. From (9) we see that

$$|f(\theta) - g_k(\theta)| \leq \|f - g_k\|_v \quad (\theta \in G).$$

In view of $g_k(\infty) = 0$, the conclusion is obtained by letting $\theta \rightarrow \infty$ and $k \rightarrow \infty$.

5. The bilinear operator B_p . From 3.2 results that $V_s \subset L^1 \cap V \subset V_\infty$; it follows from 4.2 that $L^1 \cap V$ is dense in V_∞ . Consider the bilinear operator $B = \{(x, a) \rightarrow (Ta)x\}$ which maps $L^0 \times (L^1 \cap V)$ into L^p . From 3.7 we see that B is a continuous bilinear mapping of $L^0 \times (L^1 \cap V)$ into L^p . Since L^0 and $L^1 \cap V$ are dense in L^p and V_∞ , respectively, it follows that B has a (unique) continuous extension B_p to $L^p \times V_\infty$. Accordingly, if $a \in V_\infty$, then

$$(10) \quad \|B_p(x, a)\|_p \leq 2^{-1}c_p \|a\|_v \|x\|_p \quad (\text{if } x \in L^p)$$

If $a \in L^1 \cap V$, then (from 3.5)

$$(11) \quad B_p(x, a) = x * Ya \quad (\text{if } x \in L^0).$$

5.1 NOTATION. We henceforth identify functions equal almost-everywhere on G . If the sequence $\{f_n\}$ converges in the mean of order p (i.e., in the topology of L^p), then its limit will be denoted $(L^p) \lim f_n$.

5.2 LEMMA. Let $\bar{\chi}_n$ be the function defined by

$$\bar{\chi}_n(\theta) = (\sin 2\pi n\theta)/\pi\theta \quad (\theta \in G) .$$

If $f \in L^p$, then $f = (L^p) \lim f * \bar{\chi}_n$ as $n \rightarrow \infty$.

Proof. Observe that Dunford's proof [2, p. 51, Lemma 3] for the case $p = 2$ holds without alteration whenever $1 < p < \infty$.

6. The main result. Suppose $\iota = 0, 1$. When f is a locally integrable function, we set

$$(12) \quad [({}_\iota Y_p)f] = (L^p) \lim_{n \rightarrow \infty} [{}_l Y(\chi_n \cdot f)] .$$

As in 3.1, we lighten the notation by writing $Y_p f = [({}_1 Y_p)f]$ and $\Psi_p f = [({}_0 Y_p)f]$.

6.1 REMARK. If $f \in L^1$ then $[({}_\iota Y_p)f] = [{}_l Yf]$. The following definition is an extension of the one used by Mihlin ("Multipliers of Fourier integrals"²).

6.2 DEFINITION. A locally integrable function a is called a "multiplier of type L^p " if both the following conditions hold:

$$\left\{ \begin{array}{l} \text{the transform } Y_p(a \cdot [\Psi x]) \text{ exists and belongs to } L^p \text{ whenever } x \in L^0 \\ \infty \neq \sup \{ \| Y_p(a \cdot [\Psi x]) \|_p : x \in L^0 \text{ and } \| x \|_p \leq 1 \} . \end{array} \right.$$

Let $\blacktriangle(L^p)$ denote the set of all multipliers of type L^p . When $a \in \blacktriangle(L^p)$, then $(t_p a)$ is defined as the continuous extension to all of L^p of the transformation $\{x \rightarrow Y_p(a \cdot [\Psi x])\}$ defined on L^0 .

6.3 THEOREM. If $a \in V_\infty$, then $a \in \blacktriangle(L^p)$ and $(t_p a)x = B_p(x, a)$ for all x in L^p .

Proof. Note first that $a_n = (\chi_n \cdot a) \in L^1 \cap V$. Suppose $x \in L^0$. From (11) we see that

$$[B_p(x, a_n)]_\lambda = \int d\theta \cdot x(\theta) \int dt \cdot e^{-2\pi i(\lambda - \theta)t} a_n(t) \quad (\text{when } \lambda \in G) .$$

By Fubini's theorem

$$[B_p(x, a_n)]_\lambda = \int dt \cdot a_n(t) e^{-2\pi i\lambda t} [\Psi x]_t \quad (\text{for all } \lambda \text{ in } G) .$$

Or, equivalently

$$B_p(x, a_n) = Y(\chi_n \cdot a \cdot [\Psi x]) .$$

² See [6]; in that article, Mihlin gives a condition which ensures that a differentiable function be in $\blacktriangle(L^p)$.

From (10) and 4.1 we can now infer that

$$B_p(x, a) = (L^p) \lim_{n \rightarrow \infty} Y(\chi_n \cdot \{a \cdot [\Psi x]\}) .$$

From the definition (12) now results that $B_p(x, a) = Y_p(a \cdot [\Psi x])$ for all x in L^0 . This completes the proof, in view of (10) and 6.2.

7. Hille's definition. Set $q = p/(p - 1)$. The following definition is found in [3, p. 566]: a function a is said to be a *factor function for Fourier transforms of type (L_p, L_p)* if and only if

$$a \cdot [\Psi_q x] \in \{\Psi_q z : z \in L^p\}$$

wherever $x \in L^p$. This definition seems to require the restriction $p \leq 2$, since $[\Psi_q x]$ need not exist otherwise.

7.1 THEOREM. *Suppose $1 < p \leq 2$. If a is a factor function for Fourier transforms of type (L_p, L_p) , then $a \in \blacktriangle(L^p)$.*

Proof. If a is such a factor function, there exists a bounded linear mapping $(t'_p a)$ of $L^p(G)$ into itself (see [3, Theorem 21.2.1]); this operator is defined by

$$a \cdot [\Psi_q x] = \Psi_q((t'_p a)x) \quad \text{for all } x \text{ in } L^p .$$

In view of [11, 5.17], this implies

$$(13) \quad Y_p(a \cdot [\Psi_q x]) = (t'_p a)x \quad \text{for all } x \text{ in } L^p .$$

The conclusion follows from 6.1 and 6.2.

7.2 THEOREM. *Suppose $1 < p \leq 2$ and $a \in V_\infty$. Then a is a factor function for Fourier transforms of type (L_p, L_p) ; moreover,*

$$(14) \quad \Psi_q(B_p(x, a)) = a \cdot [\Psi_q x] \quad (\text{when } x \in L^p) .$$

Proof. Since $B_p(x, a) \in L^p$ when $x \in L^p$ (see §4), it will suffice to prove (14). Consider first the case $(x, a) \in L^0 \times V_s$. From (12) we see that

$$(15) \quad \Psi_q(B_p(x, a)) = (L^q) \lim_{n \rightarrow \infty} g_n ,$$

where $g_n = \Psi[\chi_n \cdot B_p(x, a)]$. From (11):

$$g_n(\lambda) = \int_{-n}^n d\theta \cdot e^{2\pi i \lambda \theta} \int d\alpha \cdot x(\alpha) [Ya]_{\theta-\alpha} \quad (\text{when } \lambda \in G) .$$

A repeated application of the Fubini theorem yields

$$g_n(\lambda) = \int dt \cdot a(t)[\Psi x]_t \int_{-n}^n d\theta \cdot e^{2\pi i(\lambda-t)\theta} \quad (\text{when } \lambda \in G).$$

In the notation of 5.2 we accordingly have

$$g_n = \{a \cdot [\Psi x]\} * \bar{\chi}_n.$$

Since $a \cdot [\Psi x]$ is in L^q , it can be inferred from 5.2 and (15) that

$$\Psi_q(B_p(x, a)) = (L^q) \lim_{n \rightarrow \infty} (\{a \cdot [\Psi x]\} * \bar{\chi}_n) = a \cdot [\Psi x].$$

Keeping $\Psi x = \Psi_q x$ in mind (see 6.1), it is clear that (14) is now proved in the case $(x, a) \in L^0 \times V_s$. Consider the bilinear operator $R = \{(x, a) \rightarrow a \cdot \Psi_q x\}$ defined on $L^p \times V_\infty$; since $\|\Psi_q z\|_q \leq \|z\|_p$, it follows that $\|R(x, a)\|_q \leq \|x\|_p \|a\|_\infty$, and from (9) results that R is a bounded bilinear mapping of $L^p \times V_\infty$ into L^q . In view of (10), this remark also shows that the bilinear operator $J = \{(x, a) \rightarrow \Psi_q(B_p(x, a))\}$ is a bounded bilinear mapping of $L^p \times V_\infty$ into L^q .

Having shown that $R(x, a) = J(x, a)$ whenever $(x, a) \in L^0 \times V_s$, the desired conclusion $R = J$ can now be inferred from the denseness of L^0 and V_s in L^p and V_∞ , respectively (see 4.2).

8. Concluding remarks. From 6.3, 3.2 and 3.5 follows that, if $f \in L^1 \cap V$ and $x \in L^p$, then $(t_p f)x = B_p(x, f) = Tf$; hence, if F is the Fourier-Stieltjes transform of f , we have (from 3.3) the relation

$$[(t_p f)x]_\lambda = \frac{1}{2\pi i} \int_{-\infty}^{\infty} x(\theta) \frac{F(\theta - \lambda)}{\theta - \lambda} d\theta \quad (\lambda \in G)$$

which was announced in the introduction. Property (ii) of the introduction follows from (11) and 6.3. If $A \in L^1$ we denote by A_{*p} the bounded operator $\{x \rightarrow x * A\}$ defined on L^p . Let S be the set of all a in $L^1 \cap V$ such that $Ya \in L^1$, and observe that $(Ya)_{*p} = (t_p a)$ when $a \in S$. Again if $a \in S$, then $A = Ya \in L^1$ and $a = \Psi A$; from [4] it is seen that the spectrum of $(t_p a)$ is the closure of the range of a .

8.1 REMARK. Set $\hat{G} = [0, 1]$ and $G = \{0, \pm 1, \pm 2, \dots\}$. We will now sketch a proof of the properties (i')-(ii') described in §1. Denote by $\|A\|_v$ the total variation of A on \hat{G} , and suppose $\|A\|_v \neq \infty$. Observe that, since $A \in L^1(\hat{G})$, we may borrow from [5, p. 10] the following conclusion: $a = YA \in \blacktriangle(L^p(\hat{G}))$ and $t_p(YA) = A_*$ is a bounded linear operator on $L^p(\hat{G})$.

This is all of (i')-(ii') except for the inequality. The main result of [5] can be stated as follows³:

³ The definition of $V_\sigma(a)$ is given in [5, p. 8].

$$(16) \quad \|t_p(a)\| \leq 2k_p \cdot V_\sigma(a) .$$

Note also that $|[YA]_n| \leq |2\pi n|^{-1} \|A\|_p$ when $n \in G$ (this is obtained by integrating by parts, as in 3.2); consequently $V_\sigma(a) = V_\sigma(YA) \leq m_p \|A\|_p$. In view of (16), the proof of the inequality in (i') is completed.

REFERENCES

1. A. P. Calderón and A. Zygmund, *Singular integrals and periodic functions*, Studia Math. **14** (1954), 249-271.
2. N. Dunford, *Spectral theory in abstract spaces and Banach algebras*, Proceedings of the symposium on spectral theory and differential problems, Stillwater, Oklahoma, (1955), 1-65.
3. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, XXXI (1957).
4. G. L. Krabbe, *Spectral invariance of convolution operators on $L^p(-\infty, \infty)$* , Duke Math. J. **25** (1958), 131-142.
5. G. L. Krabbe, *Spectra of convolution operators on L^p and rings of factor-sequences*, Quart. J. Math. Oxford Ser. 2, **8** (1957), 1-12.
6. S. G. Mihlin, *On the multipliers of Fourier integrals*, Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 701-703.
7. H. Pollard, *Integral transforms*, Duke Math. J. **13** (1946), 307-330.
8. M. Riesz, *Sur les fonctions conjuguées*, Math. Zeit. **27** (1928), 218-244.
9. ———, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires*, Acta Math. **49** (1926), 465-497.
10. S. B. Stečkin, *On bilinear forms*, Doklady Akad. Nauk SSSR (N. S.) **71** (1950), 237-240.
11. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford, 1948.

PURDUE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG
Stanford University
Stanford, California

R. A. BEAUMONT
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

L. J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
C. E. BURGESS
E. HEWITT
A. HORN

V. GANAPATHY IYER
R. D. JAMES
M. S. KNEBELMAN
L. NACHBIN

I. NIVEN
T. G. OSTROM
H. L. ROYDEN
M. M. SCHIFFER

E. G. STRAUS
G. SZEKERES
F. WOLF
K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chivoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 9, No. 3

July, 1959

Errett Albert Bishop, <i>A minimal boundary for function algebras</i>	629
John W. Brace, <i>The topology of almost uniform convergence</i>	643
Cecil Edmund Burgess, <i>Chainable continua and indecomposability</i>	653
L. Carlitz, <i>Multiplication formulas for products of Bernoulli and Euler polynomials</i>	661
Eckford Cohen, <i>A class of residue systems (mod r) and related arithmetical functions. II. Higher dimensional analogues</i>	667
Shaul Foguel, <i>Boolean algebras of projections of finite multiplicity</i>	681
Richard Robinson Goldberg, <i>Averages of Fourier coefficients</i>	695
Seymour Goldberg, <i>Ranges and inverses of perturbed linear operators</i>	701
Philip Hartman, <i>On functions representable as a difference of convex functions</i>	707
Milton Vernon Johns, Jr. and Ronald Pyke, <i>On conditional expectation and quasi-rings</i>	715
Robert Jacob Koch, <i>Arcs in partially ordered spaces</i>	723
Gregers Louis Krabbe, <i>A space of multipliers of type $L^p(-\infty, \infty)$</i>	729
John W. Lamperti and Patrick Colonel Suppes, <i>Chains of infinite order and their application to learning theory</i>	739
Edith Hirsch Luchins, <i>On radicals and continuity of homomorphisms into Banach algebras</i>	755
T. M. MacRobert, <i>Multiplication formulae for the E-functions regarded as functions of their parameters</i>	759
Michael Bahir Maschler, <i>Classes of minimal and representative domains and their kernel functions</i>	763
William Schumacher Massey, <i>On the imbeddability of the real projective spaces in Euclidean space</i>	783
Thomas Wilson Mullikin, <i>Semi-groups of class (C_0) in L_p determined by parabolic differential equations</i>	791
Steven Orey, <i>Recurrent Markov chains</i>	805
Ernest Tilden Parker, <i>On quadruply transitive groups</i>	829
Calvin R. Putnam, <i>On Toeplitz matrices, absolute continuity, and unitary equivalence</i>	837
Helmut Heinrich Schaefer, <i>On nonlinear positive operators</i>	847
Robert Seall and Marion Wetzel, <i>Some connections between continued fractions and convex sets</i>	861
Robert Steinberg, <i>Variations on a theme of Chevalley</i>	875
Olga Taussky and Hans Zassenhaus, <i>On the similarity transformation between a matrix and its transpose</i>	893
Emery Thomas, <i>The suspension of the generalized Pontrjagin cohomology operations</i>	897
Joseph L. Ullman, <i>On Tchebycheff polynomials</i>	913
Richard Steven Varga, <i>Orderings of the successive overrelaxation scheme</i>	925
Orlando Eugenio Villamayor, Sr., <i>On weak dimension of algebras</i>	941