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**ON WEAK DIMENSION OF ALGEBRAS**

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**1. Introduction.** In this note we try to characterize algebras whose weak dimension is zero, i.e., algebras  $A$  which are flat  $A^e$ -modules.

In this direction, Theorems 1 and 2 give the corresponding results, for weak dimension, to known theorems for (strong) dimension. However, it seems to be more interesting to find relations between these two dimensions.

Theorem 3 gives such a relation for commutative algebras over a field. For the non-commutative case, only a weaker necessary condition is found in Theorem 5. However, in the case of algebras satisfying the descending chain condition for left ideals a complete picture of the 0-weak dimensional ones is given in Theorem 6.

Section 6 applies these results to group algebras. In [2] Auslander partially succeeded in characterizing (von Neumann) regular group algebras. However, concerning the group, he only proved the necessity of the group being torsion and the sufficiency of the local finiteness. The difference seemed to be related to the Burnside problem. Theorem 8, then, fills the gap and the problem is now completely solved.

In the last section we study some relations between weak dimensionality and semisimplicity (in the sense of Jacobson) in tensor products of algebras.

After this paper was written we received a copy of a paper by Prof. Harada on the same subject [4]. However, there is no overlapping of the main results.

We would like to express our thanks and indebtedness to Professor Rosenberg for this helpful advice and criticism.

**2. Notations and terminology.** Throughout this note we use the homological notation and terminology of [2].

Since we are dealing with algebras over a (fixed) ground ring  $K$ , all tensor products are supposed to be taken over the ground ring  $K$ , unless otherwise specifically expressed, so, we shall use  $\otimes$  for  $\otimes_K$ . Similarly, homological dimension of algebras are indicated by  $\dim A$  or  $w.\dim A$  if they are considered over  $K$ , or  $R\text{-dim } A$  (resp.  $R\text{-}w.\dim A$ ) if they are considered over another ring  $R$ .

For a ring, simple and semisimple mean simple and semisimple with minimum condition for one-sided ideals. Regular will always mean regular in the sense of von Neumann.

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Semisimplicity in the sense of Jacobson is called  $J$ -semisimplicity.

**3. Characterization of O-weak dimensional algebras.** Let  $A$  be an algebra over a commutative ring  $K$ . The dimension (resp. weak dimension) of  $A$  as an algebra is, following the classical definitions, the dimension (resp. w.dim) of  $A$  as an  $A^e$ -module, where  $A^e = A \otimes A^*$  ( $A^*$  the algebra anti-isomorphic to  $A$ ). Since  $A$  is a cyclic  $A^e$ -module, we shall start with some considerations on cyclic flat modules (i.e. cyclic modules  $M$  with  $w.\dim.M = 0$ ).

**LEMMA 1.** *Let  $R$  be any ring and  $A$  a cyclic left  $R$ -module. Then the following conditions are equivalent:*

- (a)  $A$  is  $R$ -flat.
- (b)  $\text{Tor}_1^R(R/I, A) = 0$  for every principal right ideal  $I$  in  $R$ .
- (c) If  $A = R/J$ ,  $a$  the image of  $1$  in  $A$  and  $x \in J$ , there exists  $y \in R$  such that  $xy = 0, ya = a$ .

*Proof.* (a)  $\Rightarrow$  (b) is obvious.

(b)  $\Rightarrow$  (c). Let  $x \in J$  and let  $I$  be the right ideal generated by  $x$ . According to ([3], VI, Ex. 19, p. 126) condition (b) implies  $I \cap J = IJ$ . Since  $x \in I \cap J$ , then  $x \in IJ$ , that is, there is a  $z \in J$  such that  $x = xz$ , hence  $za = 0$ , and  $y = 1 - z$  verifies  $xy = 0$  and  $ya = a$ . (c)  $\Rightarrow$  (a). Let  $B = R/I$ , for any right ideal  $I$ . If  $x \in I \cap J$ , condition (c) assures the existence of an element  $z \in J$  such that  $xz = x$ , so that  $x \in IJ$ , hence  $I \cap J = IJ$ . That is,  $\text{Tor}_1^R(B, A) = 0$  for every cyclic module  $B$  ([3], VI, Ex. 19), so  $A$  is flat ([5]).

As a consequence, we obtain

**LEMMA 2.** *Let  $A = R/J$  be a cyclic flat left- $R$ -module. If  $I$  is a finitely generated left ideal contained in  $J$ , there is a principal left ideal  $I'$  such that  $I \subseteq I' \subseteq J$ .*

*Proof.* We proceed by induction on the number of generators of  $I$ . If  $I$  has one generator,  $I = I'$ . Suppose the lemma is true if  $I$  has  $n - 1$  generators, and suppose  $x_1, \dots, x_n$  generate  $I$ . Let us call  $a$  the image of  $1$  in  $A$ . If  $x_1, x_2 \in I \subseteq J$ , then  $x_1a = x_2a = 0$  and there is an element  $y \in R$  such that  $x_1y = 0$  and  $ya = a$ , hence  $x_2ya = 0$ , and there is a  $z \in R$  such that  $x_2yz = 0$  and  $za = a$ , so  $yz a = a$ . If we call  $r = 1 - yz$  then  $x_1r = x_1, x_2r = x_2$  and  $ra = 0$ . This last condition implies  $r \in J$  and  $I \subseteq I_1 \subseteq J$  where  $I_1$  is the ideal generated by  $r, x_3, \dots, x_n$ .

From these lemmas, the following well known result may be immediately proved:

**COROLLARY 1.** *If a cyclic left module  $A = R/I$  is  $R$ -flat and  $I$  is*

finitely generated, then  $A$  is  $R$ -projective.

In fact, Lemma 2 implies  $I$  is generated by a single element, say  $x$ , and Lemma 1 assures the existence of  $y \in I$  with  $xy = x$ , hence  $R \rightarrow Ry$  is a projection of  $R$  onto  $I$  and  $I$  is a direct summand in  $R$ , so  $A$  is projective.

Now, we shall apply these results to characterize  $O$ -weak dimensional algebras and the following theorem corresponds to that one given in [3] (IX, Proc. 7.7, p, 179) for  $\dim A = O$ .

**THEOREM 1.** *In order that  $\text{w.dim } A = 0$  it is necessary and sufficient that, for every finite set  $\{a_1, \dots, a_n\}$  in  $A$ , there exists an element  $e$  in the two-sided  $A$ -module  $A \otimes A$  such that  $a_i e = e a_i (1 \leq i \leq n)$  and that, under the mapping  $x \otimes y \rightarrow xy$  the image of  $e$  in  $A$  is 1.*

*Proof.* Let  $a_1, \dots, a_n \in A$ . Suppose  $\text{w.dim } A = O$ , i.e.,  $A$  is  $A^e$ -flat. The elements  $1 \otimes a_i^* - a_i \otimes 1^*$  belong to  $J = \text{Ker}(A^e \rightarrow A)$ , then they are contained in a principal left-ideal  $I \subseteq J$ . If  $z$  is the generator of  $I$ ,  $z.1 = O(1 \in A)$ , then there is an element  $e$  such that  $e.1 = 1$  and  $ze = O$ , hence  $(1 \otimes a_i^* - a_i \otimes 1^*)e = O$  and the necessity of the conditions proved.

To prove the sufficiency, let us consider an element  $z \in J$ . Thus  $z = \sum y_i (1 \otimes a_i^* - a_i \otimes 1)(y_i \in A^e, a_i \in A)$ , so, there is an  $e \in A^e$  such that  $(1 \otimes a_i^* - a_i \otimes 1)e = O, e.1 = 1$ , hence  $ze = O$  and Lemma 1 implies  $A$  is  $A^e$ -flat.

As a consequence of Theorem 1 and [3], (IX, prop. 7.7) we obtain

**COROLLARY 2.** *If  $A$  is a finitely generated  $K$ -algebra, then  $\text{w.dim } A = 0$  if and only if  $\dim A = 0$ .*

Of course, this result may also be obtained from Corollary 1 and the fact that  $\text{Ker}(A^e \rightarrow A)$  is a left ideal generated by the set  $\{a_i \otimes 1^* - 1 \otimes a_i^*\}$ , where the  $a_i$ 's generate  $A$  as an algebra.

Now, following the same lines given by Rosenberg and Zelinsky ([9], Th. 1, p. 88) we prove

**THEOREM 2.** *Let  $A$  be a  $K$ -algebra which is free as a  $K$ -module. If  $\text{w.dim } A = O$ , then  $A$  is locally finite over  $K$ .<sup>1</sup>*

*Proof.* Let  $\{x_i\}$  be a  $K$ -basis of  $A$  and  $\{b_1, \dots, b_n\}$  be a finite subset in  $A$ . If  $B$  is the subalgebra generated by the set  $\{b_1, \dots, b_n\}$ , then, for every  $z \in B, 1 \otimes z^* - z \otimes 1 \in A^e$  is contained in the left-ideal generated by the set  $\{1 \otimes b_i^* - b_i \otimes 1\}$ .

<sup>1</sup> An algebra  $A$  over a ring  $K$  will be called locally finite if every finitely generated subalgebra is contained in a finitely generated free  $K$ -submodule of  $A$ .

Theorem 1 shows the existence of elements  $y_1, \dots, y_k$  such that  $(1 \otimes b_i^* - b_i \otimes 1) \sum_j x_j \otimes y_j^* = 0$  and  $\sum_j x_j y_j = 1$ . Thus

$$(1 \otimes z^* - z \otimes 1^*) \sum_j x_j \otimes y_j^* = 0$$

for every  $z \in B$ , that is,

$$\sum z x_j \otimes y_j = \sum x_j \otimes y_j z .$$

If we write  $zx_j$  as a linear combination of the  $x_i$ 's, this formula shows that  $y_j z$  is a linear combination of the  $y$ 's, that is,

$$y_j z = \sum_i k_{ij} y_i \quad (k_{ij} \in K)$$

hence,  $z = \sum_j x_j y_j z = \sum_{i,j} k_{ij} x_j y_i$ , and, then,  $B$  is contained in the  $K$  submodule generated by the set  $\{x_j y_i\}$

Finally, if we write the elements  $x_j y_i$  in terms of a basis, say  $\{x_k\}$ , since only a finite number of  $x_k$ 's appear in each  $x_j y_i$ ,  $B$  is contained in a finitely generated  $K$ -free  $K$ -submodule of  $A$ .

**4. Algebras over a field.** In the case  $K$  is a field, then, trivially,  $A$  is a  $K$ -free  $K$ -module and, if  $\text{w.dim } A = 0$ , the conditions of Theorems 1 and 2 must be satisfied.

The results of [3] (IX, 7.5 and 7.6), referred to weak dimension (i.e., starting from IX. 28 instead of IX. 2.8a) may be condensed, by using the equivalence between  $\text{w.gl.dim } R = 0$  and  $R$  being a (von Neumann) regular ring [5], in the following proposition:

**PROPOSITION 1.** *If  $A$  is a  $K$ -algebra over a regular ring  $K$ , then  $\text{w.dim } A = 0$  if and only if  $A^e$  is a regular ring.*

In the case of commutative algebras over a field a complete characterization of the case  $\text{w.dim } A = 0$  is obtained in the following result.

**THEOREM 3.** *Let  $A$  be a commutative algebra over a field  $K$ . Then, then following conditions are equivalent:*

- (i)  $A$  is locally separable<sup>2</sup>
- (ii)  $\text{w. dim } A = 0$
- (iii)  $A \otimes F$  is regular for every field  $F$  containing  $K$ .

*Proof.* (i)  $\Rightarrow$  (ii). Obviously, since  $A$  is locally separable, it satisfies the conditions of Theorem 1.

(ii)  $\Rightarrow$  (iii). This is a trivial consequence of the inequality

$$\text{w.gl.dim } A \otimes F \leq \text{w.gl.dim } F + \text{w.dim } A$$

obtained from the spectral sequences [3] (XVI, 5.5a p.347) and the equivalence between  $\text{w.gl.dim } R = 0$  and regularity obtained in [5].

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<sup>2</sup> An algebra  $A$  over a field  $K$  is called locally separable if every finitely generated subalgebra is contained in a (finitely generated) separable subalgebra.

(iii)  $\Rightarrow$  (i). If  $A$  is commutative and it is not locally finite over  $K$ , then there is at least one element  $x$  which is transcendental over  $K$ , hence  $A$  contains a subalgebra isomorphic with the polynomial ring  $K[x]$ .

For every polynomial  $p(x)$ , let  $0_p$  be the set of elements  $y \in A$  such that  $yp(x) = 0$ . Let  $I = \cup 0_p$ , then, trivially,  $I$  is an ideal in  $A$ , and no element of  $K(x)$  is in  $I$ ; otherwise, if  $q(x) \in I$ , there is a  $p(x)$  such that  $q(x)p(x) = 0$ , contradicting the transcendency of  $x$ .

If  $A$  is regular, for every  $p(x)$  there is an element  $z$  such that  $z(p(x))^2 = p(x)$ , hence  $1 - zp(x) \in I$  and, in  $A/I$ , the images of all  $p(x)$  have inverses, so  $A/I$  contains a subalgebra isomorphic with the field of rational functions  $K(X)$ .

Let us call  $B = A/I$ . If  $A$  is regular, then  $B$  is regular too, and, from the exactness of  $A \rightarrow B \rightarrow 0$  we obtain  $F \otimes A \rightarrow F \otimes B \rightarrow 0$  exact. Then, if  $F \otimes A$  is regular, so is  $F \otimes B$ .

Since  $B \supseteq K(X)$  and  $K(X)$  is a field,  $B$  is the direct sum of  $K(X)$ -modules isomorphic with  $K(X)$ , hence, from the fact that  $\otimes$  distributes on direct sums,  $F \otimes B$  is a direct sum of  $F \otimes K(X)$ -modules isomorphic with  $F \otimes K(X)$ . Applying now ([2], Prop. 2, p. 659), we obtain  $\text{w.gl.dim } F \otimes B \geq \text{w.gl.dim } F \otimes K(X)$ . Then, we must prove just that  $F \otimes K(X)$  is not a regular ring. In fact, if  $F$  is any field containing properly  $K$ , then  $F \otimes K(X)$  contains a subring isomorphic with  $F \otimes K[X] \approx F[X]$ , which is an integral domain, and  $F \otimes K(X)$  is the set of rational functions  $q(x)/p(x)$  with  $q(x) \in F[X]$  and  $p(x) \in K[X]$ , hence, it is an integral domain but not a field because it has no inverse for  $q(x) \in F[X]$  if  $q(x) \notin K[X]$ , thus  $F \otimes K(X)$  is not a regular ring.

Thus, condition (iii) implies  $A$  is locally finite.

Since  $A \otimes F$  is regular and commutative,  $B \otimes F$  has to be semi-simple for every finitely generated subalgebra  $B$  and every field  $F$  containing  $K$ , hence  $B$  is separable and so  $A$  is locally separable.

The result of Corollary 2 can be extended, in the case of algebras over a field, by using the following result of Kaplansky ([7], Lemma 1).

**LEMMA 3.** *If  $I$  is a countably generated left-ideal in a regular ring  $R$ , then  $\dim_r I = 0$ .*

A direct implication of this lemma is obtained in

**THEOREM 4.** *Let  $A$  be an algebra over a field  $K$ . If  $[A:K] = \aleph_0$  and  $\text{w.dim } A = 0$ , then  $\dim. A = 1$ .*

*Proof.* Since  $\text{w.dim } A = 0$  implies  $A \otimes A^*$  regular, and  $\text{Ker}(A^e \rightarrow A)$  is generated by the set  $\{x_i \otimes 1 - 1 \otimes x_i^*\}$  (where the  $x_i$ 's are generators of  $A$ ) and this set is countable, then  $\text{Ker}(A^e \rightarrow A)$  is projective. Thus,  $\dim A \leq 1$ . Since  $\dim A = 0$  implies  $[A;K]$  finite, then  $[A:K] = \aleph_0$  implies  $\dim A = 1$ .

We shall say that an algebra  $A$  is locally one dimensional if every finite set of elements is contained in a subalgebra  $B$  such that  $\dim B \leq 1$ . The following theorem approximates the result obtained in Theorem 3 for commutative algebras.

**THEOREM 5.** *Let  $A$  be an algebra over a field  $K$ . If  $w.\dim A = 0$ , then  $A$  is locally one dimensional.*

*Proof.* Let  $\{a_1, \dots, a_n\}$  be a finite set of elements in  $A$ . Let  $B_0$  be the subalgebra generated by this set. Since  $A^e$  is regular, there is an idempotent  $e_1$  such that  $(1 \otimes a_i^* - a_i \otimes 1)e_1 = 0$  and  $e_1$  is mapped onto 1 by the natural map  $\sigma: A^e \rightarrow A$  (In fact, the left ideal generated by the finite set  $\{1 \otimes a_i^* - a_i \otimes 1\}$  is generated by an idempotent  $1 - e_1$  which is mapped onto 0).

If  $x \in B$ , then  $1 \otimes x^* - x \otimes 1^* = \sum y_i (1 \otimes a_i^* - a_i \otimes 1)(y_i \in A^e)$ , thus  $(1 \otimes x^* - x \otimes 1^*)e_1 = 0$  for every  $x \in B_0$ .

Let  $\{b_1, \dots, b_m\}$  be the set of elements of  $A$  appearing in  $e_1$ , and  $B_1$  the subalgebra generated by  $\{a_1, \dots, a_n, b_1, \dots, b_m\}$ . Then, by the same arguments, there is an idempotent  $e_2 \in A^e$  such that  $(x \otimes 1 - 1 \otimes x^*)e_2 = 0$  for every  $x \in B_1$ , and  $\sigma(e_2) = 1$ .

By repeating the process we obtain a chain of subalgebras  $B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$ . If we call  $B = \cup B_i (i = 1, 2, \dots)$  then Theorem 1 implies  $w.\dim B = 0$ . In fact, for every finite subset  $\{x_1, \dots, x_h\}$  in  $B$  there is a finitely generated subalgebra  $B_k$  with  $x_i \in B_k (1 \leq i \leq h)$ , then  $e_{k+1}$  satisfies  $(x_i \otimes 1^* - 1 \otimes x_i^*)e_{k+1} = 0, \sigma(e_{k+1}) = 1$ , and  $e_{k+1} \in B^e$ .

Since  $B$  is, at most, countably generated, then Corollary 2 and Theorem 4 imply  $\dim B \leq 1$ .

**REMARK 1.** According to Proposition 1, if  $A$  is an algebra over a regular ring  $K$ ,  $w.\dim A = 0$  implies  $A^e$  is regular. Then, in this case, Theorem 4 may be expressed in the following way:

**THEOREM 4'.** *Let  $A$  be an algebra over a regular ring  $K$ . If  $A$  is denumerably generated and  $w.\dim A = 0$ , then  $\dim A \leq 1$ .*

Thus, Theorem 5 is valid for algebras over a regular ring  $K$ .

**5. Algebras with descending chain condition.** Theorem 3 shows that, for a commutative algebra over a field,  $w.\dim A = 0$  if and only if  $A$  is locally separable. We do not know whether this statement is true in the non commutative case.

In the case of algebras satisfying the descending chain condition for left-ideals, the following result, suggested to the author by Professor Rosenberg, characterizes completely the 0-w. dimensional case.

**THEOREM 6.** *Let  $A$  be an algebra over a field  $K$  satisfying the descending chain condition for left ideals. Then,  $\text{w.dim } A = 0$  if and only if:*

- (a)  $A$  is semisimple
- (b)  $A$  is locally finite over  $K$
- (c) The center  $C$  of  $A$  is locally separable.

*Proof.* If  $\text{w.dim } A = 0$ , condition (a) follows from the regularity of  $A$ , (b) from Theorem 1 and (c) from Theorem 3.

Suppose, now  $A$  satisfies (a), (b) and (c). Since  $A$  is semisimple, it is a direct sum of (a finite number of) simple algebras  $S_i$  satisfying conditions (b) and (c), and, because of the direct sum decomposition,  $\text{w.dim } A = \max(\text{w.dim } S_i)$ . Since each  $S_i$  is a matrix ring over a division algebra  $D_i$  satisfying (b) and (c) and  $\text{w.dim } S_i = \text{w.dim } D_i$ , it will be enough to prove the sufficiency of these conditions for division algebras.

Let  $A$  be a division algebra. Condition (c) implies  $\text{w.dim } C = 0$ . According to the sub-additivity of the dimension ([9], Th. 5, p.93) we have  $\text{w.dim } A \leq \text{w.dim } C + C\text{-w.dim } A$ , then, it is sufficient to prove that  $A \otimes_C A^*$  is regular. This is so if  $A \otimes_C S^*$  is regular for every finitely generated subalgebra  $S$  of  $A$ .

Since  $A^*$  is locally finite and  $S^*$  finitely generated, then  $[S^*: C] < \infty$  and  $S^*$  is a division ring. Thus  $A \otimes_C S^*$  satisfies the descending chain condition. Since  $A$  is central simple and  $S^*$  simple (because now we are considering  $A$  and  $S$  as algebras over  $C$ ), then  $A \otimes_C S^*$  is simple, hence regular, and the theorem is proved.

**6. Group algebras.** In [2], Auslander studies necessary and sufficient conditions for a group  $G$  and a ring  $K$  to obtain (von Neumann) regular group algebras  $K(G)$ . He proved the necessity of  $G$  being a torsion group and the sufficiency of  $G$  being locally finite, besides the conditions on  $K$ .

In Theorem 8 we prove the necessity of the local finiteness, and then regular group algebras are completely characterized.

A similar difference existed between Theorem 3 and 4 in [8], but by direct application of Theorem 2 we fill the gap obtaining the following result.

**THEOREM 7.** *Let  $G$  be a group,  $S$  a subgroup contained in the center of  $G$  and  $K$  any commutative ring, then  $K(S)\text{-w.dim } K(G) = 0$  if and only if  $G/S$  is locally finite and  $K$  is uniquely divisible by the order of each element in  $G/S$ .*

In fact, the local finiteness of  $K(G)$  as a  $K(S)$ -algebra implies the local finiteness of  $G/S$ .



**THEOREM 8.** *Let  $G$  be a group and  $K$  any commutative ring. Then  $K(G)$  is regular if and only if  $G$  is locally finite and  $K$  is a regular ring uniquely divisible by the order of each element in  $G$ .*

*Proof.* A trivial modification in the proof of ([3] X. 6.1) may be used to prove

$$\text{w.dim } K(G) = \text{r.w.dim}_{K(G)} K .$$

Thus,  $K(G)$  regular implies  $\text{w.dim } K(G) = 0$  and Theorem 2 implies  $G$  is locally finite.

The remaining part of the proof follows from Auslander's result. It also may be seen as a special case of Theorem 7.

**REMARK 2.** The proof of the necessity of the local finiteness of  $G$  for a group algebra  $K(G)$  to be regular does not need all the homological machinery. In fact, it follows immediately from the following lemma:

**LEMMA 4.** *Let  $K(G)$  be the group algebra generated by a group  $G$  over any commutative ring  $K$  and  $g_1, \dots, g_n$  be elements of  $G$ . Then the subgroup  $S$  generated by  $\{g_1, \dots, g_n\}$  is finite if and only if there is an element  $x \in K(G)$  such that  $(1 - g_i)x = 0$  ( $1 \leq i \leq n$ ). If this is the case,  $x = sy$ , where  $s$  is the sum of all elements in  $S$ .*

*Proof.* If  $S$  is finite, the sum  $s$  of all elements in  $S$  satisfies the equations  $(1 - g_i)x = 0$ , and so every product  $sy$ .

Conversely, suppose  $(1 - g_i)x = 0$  ( $1 \leq i \leq n$ ). Thus,  $x = g_1x = \dots = g_nx$ . Since every  $f \in S$  is a product of powers of the  $g_i$ 's, then  $fx = x$ . Let  $x = \sum_1^m k_j h_j$  ( $h_j \in G$ ). For every  $f \in S$ ,  $fx = x$  implies  $x$  has a term  $k_1 f h_1$  and so all elements of  $S$  appear multiplied by  $k_1 h_1$ , hence  $S$  is finite (because  $x$  is a finite sum). Furthermore, we obtained  $x = k_1 s h_1 + x'$ , with  $(1 - g_i)x' = 0$ . By induction on the number of terms in  $x$  we obtain the last result.

A complete proof of Theorem 8 may be obtained as follows: Suppose  $K(G)$  is regular. Then the ring homomorphism  $\sigma: K(G) \rightarrow K$  defined by the group homomorphism  $G \rightarrow \{1\}$  implies  $K$  is regular.

If  $K(G)$  is regular, every finitely generated proper left ideal is a direct summand, hence it is annihilated, on the right, by a non-zero element  $x \in K(G)$ . Since all  $1 - g$  are in  $\text{Ker } \sigma$ , every finite set generates a proper left ideal, and so the previous lemma implies  $G$  is locally finite.

Suppose  $g \in G$  has order  $n$ . By Lemma 1 there is an element  $x$  with  $\sigma(x) = 1$  and  $(1 - g)x = 0$ , hence Lemma 4 implies  $x = sy$  ( $s = \sum g^i$ ) and so  $\sigma(x) = \sigma(s)\sigma(y) = n\sigma(y) = 1$ , hence  $n$  has an inverse in  $K$

and the necessity of the conditions is proved.

Suppose, now,  $K$  and  $G$  satisfy the conditions of the theorem. Let  $x = \sum k_i g_i \in \text{Ker } \sigma$ , so,  $x = \sum k_i (g_i - 1)$ . Let  $S$  be the subgroup generated by  $g_1, \dots, g_n$ ,  $m$  its order and  $s$  the sum of all elements in  $S$ . Since  $m$  has an inverse  $m^{-1}$  in  $K$ , then  $y = m^{-1}s$  satisfies  $xy = O$ ,  $\sigma(y) = 1$ , and Lemma 1 implies  $K$  is  $K(G)$ -flat. So  $\text{w.dim } K(G) = \text{w.dim}_{K(G)} K = O$  ([3] X, 6.2) and  $K(G)$  is regular.

**7. Weak dimension and Jacobson semisimplicity.** A ring will be called  $J$ -semisimple if its Jacobson radical is  $(O)$ .

If  $T$  is a ring and  $M$  a left- $T$ -module, then the ring  $\text{Hom}_T(M, M)$ , with the operations defined in the classical way is a topological ring by defining the finite topology induced by  $M$ . ([6], Ch. IV).

If we are in the situation  $S \subseteq R \subseteq \text{Hom}_T(M, M)$ , where  $S$  and  $R$  are rings, we shall say  $S$  is dense in  $R$  if it is so in the finite topology induced by  $\text{Hom}_T(M, M)$ .

In this section we shall prove the following theorem:

**THEOREM 9.** *Let  $A$  be a  $K$ -projective  $K$ -algebra. If  $B$  is  $J$ -semisimple  $K$ -algebra and  $\text{w.dim } A = 0$ , then  $A \otimes B$  is  $J$ -semisimple.*

Before proving the Theorem we shall state the following lemmas:

**LEMMA 5.** *Let  $T$  be any ring and  $M$  a left- $T$ -module. If  $S, R$  are rings such that  $S \subseteq R \subseteq \text{Hom}_T(M, M)$ ,  $R$  is regular and  $S$  is dense in  $R$ , then  $S$  is  $J$ -semisimple.<sup>3</sup>*

*Proof.* Let  $x \in S$ . Since  $R$  is regular, there are elements  $y, z \in R$ ,  $z \neq O$ , such that  $z(1 - xy) = O$ . Since  $R \subseteq \text{Hom}_T(M, M)$ , there is at least one  $m \in M$  such that  $m^z \neq O$  and  $m^{z(1-xy)} = O$ , that is, there exists an  $n \in M (n = m^z)$  such that  $n \neq O$  and  $n = n^{xy}$ . Now, we have  $x \in S$ ,  $y \in R$ ,  $(n^x)^y = n$ , and  $S$  is dense in  $R$ , then there is an  $u \in S$  such that  $(n^x)^u = n$ , that is,  $n^{1-xu} = O$ , and  $1 - xu$  can not have an inverse in  $S$ , so  $xu$  is not quasi-regular, and  $S$  is  $J$ -semisimple.

**LEMMA 6.** *If  $A$  is a  $K$ -projective  $K$ -algebra and  $B$  is a  $K$ -algebra which is a subdirect sum of  $K$ -algebras  $P_i$ , then  $A \otimes B$  is a subdirect sum of  $A \otimes P_i$ .*

*Proof.*  $B$  is a subdirect sum of  $P_i$ 's if and only if the sequences  $B \rightarrow P_i \rightarrow 0$  and  $0 \rightarrow B \rightarrow \prod P_i$  are exact.

---

<sup>3</sup> The conditions of the lemma are, evidently, stronger than those which are really needed in the proof. In fact, we only need  $S$  to be 1-fold transitive in  $R$  and  $R$   $J$ -semisimple in which, for every element  $x$  there is an  $y$  such that  $1-xy$  has a left annihilator.

It may be seen that, if  $S$  is commutative, the conditions of Lemma 5 are necessary.

Now from the exact sequence  $B \rightarrow P_i \rightarrow 0$  we obtain  $A \otimes B \rightarrow A \otimes P_i \rightarrow 0$  exact. We need only to prove the exactness of  $0 \rightarrow A \otimes B \rightarrow \coprod (A \otimes P_i)$ .

Since  $A$  is  $K$ -projective, we have, from  $0 \rightarrow B \rightarrow \coprod P_i$ , the exact sequence  $0 \rightarrow A \otimes B \rightarrow A \otimes \coprod P_i$ . We have ([3], Ex. II. 2, 31) a natural homomorphism

$$A \otimes \coprod P_i \rightarrow \coprod (A \otimes P_i)$$

which is, trivially, a monomorphism if  $A = K$ . Since  $\otimes$  commutes with direct sums, it is a monomorphism if  $A$  is  $K$ -free and, a posteriori, if  $A$  is  $K$ -projective. Then the composite map gives the exact sequence

$$0 \rightarrow A \otimes B \rightarrow \coprod (A \otimes P_i)$$

and the lemma is proved.

*Proof of the theorem.* Since  $B$ , being semisimple, is a subdirect sum of primitive rings  $P_i$ , then the previous lemma implies that  $A \otimes B$  is a subdirect sum of rings  $A \otimes P_i$ ; then, to prove the theorem it is sufficient to show that the rings  $A \otimes P_i$  are  $J$ -semisimple. Now, since  $P_i$  is primitive, it is dense in a ring of linear transformations, that is,  $P_i \subseteq R_i = \text{Hom}_{F_i}(M_i, M_i)$  where the rings  $R_i$  are regular and the  $P_i$ 's are dense in the  $R_i$ 's. Since  $A$  is  $K$ -projective, we may apply the spectral sequences [3] (XVI, 5a, p. 347) and then,  $R_i$  regular and  $\text{w.dim } A = 0$  give  $A \otimes R_i$  regular.

If we show the inclusion  $A \otimes P_i \subseteq A \otimes R_i \subseteq \text{Hom}_{A \otimes F_i}(A \otimes M_i, A \otimes M_i)$  and the density of  $A \otimes P_i$  into  $A \otimes R_i$ , Lemma 5 completes the proof of the theorem.

Since  $A$  is  $K$ -projective, we have the exact sequence  $0 \rightarrow A \otimes S_i \rightarrow A \otimes R_i = A \otimes \text{Ho}_F \text{m}(M_i, M_i)$ .

If  $A$  is  $K$ -free, the natural mapping  $A \otimes \text{Hom}_F(M_i, M_i) \rightarrow \text{Hom}_F(M, A \otimes M_i)$  is the natural mapping of a direct sum into a direct product, which is a monomorphism. Since  $A$ , being projective, is a direct summand of a free module, and since both  $\otimes$  and  $\text{Hom}$  commute with finite direct sums, then the given mapping is also a monomorphism.

From the natural isomorphism  $\text{Hom}_F(M_i, A \otimes M_i) \approx \text{Hom}_{A \otimes F}(A \otimes M_i, A \otimes M_i)$  we obtain the inclusions.

$$A \otimes S_i \subseteq A \otimes R_i \subseteq \text{Hom}_{A \otimes F_i}(A \otimes M_i, A \otimes M_i)$$

Let  $x \in A \otimes R_i$ , then  $x = \sum a_j \otimes r_j$ , and  $v_k = \sum_i b_{kl} \otimes m_{kl}$  ( $b_{kl} \in A, m_{kl} \in M_i$ ) be a finite set of elements in  $A \otimes M_i$ . Then  $x(v_k) = \sum_{i,j} a_j b_{kl} \otimes r_j(m_{kl})$ . Since the set  $\{m_{kl}\}$  is finite and  $B_i$  is dense in  $R_i$ , for each  $r_j$  there is an  $s_j \in B_i$  such that  $r_j(m_{kl}) = s_j(m_{kl})$ , then  $y = \sum a_j \otimes s_j \in A \otimes P_i$  and  $y(v_k) = x(v_k)$ , so  $A \otimes P_i$  is dense in  $A \otimes R_i$ .

As a consequence of this theorem we can state the following corollary:

**COROLLARY 3.** *If  $A, B$  are algebras over a field  $K$  and  $w.\dim B = 0$ , Then  $J(A \otimes B) = J(A) \otimes B$ . (We call  $J(R)$  the Jacobson radical of a ring  $R$ ).*

In fact, since  $A/J(A)$  is semisimple, from the exact sequence

$$0 \rightarrow J(A) \otimes B \rightarrow A \otimes B \rightarrow (A/J(A)) \otimes B \rightarrow 0$$

we obtain  $J(A) \otimes B \supseteq J(A \otimes B)$ .

From Theorem 2 and ([6], V. 14, Th. 1, p 123) it follows that every element in  $J(A) \otimes B$  is quasi-regular, so it is a radical ideal in  $A \otimes B$ . Thus  $J(A) \otimes B \subseteq J(A \otimes B)$  and the corollary is proved.

This result generalizes ([1], Th. 1).

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