

# Pacific Journal of Mathematics

**ON THE SIMILARITY TRANSFORMATION BETWEEN A  
MATIRX AND ITS TRANSPOSE**

OLGA TAUSSKY AND HANS ZASSENHAUS

# ON THE SIMILARITY TRANSFORMATION BETWEEN A MATRIX AND ITS TRANSPOSE

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It was observed by one of the authors that a matrix transforming a companion matrix into its transpose is symmetric. The following two questions arise:

I. Does there exist for every square matrix with coefficients in a field a non-singular symmetric matrix transforming it into its transpose?

II. Under which conditions is every matrix transforming a square matrix into its transpose symmetric?

The answer is provided by

**THEOREM 1.** *For every  $n \times n$  matrix  $A = (\alpha_{ik})$  with coefficients in a field  $F$  there is a non-singular symmetric matrix transforming  $A$  into its transpose  $A^r$ .*

**THEOREM 2.** *Every non-singular matrix transforming  $A$  into its transpose is symmetric if and only if the minimal polynomial of  $A$  is equal to its characteristic polynomial i.e. if  $A$  is similar to a companion matrix.*

*Proof.* Let  $T = (t_{ik})$  be a solution matrix of the system  $\Sigma(A)$  of the linear homogeneous equations.

$$(1) \quad TA - A^r T = 0$$

$$(2) \quad T - T^r = 0.$$

The system  $\Sigma(A)$  is equivalent to the system

$$(3) \quad TA - A^r T^r = 0$$

$$(4) \quad T - T^r = 0$$

which states that  $T$  and  $TA$  are symmetric. This system involves  $n^2 - n$  equations and hence is of rank  $n^2 - n$  at most. Thus there are at least  $n$  linearly independent solutions of  $\Sigma(A)$ .<sup>1</sup>

On the other hand it is well known that there is a non-singular matrix  $T_0$  satisfying

$$T_0 A T_0^{-1} = A^r,$$

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This part of the proof was provided by the referee. Our own argument was more lengthy.

From (1) we derive

$$(1a) \quad T_0^{-1}TA = AT_0^{-1}T$$

and conversely, (1a) implies (1) so that there is the linear isomorphism

$$T \rightarrow T_0^{-1}T$$

of the solution space of (1) onto the centralizer ring of the matrix  $A$ .

If the minimal polynomial of  $A$  is equal to the characteristic polynomial then the centralizer of  $A$  consists only of the polynomials in  $A$  with coefficients in  $F$ . In this case the solution space of (1) is of dimension  $n$ . A fortiori the solution space of  $\Sigma(A)$  is at most of dimension  $n$  since the corresponding system involves more equations. Together with the inequality in the other direction it follows that the dimension of the solution space of  $\Sigma(A)$  is exactly  $n$ . This implies that every solution matrix of (1) is symmetric.

If the square matrix  $A$  is arbitrary then we apply first a similarity (in the field  $F$ ) which transforms it to the form

$$B = \begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & A_r \end{pmatrix}$$

where  $A_i$  is a square matrix of the form

$$\begin{pmatrix} {}_pA & & & & \\ L & {}_pA & & & \\ & L & {}_pA & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & L & {}_pA \end{pmatrix}$$

Here  ${}_pA$  is the companion matrix of the irreducible polynomial  $p$  which is a factor of the characteristic polynomial of  $A$  and  $L$  is the matrix with 1 in the bottom left corner and 0 elsewhere, of appropriate size (Reference 1, p. 94). The matrix  $A$  is derogatory if two blocks  $A_i$  corresponding to the same  $p$  appear in  $B$ . Let  $A_1$  and  $A_2$  be two such blocks.

There is a non-singular matrix  $Y$  satisfying

$$Y_pA = {}_pA^tY.$$

The matrix of matrices  $V$  that has  $Y$  in the top left corner and 0 elsewhere, of appropriate size, satisfies

$$VA_2 = A_1^t V.$$

Consider then the matrix

$$\begin{pmatrix} S_1 & V \\ & S_2 \\ & & \cdot \\ & & & \cdot \\ & & & & S_r \end{pmatrix}$$

where  $S_i$  is a non-singular matrix transforming  $A_i$  into  $A_i^t$ . It is a non-singular non-symmetric matrix which transform  $B$  into its transpose. Thus Theorem 2 is proved.

REMARK. M. Newman pointed out to us that the product of two non-singular skew symmetric matrices  $B, C$  can always be transformed into its transpose by a non-symmetric matrix, namely

$$B^{-1}BCB = (BC)^t = CB.$$

Theorem 2 shows that such a product  $BC$  must be derogatory.<sup>2</sup> This can also be shown directly in the following way:

Let  $\lambda$  be a characteristic root of  $BC$  and  $x$  a corresponding characteristic vector, then

$$BCx = \lambda x.$$

Since  $B$  is non-singular this implies

$$Cx = \lambda B^{-1}x$$

or

$$(C - \lambda B^{-1})x = 0.$$

Since  $B$  is a non-singular skew symmetric matrix, it follows that the degree of  $B$  and hence the degree of  $C - \lambda B^{-1}$  is even. Moreover, the skew symmetric matrix  $C - \lambda B^{-1}$  has even rank.

<sup>2</sup> Although Newman's comment is only significant for fields of characteristic  $\neq 2$  the remainder of this section holds generally if skew symmetric is understood to mean  $T = -T^t$  and vanishing of the diagonal elements. We observe that this definition is invariant under the transformation  $T \rightarrow X^t T X$ . This is the transformation  $T$  undergoes when the matrix  $A$  in (1), (2) undergoes the similarity transformation  $A \rightarrow X^{-1} A X$ . Since this transformation preserves linear independence, we are permitted to apply it for the purpose of finding a non 'skew symmetric' solution of (1), (2). We now extend the field of reference to include the eigenvalues of  $A$  (from the theory of homogeneous linear equations it follows that the maximal number of linear independent solutions will remain the same). It can then be observed that for a block of the Jordan canonical form of a matrix any matrix with all coefficients zero excepting the first diagonal coefficient satisfies (1), (2). Therefore

It follows that another vector  $y$  exists such that also

$$(C - \lambda B^{-1})y = 0$$

and hence also

$$BCy = \lambda y .$$

This implies that  $\lambda$  is a characteristic root of multiplicity at least two and with at least two corresponding vectors. The product of two general non-singular skew symmetric matrices  $B$ ,  $C$  has every characteristic root of multiplicity exactly 2. For, specialize to the case  $B = C$ . Then  $BC$  is a symmetric matrix whose characteristic roots are the squares of the roots of  $B$ , hence all exactly double for a general  $B$ . This shows that the general  $BC$  has all its characteristic roots double with two independent characteristic vectors. Such a matrix is derogatory and its characteristic polynomial is the square of its minimum polynomial.

#### REFERENCES

1. M. P. Drazin, *A note on skew-symmetric matrices*, Math. Gazette, **36** (1952), 253-255.
2. N. Jacobson, *Lectures on abstract algebra*, New York 1953.
3. ———, *An application of E. H. Moore's determinant of a Hermitian matrix*, Bull. Amer. Math. Soc., **45** (1939), 745-748.
4. K. Tatarkiewicz, *Sur l'orthogonalité généralisée des matrices propres*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A, **9** (1955), 5-28.<sup>3</sup>
5. A. Voss, *Symmetrische und alternierende Lösungen der Gleichung  $SX = XS'$* , S. ber, math. phys. Kl. K. B. Akademie München, **26** (1896), 273-281.

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for any matrix  $A$  we can find solutions of (1), (2) that are non 'skew-symmetric'.

<sup>3</sup> This paper which is related to our investigation was pointed out to us by the referee to whom we are indebted for other useful comments.

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