

Pacific Journal of Mathematics

**MULTIPLICATION FORMULAE FOR THE E -FUNCTIONS
REGARDED AS FUNCTIONS OF THEIR PARAMETERS**

T. M. MACROBERT

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1. **Introduction.** The formulae to be proved are

$$\begin{aligned}
 & \sum_{i,-i} \frac{1}{i} E(p; m\alpha_r; q; m\rho_s; ze^{i\pi}) \\
 &= (2\pi)^{-\frac{1}{2}(m-1)(p-q-1)} m^{m(\sum\alpha_r - \sum\rho_s) - \frac{1}{2}(p-q-1)} \\
 (1) \quad & \times \sum_{i,-i} \frac{1}{i} E \left\{ \alpha_1, \alpha_1 + \frac{1}{m}, \dots, \alpha_1 + \frac{m-1}{m}, \dots, \alpha_p + \frac{m-1}{m} : \right. \\
 & \left. \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}, \rho_1, \dots, \rho_q + \frac{m-1}{m} : \right. \\
 & \left. \left(\frac{z}{m^{p-q-1}} \right)^m e^{i\pi} \right\},
 \end{aligned}$$

where m is a positive integer, $p > q + 1$, and $|\text{amp } z| < 1/2(p - q - 1)\pi$.
If $p \leq q + 1$, both sides vanish identically.

For all values of p and q

$$\begin{aligned}
 & E(p; m\alpha_r; q; m\rho_s; ze^{\pm i\pi}) \\
 &= (2\pi)^{-\frac{1}{2}(m-1)(p-q-1)} m^{m(\sum\alpha_r - \sum\rho_s) - \frac{1}{2}(p-q+1)} \\
 (2) \quad & \times \sum_{n=0}^{m-1} \left(\frac{m^{n-q-1}}{z} \right)^n E \left\{ \alpha_1 + \frac{n}{m}, \dots, \alpha_1 + \frac{n+m-1}{m}, \dots, \alpha_p + \frac{n+m-1}{m} : \right. \\
 & \left. \frac{n+1}{m}, \frac{n+2}{m}, \dots * \dots, \frac{n+m}{m}, \rho_1 + \frac{n}{m}, \dots, \right. \\
 & \left. \rho_q + \frac{n+m-1}{m} : \left(\frac{z}{m^{p-q-1}} \right)^m e^{\pm i\pi} \right\},
 \end{aligned}$$

the asterisk indicating that the parameter m/m is omitted.

The proof of (1) is based on the formula ([1], p. 374)

$$(3) \quad E(p; \alpha_r; q; \rho_s; z) = \frac{1}{2\pi i} \int \frac{\Gamma(\xi) \prod \Gamma(\alpha_r - \xi)}{\prod \Gamma(\rho_s - \xi)} z^\xi d\xi,$$

where the integral is taken up the η -axis, with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at

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$\alpha_1, \alpha_2, \dots, \alpha_p$ to the right of the contour. Zero and negative integral values of the α 's and ρ 's are excluded, and the α 's must not differ by integral values. The contour must be modified if $p < q + 1$; and if $p = q + 1, |z| < 1$; but we are here concerned only with the case $p > q + 1$. Then z must satisfy the condition $|\text{amp } z| < 1/2(p - q + 1)\pi$.

From (3) it follows that, if $p > q + 1, |\text{amp } z| < 1/2(p - q - 1)\pi$,

$$(4) \quad \sum_{i=1}^p \frac{1}{i} E(p; \alpha_r; q; \rho_s; ze^{i\pi}) = \frac{1}{i} \int \frac{\prod \Gamma(\alpha_r - \zeta)}{\Gamma(1 - \zeta) \prod \Gamma(\rho_s - \zeta)} z^\zeta d\zeta.$$

For, on substituting on the left from (3), a factor $(e^{i\pi\zeta} - e^{-i\pi\zeta})$ appears in the integral, and

$$\Gamma(\zeta) \sin \pi\zeta = \pi/\Gamma(1 - \zeta).$$

The three following formulae ([1], pp. 154, 406, 407) are also required.

If m is a positive integer,

$$(5) \quad \Gamma(mz) = (2\pi)^{\frac{1}{2} - \frac{1}{2}m} m^{mz - \frac{1}{2}} \Gamma(z) \Gamma\left(z + \frac{1}{m}\right) \dots \Gamma\left(z + \frac{m-1}{m}\right);$$

$$(6) \quad \int_0^\infty e^{-\lambda} \lambda^{k-1} E(p; \alpha_r; q; \rho_s; z/\lambda^m) d\lambda \\ = (2\pi)^{\frac{1}{2} - \frac{1}{2}m} m^{k - \frac{1}{2}} E(p + m; \alpha_r; q; \rho_s; z/m^m),$$

where $R(k) > 0, \alpha_{p+1+\nu} = (k + \nu)/m, \nu = 0, 1, 2, \dots, m - 1$;

$$(7) \quad \frac{1}{2\pi i} \int e^{\xi\zeta - \rho} E(p; \alpha_r; q; \rho_s; \zeta^m z) d\zeta \\ = (2\pi)^{\frac{1}{2}m - \frac{1}{2}} m^{\frac{1}{2} - \rho} E(p; \alpha_r; q + m; \rho_s; zm^m),$$

where the contour of integration starts from $-\infty$ on the ξ -axis, passes round the origin in the positive direction, and ends at $-\infty$ on the ξ -axis, $\text{amp } \zeta$ being $-\pi$ initially, and $\rho_{q+1+\nu} = (\rho + \nu)/m, \nu = 0, 1, 2, \dots, m - 1$.

2. Proofs of the formulae. On applying (4) on the left of (1) and replacing ζ by $m\zeta$ the left hand side becomes

$$\frac{m}{i} \int \frac{\pi \Gamma(m\alpha_r - m\zeta)}{\Gamma(1 - m\zeta) \pi \Gamma(m\rho_s - m\zeta)} z^{m\zeta} d\zeta.$$

Here apply (5) and get

$$\begin{aligned}
 & (2\pi)^{-\frac{1}{2}(m-1)(p-q-1)} m^{m(\sum \alpha_r - \sum \rho_s) - \frac{1}{2}(p-q-1)} \\
 & \times \frac{1}{i} \int \frac{\prod \left\{ \Gamma(\alpha_r - \zeta) \Gamma\left(\alpha_r + \frac{1}{m} - \zeta\right) \cdots \Gamma\left(\alpha_r + \frac{m-1}{m} - \zeta\right) \right\}}{\Gamma(1-\zeta) \Gamma\left(\frac{1}{m} - \zeta\right) \cdots \Gamma\left(\frac{m-1}{m} - \zeta\right) \prod \left\{ \Gamma(\rho_s - \zeta) \cdots \Gamma\left(\rho_s + \frac{m-1}{m} - \zeta\right) \right\}} \\
 & \times \left(\frac{z}{m^{p-q-1}} \right)^{m\zeta} d\zeta,
 \end{aligned}$$

and from (4), this is equal to the right hand side of (1).

Formula (2) can be obtained by showing that

$$\begin{aligned}
 E(: e^{\pm i\pi z}) &= e^{1/z} \\
 &= \sum_{n=0}^{m-1} \frac{(1/z)^n}{n!} F\left\{ ; \frac{n+1}{m}, \dots * \dots, \frac{n+m}{m}; (mz)^{-m} \right\} \\
 &= (2\pi)^{\frac{1}{2}m - \frac{1}{2}} m^{-\frac{1}{2}} \sum_{n=0}^{m-1} \left(\frac{1}{mz} \right)^n E\left\{ : \frac{n+1}{m}, \dots * \dots, \frac{n+m}{m}; e^{\pm i\pi} (mz)^m \right\},
 \end{aligned}$$

and then generalizing by employing (6) and (7).

Note 1. Ragab's formula [2]

$$\begin{aligned}
 (8) \quad & \sum_{i=-i} \frac{1}{i} \int_0^\infty e^{-pt} E\left(\alpha, \alpha + \frac{1}{m}, \dots, \alpha + \frac{m-1}{m} : : e^{i\pi} z m^{-m} / t\right) dt \\
 &= (2\pi)^{\frac{1}{2} + \frac{1}{2}m} m^{-m\alpha - \frac{1}{2}} p^{\alpha-1} z^\alpha \exp(-p^{1/m} z^{1/m}),
 \end{aligned}$$

where m is a positive integer greater than 1, p is positive, $|\text{amp } z| < 1/2(m-1)\pi$, can be derived by substituting on the left from (4), changing the order of integration, evaluating the inner integral, applying (5), replacing ζ by $\alpha - \zeta/m$, and applying (3).

Note 2. It has been pointed out by a referee that there seems to be some connection between the formulae of this paper and certain formulae of Meijer's for the G -function which are reproduced on pages 209, 210 of the first volume of Higher Transcendental Functions [McGraw Hill Book Co., 1953].

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2. F. M. Ragab, *The inverse Laplace transform of an exponential function*, New York University, Institute of Mathematical Sciences, Astia Document No. AD 133670.

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