ON THE SIMILARITY TRANSFORMATION BETWEEN A MATRIX AND ITS TRANSPOSE

OLGA TAUSKY AND HANS ZASSENHAUS
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It was observed by one of the authors that a matrix transforming a companion matrix into its transpose is symmetric. The following two questions arise:

I. Does there exist for every square matrix with coefficients in a field a non-singular symmetric matrix transforming it into its transpose?

II. Under which conditions is every matrix transforming a square matrix into its transpose symmetric?

The answer is provided by

**THEOREM 1.** For every $n \times n$ matrix $A = (a_{ik})$ with coefficients in a field $F$ there is a non-singular symmetric matrix transforming $A$ into its transpose $A^\tau$.

**THEOREM 2.** Every non-singular matrix transforming $A$ into its transpose is symmetric if and only if the minimal polynomial of $A$ is equal to its characteristic polynomial i.e. if $A$ is similar to a companion matrix.

**Proof.** Let $T = (t_{ik})$ be a solution matrix of the system $\Sigma(A)$ of the linear homogeneous equations.

1. $TA - A^\tau T = 0$
2. $T - T^\tau = 0$

The system $\Sigma(A)$ is equivalent to the system

1. $TA - A^\tau T = 0$
2. $T - T^\tau = 0$

which states that $T$ and $TA$ are symmetric. This system involves $n^2 - n$ equations and hence is of rank $n^2 - n$ at most. Thus there are at least $n$ linearly independent solutions of $\Sigma(A)$.\(^1\)

On the other hand it is well known that there is a non-singular matrix $T_0$ satisfying

$$T_0 A T_0^{-1} = A^\tau,$$

Received December 18, 1958.
This part of the proof was provided by the referee. Our own argument was more lengthy.

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From (1) we derive

\[(1a) \quad T_0^{-1}TA = AT_0^{-1}T\]

and conversely, (1a) implies (1) so that there is the linear isomorphism

\[T \to T_0^{-1}T\]

of the solution space of (1) onto the centralizer ring of the matrix \(A\).

If the minimal polynomial of \(A\) is equal to the characteristic polynomial then the centralizer of \(A\) consists only of the polynomials in \(A\) with coefficients in \(F\). In this case the solution space of (1) is of dimension \(n\). A fortiori the solution space of \(\Sigma(A)\) is at most of dimension \(n\) since the corresponding system involves more equations. Together with the inequality in the other direction it follows that the dimension of the solution space of \(\Sigma(A)\) is exactly \(n\). This implies that every solution matrix of (1) is symmetric.

If the square matrix \(A\) is arbitrary then we apply first a similarity (in the field \(F\)) which transforms it to the form

\[
B = \begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_r
\end{pmatrix}
\]

where \(A_i\) is a square matrix of the form

\[
\begin{pmatrix}
\nu A \\
L \nu A \\
\vdots \\
L \nu A
\end{pmatrix}
\]

Here \(\nu A\) is the companion matrix of the irreducible polynomial \(\nu\) which is a factor of the characteristic polynomial of \(A\) and \(L\) is the matrix with 1 in the bottom left corner and 0 elsewhere, of appropriate size (Reference 1, p. 94). The matrix \(A\) is derogatory if two blocks \(A_i\) corresponding to the same \(\nu\) appear in \(B\). Let \(A_1\) and \(A_2\) be two such blocks.

There is a non-singular matrix \(Y\) satisfying

\[Y_\nu A = \nu A^\nu Y.\]

The matrix of matrices \(V\) that has \(Y\) in the top left corner and 0 elsewhere, of appropriate size, satisfies
Consider then the matrix
\[
\begin{pmatrix}
S_1 & V \\
S_2 & \\
& \ddots \\
& & \ddots \\
& & & S_r
\end{pmatrix}
\]
where \(S_i\) is a non-singular matrix transforming \(A_i\) into \(A^T_i\). It is a non-singular non-symmetric matrix which transform \(B\) into its transpose. Thus Theorem 2 is proved.

REMARK. M. Newman pointed out to us that the product of two non-singular skew symmetric matrices \(B, C\) can always be transformed into its transpose by a non-symmetric matrix, namely
\[
B^{-1}BCB = (BC)^T = CB.
\]

Theorem 2 shows that such a product \(BC\) must be derogatory.\(^2\) This can also be shown directly in the following way:

Let \(\lambda\) be a characteristic root of \(BC\) and \(x\) a corresponding characteristic vector, then
\[
BCx = \lambda x.
\]
Since \(B\) is non-singular this implies
\[
Cx = \lambda B^{-1}x
\]
or
\[
(C - \lambda B^{-1})x = 0.
\]

Since \(B\) is a non-singular skew symmetric matrix, it follows that the degree of \(B\) and hence the degree of \(C - \lambda B^{-1}\) is even. Moreover, the skew symmetric matrix \(C - \lambda B^{-1}\) has even rank.

\(^2\) Although Newman's comment is only significant for fields of characteristic \(\neq 2\) the remainder of this section holds generally if skew symmetric is understood to mean \(T = -T^T\) and vanishing of the diagonal elements. We observe that this definition is invariant under the transformation \(T \rightarrow X^TX\). This is the transformation \(T\) undergoes when the matrix \(A\) in (1), (2) undergoes the similarity transformation \(A \rightarrow X^{-1}AX\). Since this transformation preserves linear independence, we are permitted to apply it for the purpose of finding a non 'skew symmetric' solution of \(1\), \(2\). We now extend the field of reference to include the eigenvalues of \(A\) (from the theory of homogeneous linear equations it follows that the maximal number of linear independent solutions will remain the same). It can then be observed that for a block of the Jordan canonical form of a matrix any matrix with all coefficients zero excepting the first diagonal coefficient satisfies \(1\), \(2\). Therefore
It follows that another vector $y$ exists such that also
\[(C - \lambda B^{-1})y = 0\]
and hence also
\[BCy = \lambda y.\]
This implies that $\lambda$ is a characteristic root of multiplicity at least two and with at least two corresponding vectors. The product of two general non-singular skew symmetric matrices $B$, $C$ has every characteristic root of multiplicity exactly 2. For, specialize to the case $B = C$. Then $BC$ is a symmetric matrix whose characteristic roots are the squares of the roots of $B$, hence all exactly double for a general $B$. This shows that the general $BC$ has all its characteristic roots double with two independent characteristic vectors. Such a matrix is derogatory and its characteristic polynomial is the square of its minimum polynomial.

REFERENCES


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\(^3\) This paper which is related to our investigation was pointed out to us by the referee to whom we are indebted for other useful comments.
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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chivoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
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