

# Pacific Journal of Mathematics

**A NOTE ON KATO'S UNIQUENESS CRITERION FOR  
SCHRÖDINGER OPERATOR SELF-ADJOINT EXTENSIONS**

FRANK HERBERT BROWNELL, III

# A NOTE ON KATO'S UNIQUENESS CRITERION FOR SCHRÖDINGER OPERATOR SELF-ADJOINT EXTENSIONS

F. H. BROWNELL

**1. Introduction.** Kato [2] has shown local square integrability with boundedness at  $\infty$  of the potential coefficient function to be a sufficient condition for the Schrödinger operator in  $L_2(R_n)$  to have a unique self-adjoint extension in case dimension  $n = 3$ . His statement is for  $n = 3p$ , thus with  $p$  factors  $R_3$ , but with the condition on  $V$  stated separately for each  $R_3$  factor as is natural for application to quantum mechanics; this in essence amounts to  $n = 3$  from our standpoint. Using the Young-Titchmarsh theorem on Fourier transforms, we generalize Kato's argument to general dimension  $n \geq 1$ . We show the connection of the resulting criterion with our earlier construction [1] of a self-adjoint extension as the inverse of a modified Green function integral operator. We also give a variational characterization of the spectrum here.

**2. Uniqueness condition.** Let  $V(\mathbf{x})$  be a given, real-valued, measurable function over  $\mathbf{x} \in R_n$ , euclidean  $n$ -space. We consider the following additional conditions upon  $V$ , using the notation  $(\mathbf{x} \cdot \mathbf{y}) = \sum_{j=1}^n x_j y_j$  and  $|\mathbf{x}| = \sqrt{(\mathbf{x} \cdot \mathbf{x})}$  for  $\mathbf{x}$  and  $\mathbf{y} \in R_n$ , and also denoting  $n$  dimensional Lebesgue measure on  $R_n$  by  $\mu_n$ .

CONDITION I. For some  $b < +\infty$  let  $V(\mathbf{x})$  be essentially bounded ( $A = [\text{ess sup } |V(\mathbf{x})|] < +\infty$ ) over  $\{\mathbf{x} \in R_n \mid |\mathbf{x}| \geq b\}$ , and let

$$(1) \quad \int_{\{\mathbf{x} \mid |\mathbf{x}| \leq b\}} |V(\mathbf{x})|^{(1/2)(n+\rho)} d\mu_n(\mathbf{x}) = M_\rho < +\infty$$

for some  $\rho > 0$  satisfying also  $n + \rho \geq 2$ .

CONDITION II. Let  $V(\mathbf{x})$  satisfy Condition I with in addition  $n + \rho = 4$  in (1) if dimension  $n < 4$ .

Condition II is our generalization of Kato's uniqueness criterion, our following Theorem T. 1 in the special case  $n = 3$  thus being due to Kato [2]. Following Kato, we define  $\mathcal{D}_1 \subseteq L_2(R_n)$  as the linear manifold of Hermite functions, polynomials in the coordinates  $x_j$  multiplied by  $\exp(-1/2|\mathbf{x}|^2)$ . Assuming Condition II), clearly the pointwise product  $Vu \in L_2(R_n)$  for all  $u \in \mathcal{D}_1$ . Hence

---

Received December 30, 1958, amalgamation with addendum May 29, 1959. This work was supported by an Office of Naval Research Contract, and reproduction in whole or in part by U.S. Federal agencies is permitted.

$$(2) \quad [H_1 u](\mathbf{x}) = -\nabla^2 u(\mathbf{x}) + V(\mathbf{x})u(\mathbf{x})$$

with  $\nabla^2 = \sum_{j=1}^n (\partial^2/\partial x_j^2)$  the Laplacian, defines  $H_1$  as a linear operator in  $L_2(R_n)$  with dense domain  $\mathcal{D}_1$ . Also the easily established Green's identity for  $u$  and  $w \in \mathcal{D}_1$  shows that  $H_1$  is symmetric (see [3], p. 28-41, p. 48-50 for terminology and theorems used hereafter).

Next for  $u \in L_2(R_n)$  we have existent (see [4]) the Fourier-Plancherel transform  $\hat{u} \in L_2(R_n)$  defined by

$$(3) \quad \hat{u}(\mathbf{y}) = \lim_{N \rightarrow \infty} \left( \frac{1}{2\pi} \right)^{n/2} \int_{\{\mathbf{x} \mid |\mathbf{x}| \leq N\}} e^{-i(\mathbf{x} \cdot \mathbf{y})} u(\mathbf{x}) d\mu_n(\mathbf{x}),$$

with the limit in the  $L_2(R_n)$  norm sense over  $\mathbf{y} \in R_n$ ; similarly

$$(4) \quad u(\mathbf{x}) = \lim_{N \rightarrow \infty} \left( \frac{1}{2\pi} \right)^{n/2} \int_{\{\mathbf{y} \mid |\mathbf{y}| \leq N\}} e^{i(\mathbf{x} \cdot \mathbf{y})} \hat{u}(\mathbf{y}) d\mu_n(\mathbf{y}),$$

with the limit also in the  $L_2(R_n)$  norm sense. In terms of (3) and (4), define  $\mathcal{D}$  as the set of  $u \in L_2(R_n)$  such that  $|\mathbf{y}|^2 \hat{u}(\mathbf{y})$  is also in  $L_2(R_n)$  over  $\mathbf{y}$ . Define  $T$  as a linear operation in  $L_2(R_n)$  with domain  $\mathcal{D}$  by  $Tu = w$ ,  $\hat{w}(\mathbf{y}) = |\mathbf{y}|^2 \hat{u}(\mathbf{y})$  for  $u \in \mathcal{D}$ ,  $w \in L_2(R_n)$  existing uniquely for such  $u$  since (3) and (4) define a unitary operator and its inverse on  $L_2(R_n)$ .

We may now state the main theorem of this section as follows. Actually, since Condition II will be seen at the end of the next section to imply Condition S stated there, this theorem is a consequence of Stummel's theorem ([5], Th 4.2), p. 171), except for an awkward but essentially trivial change of basic domain. Also our proof is rather different, being much closer to Kato's original argument. See also [6].

**THEOREM T.1.** *Let  $V$  satisfy condition II. Then the pointwise product  $[Vu](\mathbf{x}) = V(\mathbf{x})u(\mathbf{x})$  has  $Vu \in L_2(R_n)$  for  $u \in \mathcal{D}$ , and  $Hu = Tu + Vu$  for  $u \in \mathcal{D}$  has  $H$  to be a self-adjoint operator in  $L_2(R_n)$  with dense domain  $\mathcal{D}$ . Furthermore,  $\mathcal{D}_1 \subseteq \mathcal{D}$ ,  $H_1 \subseteq H$ , and  $H$  is the unique self-adjoint extension of  $H_1$ .*

Here  $\mathcal{D}_1 \subseteq \mathcal{D}$ , and hence  $\mathcal{D}$  is dense, follows clearly from the fact ([4], p. 81, Theorem 57) that  $S^*(\mathcal{D}_1) \subseteq \mathcal{D}_1$ , where  $S$  denotes the unitary operator from  $L_2(R_n)$  onto itself given by (4),  $S\hat{u} = u$ , and where  $S^*u = \hat{u}$  in (3) represents the adjoint and inverse  $S^*$ . Thus  $T_1u = -V^2u$  for  $u \in \mathcal{D}_1$  has  $T_1u = Tu$  for  $u \in \mathcal{D}_1$  from  $[S^*(T_1u)](\mathbf{y}) = |\mathbf{y}|^2 \hat{u}(\mathbf{y})$  by integration by parts; hence  $T_1 \subseteq T$ . Thus  $H_1 \subseteq H$  follows from the following lemma (Lemma 4 of Kato [2]), which represents the heart of our argument.

LEMMA T. 2. *Let  $V$  satisfy Condition II. Then for  $u \in \mathcal{D}$  follows both  $Vu \in L_2(R_n)$  and the  $L_2(R_n)$  norm inequality*

$$(5) \quad \|Vu\| \leq \alpha \|Tu\| + \beta \|u\|$$

for some  $\alpha$  and  $\beta$  positive and finite, for which  $\alpha$  may be chosen as small as desired with  $\beta$  depending on  $\alpha$ .

To prove this lemma, we will first establish that  $\mathcal{D} \subseteq L_{r'}(R_n)$  with  $r' = 2(n + \rho)/(n - 4 + \rho)$  and  $\rho > 0$  given in Condition II if dimension  $n \geq 4$ , and that  $\mathcal{D} \subseteq L_\infty(R_n)$  if  $n = 1, 2$ , or  $3$ . For this purpose we start, for  $u \in \mathcal{D}$  and arbitrary  $\omega > 0$  and with  $\rho > 0$  as in Condition II, with the Schwarz-Hölder estimate

$$\begin{aligned} (6) \quad & \int_{R_n} |\hat{u}(\mathbf{y})|^{2(n+\rho)/(n+\rho+4)} d\mu_n(\mathbf{y}) \\ & \leq \left[ \int_{R_n} |\hat{u}(\mathbf{y})|^2 (1 + \omega^4 |\mathbf{y}|^4) d\mu_n(\mathbf{y}) \right]^{1/p} \left[ \int_{R_n} (1 + \omega^4 |\mathbf{y}|^4)^{-(p'/p)} d\mu_n(\mathbf{y}) \right]^{1/p'} \\ & = [\|u\|^2 + \omega^4 \|Tu\|^2]^{1/p} \left[ \sigma_n \int_0^\infty \frac{r^{n-1}}{(1 + \omega^4 r^4)^{(n+\rho)/4}} dr \right]^{1/p'} \\ & = [\|u\|^2 + \omega^4 \|Tu\|^2]^{1/p} C_{n,\rho} \omega^{-\langle n/p' \rangle} \\ & = C_{n,\rho} [\omega^{-\langle 4n/n+\rho \rangle} \|u\|^2 + \omega^{4p/(n+\rho)} \|Tu\|^2]^{1/p} \end{aligned}$$

where

$$C_{n,\rho} = \left[ \sigma_n \int_0^\infty \frac{t^{n-1}}{(1+t^4)^{(n+\rho)/4}} dt \right]^{1/p'} < +\infty,$$

where  $\sigma_n = 2\pi^{n/2}[\Gamma(n/2)]^{-1}$  is  $n - 1$  dimensional ‘‘area’’ measure of the unit spherical shell in  $R_n$ , where  $1/p + 1/p' = 1$  with  $2 = p[2(n + \rho)/(n + \rho + 4)]$  and thus  $1 < p = 1 + 4/(n + \rho) \leq 2$ ,  $p'/p = 1/(p - 1) = (n + \rho)/4$ ,  $-np/p' = -4n/(n + \rho)$ , and  $4 - np/p' = 4\rho/(n + \rho)$ .

Now if dimension  $n = 1, 2$ , or  $3$ , then  $n + \rho = 4$  in Condition II and (6) yields for  $u \in \mathcal{D}$

$$\begin{aligned} (7) \quad & \left( \operatorname{ess\,sup}_{\mathbf{x} \in R_n} |u(\mathbf{x})| \right) \leq \left( \frac{1}{2\pi} \right)^{n/2} \int_{R_n} |\hat{u}(\mathbf{y})| d\mu_n(\mathbf{y}) \\ & \leq \left( \frac{1}{2\pi} \right)^{n/2} C_{n,\rho} [\omega^{-n} \|u\|^2 + \omega^{4-n} \|Tu\|^2]^{1/2} \end{aligned}$$

using also (4) with convergence almost ( $\mu_n$ ) everywhere for a subsequence from  $L_2(R_n)$  norm convergence.

Now if dimension  $n \geq 4$ , then in (6) define  $r = 2(n + \rho)/(n + \rho + 4) = 2/p$ , and hence  $1 < r < 2$  from  $1 < p = 1 + 4/(n + \rho) < 2$ . Now  $1/r + 1/r' = 1$  has

$$r' = \frac{1}{1 - \frac{1}{r}} = \frac{1}{1 - \left(\frac{1}{2} + \frac{2}{n + \rho}\right)} = \frac{2(n + \rho)}{n - 4 + \rho}.$$

Hence the Young-Hausdorff-Titchmarsh theorem ([4], Theorem 74), p. 96), generalizing with negligible changes in proof from  $R_1$  to  $R_n$ , using subsequences convergent almost everywhere to show that the known existent  $L_2(R_n)$  and  $L_{r'}(R_n)$  norm limits in (4) must agree, yields in (6) for  $u \in \mathcal{D}$  if  $n \geq 4$

$$\begin{aligned} (8) \quad & \left[ \int_{R_n} |u(\mathbf{x})|^{r'} d\mu_n(\mathbf{x}) \right]^{1/r'} \\ & \leq \left(\frac{1}{2\pi}\right)^{n(1/2-1/r')} \left[ \int_{R_n} R_n |\hat{u}(\mathbf{y})|^r d\mu_n(\mathbf{y}) \right]^{1/r} \\ & \leq \left(\frac{1}{2\pi}\right)^{n(1/2-1/r')} (c_{n,\rho})^{1/r} [\omega^{-4n/(n+\rho)} \|u\|^2 + \omega^{4\rho/(n+\rho)} \|Tu\|^2]^{1/2} \end{aligned}$$

Thus we see if dimension  $n = 1, 2,$  or  $3$  that (7) with Condition II,  $n + \rho = 4$ , yields for  $u \in \mathcal{D}$

$$(9) \quad \|Vu\|^2 \leq \left(\frac{1}{2\pi}\right)^n (c_{n,\rho})^2 M_\rho [\omega^{4-n} \|Tu\|^2 + \omega^{-n} \|u\|^2 + A^2 \|u\|^2]$$

over all  $\omega > 0$ . Thus, since  $\sqrt{|a|^2 + |b|^2} \leq |a| + |b|$ , (5) follows with  $\alpha$  arbitrarily small as desired for Lemma T. 2, since  $4 - n \geq 1$  here.

If dimension  $n \geq 4$ , then we use (8), Condition II, and over the  $|\mathbf{x}| \leq b$  portion of the integral a Schwarz-Hölder estimate with  $2\tilde{r} = r' = 2(n + \rho)/(n - 4 + \rho) > 2$  from  $1 < r < 2, 1/\tilde{r} + 1/\tilde{r}' = 1$ , and thus

$$2\tilde{r} = \frac{2}{1 - \left(\frac{n - 4 + \rho}{n + \rho}\right)} = \frac{2(n + \rho)}{4} = \frac{1}{2}(n + \rho).$$

Hence, if  $n \geq 4$ , for  $u \in \mathcal{D}$

$$\begin{aligned} (10) \quad & \|Vu\|^2 \leq (M_\rho)^{4/(n+\rho)} \left(\frac{1}{2\pi}\right)^{n(1-(2/\tilde{r}'))} (c_{n,\rho})^{2/\tilde{r}} \\ & \times [\omega^{4\rho/(n+\rho)} \|Tu\|^2 + \omega^{-4n/(n+\rho)} \|u\|^2] + A^2 \|u\|^2 \end{aligned}$$

for all  $\omega > 0$ . Thus again (5) follows in this case  $n \geq 4$  with  $\alpha$  arbitrarily small, since  $\omega^{4\rho/(n+\rho)} \rightarrow 0$  as  $\omega \rightarrow 0^+$ . Thus the proof of Lemma T. 2 is complete.

Returning to the proof of our Theorem T. 1, from the remarks preceding Lemma T. 2 we see this lemma permits  $H$  to be defined on  $\mathcal{D}$  dense, and  $H_1 \subseteq H$  from  $T_1 \subseteq T$ . Also  $T$  is self-adjoint with domain  $\mathcal{D}$ .

For by definition  $S^*TS$  is a purely multiplicative operator,  $[S^*TS\hat{u}](\mathbf{y}) = |\mathbf{y}|^2\hat{u}(\mathbf{y})$ , with the natural domain of all  $\hat{u} \in L_2(R_n)$  such that  $|\mathbf{y}|^2\hat{u}(\mathbf{y})$  is in  $L_2(R_n)$ . It is well known and easy to see that this makes  $S^*TS$  self-adjoint, and hence so is  $T$  since  $S$  is unitary.

Next  $(Tu, u) = \int_{R_n} |\mathbf{y}|^2 |\hat{u}(\mathbf{y})|^2 d\mu_n(\mathbf{y}) > 0$  for  $\|u\| > 0$  shows that the spectrum of  $T$  is confined to  $[0, +\infty]$ . Hence  $(T + \lambda^2 I)^{-1}$  is for real  $\lambda > 0$  a bounded Hermitian operator on  $L_2(R_n)$  with range  $\mathcal{D}$ ,  $(T + \lambda^2 I)\mathcal{D} = L_2(R_n)$  following from the spectral theorem for self-adjoint  $T$ . Thus (much as in Kato [2], Lemma 5), from (5), we have for all  $u \in L_2(R_n)$

$$(11) \quad \|V(T + \lambda^2 I)^{-1}u\| \leq \alpha \|T(T + \lambda^2 I)^{-1}u\| + \beta \|(T + \lambda^2 I)^{-1}u\| \\ \leq \left(\alpha + \frac{\beta}{\lambda^2}\right)\|u\|,$$

since  $\|T(T + \lambda^2 I)^{-1}\| < 1$  and  $\|(T + \lambda^2 I)^{-1}\| \leq 1/\lambda^2$  are clear from the spectral representation of  $T$ . Thus choosing  $\alpha < 1/2$  in (5), and then  $\lambda$  sufficiently positive so that  $\frac{\beta}{\lambda^2} < \frac{1}{2}$ , we see from (11) that the operator  $\tilde{V}$  defined on  $\mathcal{D}$  by  $[\tilde{V}u](\mathbf{x}) = V(\mathbf{x})u(\mathbf{x})$  satisfies

$$(12) \quad \|\tilde{V}(T + \lambda^2 I)^{-1}\| \leq \left(\alpha + \frac{\beta}{\lambda^2}\right) < 1.$$

Hence  $I + \tilde{V}(T + \lambda^2 I)^{-1}$  is a bounded linear operator on  $L_2(R_n)$  with range  $L_2(R_n)$ , since

$$[I + \tilde{V}(T + \lambda^2 I)^{-1}]^{-1} = I + \sum_{p=1}^{\infty} (-1)^p [\tilde{V}(T + \lambda^2 I)^{-1}]^p$$

also exists bounded. Thus, for  $\lambda$  large so (12) holds,

$$(13) \quad H + \lambda^2 I = T + \lambda^2 I + \tilde{V} = [I + \tilde{V}(T + \lambda^2 I)^{-1}](T + \lambda^2 I)$$

takes  $\mathcal{D}$  onto  $L_2(R_n)$ , since  $T + \lambda^2 I$  has already been seen to do so. Since  $T = T^*$  has been shown and since  $\tilde{V}$  is obviously symmetric, it follows that  $H = T + \tilde{V}$  and  $H + \lambda^2 I$  are symmetric,  $H + \lambda^2 I \subseteq (H + \lambda^2 I)^*$ . But  $(H + \lambda^2 I)\mathcal{D} = L_2(R_n)$  in (13) thus makes  $H + \lambda^2 I = (H + \lambda^2 I)^* = H^* + \lambda^2 I$ ,  $H = H^*$ , and hence  $H$  is self-adjoint (see [3], p. 35).

In order to complete the proof of Theorem T. 1, it remains only to show that the self-adjoint extension  $H$  of  $H_1$  is the unique self-adjoint extension. Since here  $H_1 \subseteq H_1^{**} \subseteq H = H^* \subseteq H_1^*$  is well-known [3], and likewise  $H_1^{**} \subseteq \tilde{H} \subseteq H_1^*$  for any other self-adjoint extension  $\tilde{H}$ , since  $H = H_1^{**}$  will make  $H_1^* = (H_1^*)^{**} = (H_1^{**})^* = H^* = H = H_1^{**}$ , and since  $H_1^{**} = \bar{H}_1$  the closure of  $H_1$ , it suffices for this uniqueness to show  $H \subseteq \bar{H}_1$ .

In order to do so, we first (Lemma 1, Kato [2]) notice that orthogonality of nonzero  $u_0 \in L_2(R_n)$  to  $(I + T_1)u = (I + T)u$  for all  $u \in \mathcal{D}_1$  would require  $\hat{u}_0$  to be orthogonal to all  $(1 + |\mathbf{y}|^2)\hat{u}(\mathbf{y})$ ; equivalently, since  $S^*(\mathcal{D}_1) \subseteq \mathcal{D}_1$  and  $S(\mathcal{D}_1) \subseteq \mathcal{D}_1$  makes  $S^*(\mathcal{D}_1) = \mathcal{D}_1 = S(\mathcal{D}_1)$ , this would require  $\hat{u}_0(\mathbf{y})(1 + |\mathbf{y}|^2) \exp(-(1/4)|\mathbf{y}|^2)$ , an element of  $L_2(R_n)$ , to be orthogonal to all polynomials in  $y_j$  multiplied by  $\exp(-(1/4)|\mathbf{y}|^2)$ . But the density of  $\mathcal{D}_1$  in  $L_2(R_n)$  and a change of scale by the factor  $\sqrt{2}$  shows this to be impossible. Hence  $(I + T)\mathcal{D}_1$  is dense in  $L_2(R_n)$ .

Thus given  $u \in \mathcal{D}$  and  $\delta > 0$  there exists  $u_1 \in \mathcal{D}_1$  such that

$$\begin{aligned} \delta > \| (I + T)u - (I + T)u_1 \| &= \| (I + S^*TS)(\hat{u} - \hat{u}_1) \| \\ &= \left[ \int_{R_n} (1 + |\mathbf{y}|^2)^2 | \hat{u}(\mathbf{y}) - \hat{u}_1(\mathbf{y}) |^2 d\mu_n(\mathbf{y}) \right]^{1/2} \\ &\geq (\max \| u - u_1 \|, \| T(u - u_1) \|) . \end{aligned}$$

Thus by (5),

$$\begin{aligned} \| Hu - H_1u_1 \| &= \| H(u - u_1) \| \leq \| T(u - u_1) \| + \| V(u - u_1) \| \\ &\leq (1 + \alpha) \| T(u - u_1) \| + \beta \| u - u_1 \| < (1 + \alpha + \beta)\delta . \end{aligned}$$

Hence the graph of  $H$  is contained in the closure of the graph of  $H_1$ ,  $H \subseteq \bar{H}_1$ , and  $H$  is the unique self-adjoint extension of  $H_1$  as desired. Thus Theorem T. 1 is completely proved.

**3. Connection with other conditions.** We will show in this section that Condition I, which is always implied by (and for  $n \geq 4$  coincides with) Condition II, implies our earlier one (Condition III, see eq. 19) for the construction of a self-adjoint extension as the inverse of a modified Green function integral operator. In fact, it is easy to verify for  $V(\mathbf{x}) = |\mathbf{x}|^{-\eta}$  that Condition I and Condition III are each equivalent to  $0 \leq \eta < (\min 2, n)$ , so that in this sense they have the same strength. We remark that Condition I is the natural one, used in a forthcoming joint paper, for an asymptotic formula for the distribution of eigenvalues of the bottom part of the Schrödinger operator spectrum. Finally we will show, as noted before T. 1, that

$$\text{Condition II} \Rightarrow \text{Condition S} \Rightarrow \text{Condition III} .$$

In order to give this connection with the modified Green function, we need to introduce the fundamental singularity  ${}_nK_\omega(r)$  for  $-\nabla^2 + \omega^2 I$  with constant  $\omega > 0$ . This may be defined (see [1], p. 555) uniquely by the requirements that  ${}_nK_\omega(r)$  be continuous over  $r > 0$ , that  ${}_nK_\omega(|\mathbf{x}|) \in L_1(R_n)$  over  $\mathbf{x}$ , and that  $[\omega^2 + |\mathbf{y}|^2]^{-1} = \int_{R_n} {}_nK_\omega(|\mathbf{x}|) e^{i(\mathbf{x}\cdot\mathbf{y})} d\mu_n(\mathbf{x})$  over  $y \in R_n$ . Such  ${}_nK_\omega(r) > 0$  over  $r > 0$  and  $\omega > 0$ . We define

$${}_n\tilde{K}_\omega(r) = M_n r^{-(n-2)} \exp\left(-\frac{\omega r}{4}\right) \quad \text{for } n \geq 3,$$

$${}_2\tilde{K}_\omega(r) = M_2[1 + \sqrt{\omega r}]^{-1}[1 + \ln(1 + (\omega r)^{-1})]e^{-\omega r},$$

and

$${}_1\tilde{K}_\omega(r) = (2\omega)^{-1}e^{-\omega r} = {}_1K_\omega(r),$$

with  $M_n$  the least possible real constant having  ${}_nK_\omega(r) \leq {}_n\tilde{K}_\omega(r)$  over all  $r > 0$  and  $\omega > 0$ , such positive finite  $M_n$  always existing. Finally define for  $\omega > 0$ ,

$$(14) \quad |\overline{V}|_\omega = \text{ess sup}_{\mathbf{x} \in R_n} \int_{R_n} {}_n\tilde{K}_\omega(|\mathbf{x} - \mathbf{y}|) |V(\mathbf{y})| d\mu_n(\mathbf{y}).$$

**THEOREM T. 3.** *Let  $V$  satisfy Condition I. Then  $|\overline{V}|_\omega < +\infty$  for all  $\omega > 0$  and*

$$(15) \quad \lim_{\omega \rightarrow +\infty} |\overline{V}|_\omega = 0.$$

Moreover for all  $\omega > 0$

$$(16) \quad \lim_{p \rightarrow +\infty} |\overline{V - V_p}|_\omega = 0,$$

where  $V_p(\mathbf{x}) = V(\mathbf{x})$  if  $|V(\mathbf{x})| \leq p$ ,  $V_p(\mathbf{x}) = p$  if  $V(\mathbf{x}) > p$ , and  $V_p(\mathbf{x}) = -p$  if  $V(\mathbf{x}) < -p$ .

The proof is rather elementary, using for  $n \geq 2$  the Schwarz-Hölder inequality with  $r = (1/2)(n + \rho) > 1$  and  $1/r + 1/r' = 1$ , and hence

$$r' = \frac{1}{1 - \frac{2}{n + \rho}} = \frac{n + \rho}{n - 2 + \rho}.$$

Thus Condition I yields in (14) for  $n \geq 2$ , the Schwarz-Hölder inequality being used on the  $|\mathbf{y}| \leq b$  portion, and also  ${}_n\tilde{K}_\omega(t/\omega) = \omega^{n-2} {}_n\tilde{K}_1(t)$  and  $(n - 2)(n + \rho)/(n - 2 + \rho) - n = -2\rho/(n - 2 + \rho) < 0$ ,

$$(17) \quad |\overline{V}|_\omega \leq (M_\rho)^{1/r} \omega^{-2\rho/(n+\rho)} \left[ \sigma_n \int_0^\infty \{ {}_n\tilde{K}_1(t) \}^{(n+\rho)/(n-2+\rho)} t^{n-1} dt \right]^{1/r'} + A\omega^{-2} \sigma_n \int_0^\infty {}_n\tilde{K}_1(t) t^{n-1} dt.$$

In (17) the second integral is obviously finite, and so is the first for  $n = 2$ . For  $n > 2$  we see in the first integral that only the portion  $0 < t < 1$  is in doubt, and here we have to consider the integrand factor  $t$  raised to the exponent

$$-(n - 2) \frac{n + \rho}{n - 2 + \rho} + n - 1 = \frac{2\rho}{n - 2 + \rho} - 1 > -1.$$



Thus the first integral in (17) is also finite for  $n > 2$  as well as for  $n=2$ , and (17) shows  $|\overline{V}|_\omega < +\infty$  for all  $\omega > 0$  and also that (15) follows for  $n \geq 2$ .

Finally for (16), taking  $p > A$  so that  $V(\mathbf{x}) - V_p(\mathbf{x}) = 0$  almost ( $\mu_n$ ) everywhere over  $|\mathbf{x}| \geq b$  by Condition I, we see that in place of (17) we have, with  $c_n < +\infty$  by the finiteness of the first integral in (17), for  $n \geq 2$

$$(18) \quad |\overline{V - V_p}|_\omega \leq c_n \omega^{-2\rho/(n+\rho)} \left[ \int_{\{|\mathbf{x}||\mathbf{x}| \leq b\}} |V(\mathbf{x}) - V_p(\mathbf{x})|^{(1/2)(n+\rho)} d\mu_n(\mathbf{x}) \right]^{1/x}.$$

Since  $\lim_{p \rightarrow \infty} |V(\mathbf{x}) - V_p(\mathbf{x})| = 0$  for all  $\mathbf{x} \in R_n$ , and since  $|V(\mathbf{x}) - V_p(\mathbf{x})| \leq |V(\mathbf{x})|$ , we see Condition I and dominated convergence in (18) yields (16) as desired for  $n \leq 2$ .

Finally consider  $n = 1$ ,  ${}_1\tilde{K}_\omega(r) = {}_1K_\omega(r) = (2\omega)^{-1}e^{-\omega r}$ . Notice that Condition I with  $1 + \rho \geq 2$  clearly implies itself with  $\rho$  replaced by  $\rho' = 1$ . Thus in place of (17) and (18) we have for  $n = 1$

$$(17)' \quad |\overline{V}|_\omega \leq M_1(2\omega)^{-1} + A\omega^{-2},$$

$$(18)' \quad |\overline{V - V_p}|_\omega \leq (2\omega)^{-1} \int_{\{|\mathbf{x}||\mathbf{x}| \leq b\}} |V(\mathbf{x}) - V_p(\mathbf{x})| d\mu_n(\mathbf{x}),$$

which clearly yield (15) and (16) in the same way as above. Thus the proof of Theorem T.3 is complete.

Now consider the following condition on  $V$ . As stated in Corollary T.4 immediately thereafter, this condition is implied by Condition I, as we see from (15) above.

CONDITION III. *There exists some  $\omega, 0 < \omega < +\infty$ , such that*

$$(19) \quad |\overline{V}|_\omega < 1.$$

COROLLARY T.4. *If Condition I is satisfied, then so is Condition III.*

Condition III is our earlier condition in [1] mentioned above. For our modified Green function, consider the formulae

$$(20) \quad G_\omega(\mathbf{x}, \mathbf{y}) = {}_nK_\omega(|\mathbf{x} - \mathbf{y}|) + \sum_{p=1}^\infty (-1)^p \int_{R_n} \int_{R_n} \dots$$

$$\int_{R_n} {}_nK_\omega(|\mathbf{x} - {}_1z|) V({}_1z) {}_nK_\omega(|{}_1z - {}_2z|) V({}_2z) \dots$$

$$V({}_p z) {}_nK_\omega(|{}_p z - \mathbf{y}|) d\mu_n({}_1z) \dots d\mu_n({}_p z),$$

$$(21) \quad [G_\omega u](\mathbf{x}) = \int_{R_n} G_\omega(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mu_n(\mathbf{y}).$$

By virtue of our earlier work ([1], p. 560, 567, Lemma 3.4, Theorem 3.5,

Theorem 4.5), we have the following theorem, using  $|\bar{V}|_\omega \leq |\bar{V}|_{\omega'}$  from  ${}_n\tilde{K}_\omega(r) \leq {}_n\tilde{K}_{\omega'}(r)$  in (14) for  $\omega \geq \omega'$ .

**THEOREM T.5.** *Let the conditions of Theorem T.3 hold and let  $\omega_1, 0 < \omega_1 < +\infty$ , be chosen so that (19) holds. Then for  $\omega \geq \omega_1$  the right side of (20) converges almost  $(\mu_n \times \mu_n)$  everywhere as a definition of  $G_\omega(\mathbf{x}, \mathbf{y}), G_\omega(\mathbf{x}, \mathbf{y}) = G_\omega(\mathbf{y}, \mathbf{x})$  almost  $(\mu_n \times \mu_n)$  everywhere, in (21) the right side exists finite almost  $(\mu_n)$  everywhere and is in  $L_2(R_n)$  for  $u \in L_2(R_n)$ , and the operator  $G_\omega$  on  $L_2(R_n)$  so defined is bounded Hermitian  $\|G_\omega\| \leq \omega^{-2}(1 - |\bar{V}|_\omega)^{-1}$ . Moreover the operator  $H_2$  defined by*

$$(22) \quad H_2 = G_\omega^{-1} - \omega^2 I$$

exists as a self-adjoint operator in  $L_2(R_n)$  independent of  $\omega \geq \omega_1$ .

Now under Condition I here, which is less than Condition II if  $n \leq 3$ , the linear manifold  $\mathcal{N} = \{u \in L_2(R_n) | \forall v \in L_2(R_n)\}$  need no longer contain  $\mathcal{D}_1$ , and hence  $H_1$  may not exist as an operator in  $L_2(R_n)$ . Thus define  $\tilde{\mathcal{D}}_1 = \mathcal{N} \cap \mathcal{D}_1$ , and as in (2)

$$(23) \quad [\tilde{H}_1 u](\mathbf{x}) = -V^2 u(\mathbf{x}) + V(\mathbf{x})u(\mathbf{x})$$

for  $u \in \tilde{\mathcal{D}}_1$ ; thus  $\tilde{H}_1$  satisfies  $(\tilde{H}_1 u, w) = (u, \tilde{H}_1 w)$  for  $u, w \in \tilde{\mathcal{D}}_1$ . Note  $\tilde{\mathcal{D}}_1 = \mathcal{D}_1$  and  $\tilde{H}_1 = H_1$  if  $n \geq 4$ , Condition I and II coinciding. Hence, after proving the following theorem,  $H_2 = H$  follows for  $n \geq 4$ .

**THEOREM T.6.** *Let  $V$  satisfy Condition I. Then the self-adjoint operator  $H_2$  defined by (22), known existent by Corollary T.4 and Theorem T.5, is an extension of  $\tilde{H}_1, \tilde{H}_1 \subseteq H_2$*

We note here that  $\tilde{\mathcal{D}}_1$  need not be dense in  $L_2(R_n)$  if  $n \leq 3$ , although  $\tilde{H}_1$  will not be a very respectable operator from the Hilbert space viewpoint if  $\tilde{\mathcal{D}}_1$  is not dense, in particular not being symmetric. This theorem is the same as our earlier one ([1], Theorem 5.3, p. 572) except for change in the initial domain from  $\mathcal{E}_0 = \mathcal{N} \cap \mathcal{E}$  there to  $\tilde{\mathcal{D}}_1 = \mathcal{N} \cap \mathcal{D}_1$  here. Merely sketching the proof, we first see

$$(24) \quad (u, \varphi) = \int_{R_n} [G_\omega u](\mathbf{x}) \{(\omega^2 + V(\mathbf{x}))\overline{\varphi(\mathbf{x})} - \nabla^2 \overline{\varphi(\mathbf{x})}\} d\mu_n(\mathbf{x})$$

follows for  $\varphi \in \mathcal{D}_1$  and  $u \in L_1(R_n) \cap L_2(R_n)$ , the proof being unchanged from the earlier one ([1], Theorem 5.1, p. 568) for  $\varphi$  having continuous second partials and vanishing outside a bounded set. Taking  $\varphi \in \tilde{\mathcal{D}}_1 = \mathcal{N} \cap \mathcal{D}_1$  in (24) and using the facts that  $G_\omega$  is bounded Hermitian and that  $L_1(R_n) \cap L_2(R_n)$  is dense in  $L_2(R_n)$ , we obtain from (23)

$$(25) \quad G_\omega(\omega^2 I + \tilde{H}_1)\varphi = \varphi$$

for  $\varphi \in \tilde{\mathcal{D}}_1$ . Thus  $\tilde{\mathcal{D}}_1 \subseteq (\text{range of } G_\omega) = (\text{domain of } G_\omega^{-1})$ , and  $\omega^2 I + \tilde{H}_1 \subseteq G_\omega^{-1}$ ,  $\tilde{H}_1 \subseteq G_\omega^{-1} - \omega^2 I = H_2$  as desired, proving T.6.

**THEOREM T.7.** *Let  $V$  satisfy Condition I and define  $h_1 = \lim_{r \rightarrow \infty} (\text{ess inf}_{|x| \geq r} V(\mathbf{x}))$ . Then this limit exists satisfying  $-A \leq h_1 \leq A$  and the spectrum  $\Sigma$  of the self-adjoint operator  $H_2$  defined by (22), known existent by Corollary T.4 and Theorem T.5, has  $(-\infty, h_1) \cap \Sigma$  to consist of pure point spectra with  $(-\infty, h) \cap \Sigma$  finite and having a finite dimensional eigenspace for all  $h < h_1$ . If also  $h_0 = [\text{ess inf}_{x \in R_n} V(\mathbf{x})] > -\infty$ , then  $(-\infty, h_0) \cap \Sigma$  is empty.*

Since (19) and (16) follow from Condition I for large  $\omega$  by Theorem T.3, this theorem follows from our earlier one ([I], Theorem 6.4, p. 579).

Finally we finish this section by proving in the following Theorems T.8 and T.9 the implications asserted before, namely  $\text{II} \Rightarrow \text{S} \Rightarrow \text{III}$ . Since Condition S, as noted before Theorem T.1, implies the conclusion of that theorem, from  $\text{II} \Rightarrow \text{S}$  we have an alternate proof of Theorem T.1. For knowledge of this work of Stummel [5] we are indebted to the referee. Although Theorems T.8 and T.9 seem of sufficient interest to record, their proofs are simple exercises in the use of the Schwarz-Hölder inequality.

We start by stating Stummel's Condition S.

**CONDITION S.**

$$(26) \quad \left\{ \sup_{x \in R_n} \int_{\{y \mid |x-y| \leq 1\}} |V(y)|^2 |x - y|^{-\gamma} d\mu_n(y) \right\} < +\infty$$

for some real  $\gamma$  satisfying  $\gamma > n - 4$  and  $\gamma \geq 0$ .

**THEOREM T.8.** *If  $V$  satisfies Condition II, then it also satisfies Condition S.*

**THEOREM T.9.** *If  $V$  satisfies Condition S, then equation (15) and hence Condition III are satisfied by  $V$ .*

To prove T.8 first, Condition II clearly yields (26) with  $\gamma=0$ , which thus takes care of the trivial case  $1 \leq n < 4$ .

Now consider dimension  $n \geq 4$ . Then for the  $\rho > 0$  in (1) of the given Condition II, we may choose real  $\gamma$  to satisfy

$$(27) \quad n - 4 \left( \frac{n}{n + \rho} \right) > \gamma > n - 4$$

and must then verify (26). Take  $p = (1/4)(n + \rho) > 1$  and then  $1/p + 1/p' = 1$ , for which

$$\begin{aligned}
 n[(n + \rho) - 4] &> (n + \rho)\gamma, \quad n + \rho > 4 + (n + \rho)\frac{\gamma}{n}, \\
 \left(1 - \frac{\gamma}{n}\right)(n + \rho) &> 4, \quad p = (1/4)(n + \rho) > \left(1 - \frac{\gamma}{n}\right)^{-1}, \\
 1 - \frac{1}{p'} &= \frac{1}{p} < 1 - \frac{\gamma}{n},
 \end{aligned}$$

and hence  $\gamma p' < n$ . Thus for (26) we have the Schwarz-Hölder estimate

$$\begin{aligned}
 (28) \quad &\int_{\{|\mathbf{y}||\mathbf{x}-\mathbf{y}|\leq 1\}} |V(\mathbf{y})|^2 |\mathbf{x}-\mathbf{y}|^{-\gamma} d\mu_n(\mathbf{y}) \\
 &\leq \left[ \int_{\{|\mathbf{y}||\mathbf{x}-\mathbf{y}|\leq 1\}} |V(\mathbf{y})|^{2p} d\mu_n(\mathbf{y}) \right]^{1/p} \left[ \sigma_n \int_0^1 r^{-p'} \gamma^{n-1} dr \right]^{1/p'}
 \end{aligned}$$

with  $2p = (1/2)(n + \rho)$ ,  $\gamma p' < n$ , and  $\sigma_n$  as in (6). Thus the second factor on the right of (28) is a finite constant, Condition II assures that the first factor is bounded over  $\mathbf{x} \in R_n$ , and (27) and (28) yield (26) for Condition S. This completes the proof of T.8.

Now for Theorem T.9 it suffices to prove that Condition S implies  $\lim_{\omega \rightarrow +\infty} |\overline{V}|_\omega = 0$ , since equation (15) yields the conclusion of Corollary T.4 as noted there. Considering first the general case  $n \geq 4$ , and taking  $\beta = n - 2 - \gamma/2 < n - 2 - (n - 4)(1/2) = n/2$  so that  $2\beta < n$  from  $\gamma > n - 4 \geq 0$ , the Schwarz inequality yields

$$\begin{aligned}
 (29) \quad &\int_{\{|\mathbf{y}||\mathbf{x}-\mathbf{y}|\leq 1\}} |V(\mathbf{y})| \frac{e^{-\omega|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{n-2}} d\mu_n(\mathbf{y}) \\
 &\leq \left[ \int_{\{|\mathbf{y}||\mathbf{x}-\mathbf{y}|\leq 1\}} |V(\mathbf{y})|^2 |\mathbf{x}-\mathbf{y}|^{-\gamma} d\mu_n(\mathbf{y}) \right]^{1/2} \left[ \sigma_n \int_0^1 e^{-\omega r} \gamma^{n-1-2\beta} dr \right]^{1/2}.
 \end{aligned}$$

On the right here the second factor is  $\leq [\omega^{-(n-2\beta)} \sigma_n \Gamma(n - 2\beta)]^{1/2} \rightarrow 0$  as  $\omega \rightarrow +\infty$  since  $n - 2\beta > 0$ ; the first factor is independent of  $\omega$  and bounded over  $\mathbf{x} \in R_n$  according to Condition S. Hence we see that the left side of (29) converges to zero uniformly over  $\mathbf{x} \in R_n$  as  $\omega \rightarrow +\infty$  for  $n \geq 4$ .

In order to estimate  $|\overline{V}|_\omega$ , we must also consider the left side of (29) with the range of integration replaced by its complement in  $R_n$ . For this we define

$B(\mathbf{j}) = \{\mathbf{x} \in R_n \mid |x_i - 2j_i(n)^{-1/2}| \leq (n)^{-1/2} \text{ for } 1 \leq i \leq n\}$ ,  $\mathbf{j} = (j_1, j_2, \dots, j_n)$  for integer  $j_i$ , and also  $r(\mathbf{j}) = \inf_{\mathbf{x} \in B(\mathbf{j})} |\mathbf{x}|$ . Noting that  $B(\mathbf{0}) \subseteq \{\mathbf{x} \mid |\mathbf{x}| \leq 1\}$  makes  $\{\mathbf{x} \mid |\mathbf{x}| > 1\} \subseteq \bigcup_{\mathbf{j} \neq \mathbf{0}} B(\mathbf{j})$ , we see with  $n \geq 4$

$$\begin{aligned}
 (30) \quad & \int_{\{\mathbf{y} \mid |\mathbf{x} - \mathbf{y}| > 1\}} |V(\mathbf{y})| \frac{e^{-\omega|\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|^{n-2}} d\mu_n(\mathbf{y}) \\
 & \leq \int_{\{z \mid |z| > 1\}} e^{-\omega|z|} |V(\mathbf{x} - z)| d\mu_n(z) \\
 & \leq \sum_{j \neq 0} e^{-\omega r(j)} \int_{B(j)} |V(\mathbf{x} - z)| d\mu_n(z) \\
 & \leq \left(\frac{2}{\sqrt{n}}\right)^{n/2} \left[ \sup_{\mathbf{x} \in R_n} \int_{\{\mathbf{y} \mid |\mathbf{x} - \mathbf{y}| \leq 1\}} |V(\mathbf{y})|^2 d\mu_n(\mathbf{y}) \right]^{1/2} \left\{ \sum_{j \neq 0} e^{-\omega r(j)} \right\}.
 \end{aligned}$$

Since  $|\mathbf{x} - \mathbf{y}|^{-\gamma} \geq 1$  in (26), we see that Condition S assures that the first factor on the far right side of (30) is a finite constant. Moreover, we see that the second factor

$$\left\{ \sum_{j \neq 0} e^{-\omega r(j)} \right\} \rightarrow 0 \quad \text{as } \omega \rightarrow +\infty,$$

using

$$r(j) \geq \left( \sup_{\mathbf{x} \in B(j)} |\mathbf{x}| \right) - 2$$

to estimate the portion of this sum where

$$r(j) \geq 3 \quad \text{by} \quad \left(\frac{2}{\sqrt{n}}\right)^{-n} \sigma_n \int_3^\infty e^{-\omega(r-2)} r^{n-1} dr$$

which  $\rightarrow 0$  by dominated convergence, and using  $r(j) \geq 1/\sqrt{n} > 0$  for  $j \neq 0$  to estimate the remaining finite sum portion. Thus the left side of (30) converges to zero uniformly over  $\mathbf{x} \in R_n$  as  $\omega \rightarrow +\infty$ , which when combined with the same conclusion about (29) proved above yields  $|\overline{V}|_\omega \rightarrow 0$  and completes the proof of T.9 for dimension  $n \geq 4$ .

For dimension  $n < 4$ , we see Condition S becomes just (26) with  $\gamma = 0$ . Hence  $\int_{R_n} [{}_n\tilde{K}_\omega(|\mathbf{x}|)]^2 d\mu_n(\mathbf{x}) = c_n \omega^{-(4-n)}$ , easily seen with  $c_n < +\infty$  for  $n < 4$  from the definition preceding (14), gives in place of (29)

$$(31) \quad \left[ \sup_{\mathbf{x} \in R_n} \int_{\{\mathbf{y} \mid |\mathbf{x} - \mathbf{y}| \leq 1\}} |V(\mathbf{y})| {}_n\tilde{K}_\omega(|\mathbf{x} - \mathbf{y}|) d\mu_n(\mathbf{y}) \right] = 0 \left( \omega^{-(4-n)1/2} \right)$$

as  $\omega \rightarrow +\infty$ . Also (30) still shows the integral over the complimentary region to converge to zero uniformly over  $\mathbf{x} \in R_n$  as  $\omega \rightarrow +\infty$  if  $n = 3$ , and a very similar computation gives the same result if  $n = 1$  or  $2$ . Hence  $\lim_{\omega \rightarrow +\infty} |\overline{V}|_\omega = 0$  follows from Condition S when dimension  $n < 4$  as well as when  $n \geq 4$ , and the proof of T.9 is complete.

**4. Variational characterization of the spectrum.** In this section we will show (see T.13 following) that a variational characterization of the spectrum, well-known at least for continuous  $V$  and bounded domains,

also holds for  $H_2$  with  $V$  subject only to Condition I. This is rather easy to obtain ([2], p. 209, eq. (23)) under Condition II, and the major effort in our argument amounts to showing that Condition I, which is weaker for  $1 \leq n \leq 3$ , actually suffices.

We start with the following theorem, where by the  $L_1$  sense of the Fourier transform  $\hat{u}$  for  $u \in L_1(R_n)$  we mean (3) with no limit and  $\int_{\{\mathbf{x} \mid |\mathbf{x}| \leq N\}}$  replaced by the ordinary Lebesgue integral  $\int_{R_n}$ . Notice that if  $u \in L_1(R_n) \cap L_2(R_n)$ , then by taking subsequences we may be sure that the two definitions of  $\hat{u}(\mathbf{y})$  are equal almost ( $\mu_n$ ) everywhere. Hereafter  $\|u\|_r$  denotes the  $L_r(R_n)$  norm of  $u$ , and  $\|u\|$  or  $\|u\|_2$  the  $L_2(R_n)$  norm.

**THEOREM T.10.** *Let  $V$  satisfy Condition I and let  $\lambda_0$  be in the point spectrum of  $H_2$ , defined by (22), with eigenvector  $u_0 \in \mathcal{D}_2 = G_\omega(L_2(R_n))$ ,  $H_2 u_0 = \lambda_0 u_0$  and  $\|u_0\| = 1$ . Then  $Vu_0 \in L_1(\{\mathbf{x} \mid |\mathbf{x}| \leq b\})$  and over  $\mathbf{y} \in R_n$*

$$(32) \quad |\mathbf{y}|^2 \hat{u}_0(\mathbf{y}) + \psi_0(\mathbf{y}) = \lambda_0 \hat{u}_0(\mathbf{y})$$

where  $\psi_0 = \hat{f}_0 + \hat{g}_0$ ,  $\hat{f}_0$  is the  $L_1$  sense transform of  $f_0(\mathbf{x}) = V(\mathbf{x})u_0(\mathbf{x})\chi_b(\mathbf{x})$  with  $\chi_b(\mathbf{x})$  the characteristic function of  $\{\mathbf{x} \in R_n \mid |\mathbf{x}| \leq b\}$ , and  $\hat{g}_0$  is the usual  $L_2$  transform of  $g_0 = Vu_0 - f_0$ .

If  $(n + \rho) \geq 4$ , then Condition II follows from Condition I,  $H_2 = H$  and  $u_0 \in \mathcal{D}_2 = \mathcal{D}$  by Theorems T.1 and T.6,  $Vu_0 \in L_2(R_n)$  by Lemma T.2 and hence  $\varepsilon L_1(\{\mathbf{x} \mid |\mathbf{x}| \leq b\})$ ,  $\psi_0$  exists as defined and  $= \widehat{Vu_0}$  defined in the usual  $L_2$  sense, and (32) follows from  $Hu_0 = \lambda_0 u_0$  and the definition of  $H$ .

The proof of T. 10 thus being complete for  $(n + \rho) \geq 4$  and hence for  $n \geq 4$ , we now consider the remaining case  $2 \leq n + \rho < 4$ , for which

$1 \leq n \leq 3$ . Since  $G_\omega u_0 = (\lambda_0 + \omega^2)^{-1}u_0$  with  $\lambda_0 + \omega^2 > 0$  for  $\omega \geq \omega_1$  follows from (22) and  $H_2 u_0 = \lambda_0 u_0$ , we see ([1], (3.5), (3.6), and (3.21), p. 558 and 562) by using the Schwarz inequality that  $u_0$  is essentially bounded,  $u_0 \in L_\infty(R_n)$  and  $\|u_0\|_\infty = \text{ess sup}_{\mathbf{x} \in R_n} |u_0(\mathbf{x})| < +\infty$ . Thus by Condition I,  $Vu_0 \in L_r(\{\mathbf{x} \mid |\mathbf{x}| \leq b\}) \subseteq L_1(\{\mathbf{x} \mid |\mathbf{x}| \leq b\})$  with  $r = \frac{1}{2}(n + \rho)$  satisfying  $1 \leq r < 2$ , and  $\psi_0$  exists as defined.

Now,  $L_1 \cap L_2$  being dense in  $L_2$ , there exists a sequence  $u'_k \in L_1(R_n) \cap L_2(R_n)$  such that the  $L_2$  norm  $\|u_0 - u'_k\|_2 \rightarrow 0$ . Hence as above,  $u_k = (\lambda_0 + \omega^2)G_\omega u'_k$  has  $u_k \in L_\infty \cap L_2$  and both  $\|u_0 - u_k\|_2 \rightarrow 0$  and also  $\|u_0 - u_k\|_\infty \rightarrow 0$ . Actually ([1], Lemma 4.1, p. 565),  $u_k$  and  $Vu_k \in L_1(R_n)$  also, and

$$(33) \quad (|\mathbf{y}|^2 - \lambda_0) \hat{u}_k(\mathbf{y}) + \psi_k(\mathbf{y}) = (\lambda_0 + \omega^2) \{\hat{u}'_k(\mathbf{y}) - \hat{u}_k(\mathbf{y})\}$$

with  $\psi_k = \widehat{Vu_k}$  in the  $L_1$  sense. Defining  $f_k$  and  $g_k$  from  $u_k$  analogously to  $f_0$  and  $g_0$  from  $u_0$ ,  $\psi_k = \hat{f}_k + \hat{g}_k$  defined in the  $L_1$  sense. Moreover,

$$\|\hat{f}_0 - \hat{f}_k\|_\infty \leq (2\pi)^{-n/2} \|f_0 - f_k\|_1 \leq (2\pi)^{-n/2} \|V\|_{1,b} \|u_0 - u_k\|_\infty \rightarrow 0$$

with

$$\|V\|_{1,b} = \int_{\{x \mid |x| \leq b\}} |V(x)| d\mu_n(x),$$

and  $\|g_0 - g_k\|_2 \leq A\|u_0 - u_k\|_2 \rightarrow 0$  by using Condition I. Thus, after taking subsequences, we may assume almost  $(\mu_n)$  everywhere that

$$\psi_k(\mathbf{y}) = \hat{f}_k(\mathbf{y}) + \hat{g}_k(\mathbf{y}) \rightarrow \hat{f}_0(\mathbf{y}) + \hat{g}_0(\mathbf{y}) = \psi_0(\mathbf{y}), \quad \hat{u}_k(\mathbf{y}) \rightarrow \hat{u}_0(\mathbf{y}),$$

and  $\hat{u}'_k(\mathbf{y}) \rightarrow \hat{u}'_0(\mathbf{y})$ , since  $\|\hat{u}_k - \hat{u}_0\|_2 = \|u_k - u_0\|_2 \rightarrow 0$  and  $\|\hat{u}'_k - \hat{u}'_0\|_2 = \|u'_k - u'_0\|_2 \rightarrow 0$ . Thus (33) yields (32), and the proof of theorem T. 10 is complete.

We next give some approximation lemmas.

LEMMA T. 11. *Let  $V$  satisfy Condition I with  $n + \rho \geq 4$ , and hence Condition II also; let  $u_0 \in \mathcal{D}$ . Then there exists a sequence of  $u_k \in \mathcal{D}$ , satisfying simultaneously  $\|u_0 - u_k\| \rightarrow 0$ ,  $\|T(u_0 - u_k)\| \rightarrow 0$ ,  $\|V(u_0 - u_k)\| \rightarrow 0$  for these  $L_2(R_n)$  norms.*

This was proved in the last two paragraphs of § 2. In the following we denote  $(z \cdot \xi) = \sum_{j=1}^n z_j \bar{\xi}_j$ ,  $|z| = \sqrt{(z \cdot z)}$  for  $z$  and  $\xi \in C_n$ , unitary  $n$  space.  $\mathcal{D}_2 = G_\omega(L_2(R_n))$  for  $\omega \geq \omega_1$  is the domain of  $H_2$  as usual.

LEMMA T. 12. *Let  $V$  satisfy Condition I with  $2 \leq n + \rho < 4$  and let  $u_0 \in \mathcal{D}_2$  satisfy  $H_2 u_0 = \lambda_0 u_0$  and  $\|u_0\| = 1$ . Then  $|\mathbf{y}| \hat{u}_0(\mathbf{y}) \in L_2(R_n)$  and  $u_0 \in L_\infty(R_n)$  and  $\hat{u}_0 \in L_1(R_n)$ , and there exists a sequence of  $u_k \in \mathcal{D}_1$  such that simultaneously  $\|u_0 - u_k\|_2 \rightarrow 0$ ,  $\|u_0 - u_k\|_\infty \rightarrow 0$ ,*

$$\int_{R_n} |V(\mathbf{x})| |u_0(\mathbf{x}) - u_k(\mathbf{x})|^2 d\mu_n(\mathbf{x}) \rightarrow 0,$$

and

$$\int_{R_n} |\nabla_{\text{gen}} u_0(\mathbf{x}) - \nabla u_k(\mathbf{x})|^2 d\mu_n(\mathbf{x}) \rightarrow 0,$$

where  $\nabla$  denotes the ordinary gradient differential operator and  $\nabla_{\text{gen}} u$  the  $C_n$  vector valued function whose components are in  $L_2(R_n)$  and have the components of  $i\mathbf{y}\hat{u}(\mathbf{y})$  as their  $L_2$  sense Fourier transforms.

To prove T. 12, first notice  $2 \leq n + \rho < 4$  makes  $1 \leq n \leq 3$ , and hence, as shown in proving T. 10,  $u_0 \in L_\infty(R_n)$  and  $f_0 \in L_r(R_n)$  with  $r = \frac{1}{2}(n + \rho)$ ,  $1 \leq r < 2$ . Thus, using the Young-Hausdorff-Titchmarsh theorem as in (8), the  $L_1$  sense  $\hat{f}_0 \in L_{r'}(R_n)$  with

$$r' = \frac{1}{1 - 1/r} = \frac{n + \rho}{n + \rho - 2} > 2, \text{ and } r' = \infty$$

if  $n + \rho = 2$ .

Next notice that for  $0 < \nu < 2$  we have from (32)

$$(34) \quad \left(\frac{|\mathbf{y}|}{1 + |\mathbf{y}|}\right)^{2-\nu} |\mathbf{y}|^\nu \hat{u}_0(\mathbf{y}) = (1 + |\mathbf{y}|)^{-2+\nu} [\lambda_0 \hat{u}_0(\mathbf{y}) - \hat{g}_0(\mathbf{y})] \\ - (1 + |\mathbf{y}|)^{-2+\nu} \hat{f}_0(\mathbf{y}) .$$

Thus we may conclude  $|\mathbf{y}|^\nu \hat{u}_0(\mathbf{y}) \in L_2(R_n)$ , as desired, whenever this holds for both terms on the right of (34). The first term is obviously in  $L_2(R_n)$ . For the second term we use  $\hat{f}_0 \in L_{r'}(R_n)$  and the Schwarz-Hölder inequality with

$$2\alpha = r' = \frac{n + \rho}{n + \rho - 2} > 2, \quad \alpha' = \frac{1}{1 - 1/\alpha} = \frac{n + \rho}{4 - (n + \rho)}$$

holding even for  $n + \rho = 2$ , for which  $\alpha = \infty$  and  $\alpha' = 1$ . Thus, with  $\sigma_n$  as in (6),

$$(35) \quad \int_{R_n} \frac{|\hat{f}_0(\mathbf{y})|^2}{(1 + |\mathbf{y}|)^{2(2-\nu)}} d\mu_u(\mathbf{y}) \leq \|\hat{f}_0\|_{r'}^2 \left[ \sigma_n \int_0^\infty \frac{t^{n-1}}{(1 + t)^{2(2-\nu)\alpha'}} dt \right]^{1/\alpha'} < +\infty$$

provided that

$$n < (2 - \nu)2\alpha' = \frac{2(n + \rho)(2 - \nu)}{4 - (n + \rho)} .$$

This last inequality is equivalent to

$$2 - \nu > \frac{[4 - (n + \rho)]n}{2(n + \rho)} ,$$

and this to

$$\nu < \frac{4\rho + n(n + \rho)}{2(n + \rho)} = \frac{n}{2} + \frac{2\rho}{n + \rho} .$$

We see for our  $n = 1, 2$ , or  $3$ ,  $\rho > 0$ ,  $2 \leq n + \rho < 4$ , that this last inequality is always satisfied for  $\nu = 1$  and for  $\nu = \nu_1 = n/2 + \rho/(n + \rho)$ . Note  $n/2 < \nu_1 < 2$ . Thus we have shown  $|\mathbf{y}| \hat{u}_0(\mathbf{y})$  and  $|\mathbf{y}|^{\nu_1} \hat{u}(\mathbf{y})$  to be  $\in L_2(R_n)$ .

Next for any finite set of  $\nu_p > 0$ , define

$$[L\hat{u}](\mathbf{y}) = \left(1 + \sum_p |\mathbf{y}|^{\nu_p}\right) \hat{u}(\mathbf{y}) .$$

As in the last two paragraphs of § 2,  $L\mathcal{D}_1$  is dense in  $L_2(R_n)$ , since any  $\hat{u} \in L_2(R_n)$  has

$$\left(1 + \sum_p |\mathbf{y}|^{\nu_p}\right) \exp(-\frac{1}{4}|\mathbf{y}|^2) \hat{u}(\mathbf{y}) \in L_2(R_n)$$



and therefore is not orthogonal to all  $Q(\mathbf{y}) \exp(-\frac{1}{4}|\mathbf{y}|^2)$  with polynomial  $Q$ , and thus  $\hat{u}$  cannot be orthogonal to  $L_{\mathcal{D}_1}$ . Hence, for any  $u \in L_2(R_n)$  such that  $L\hat{u} \in L_2(R_n)$  there exists (since  $\mathcal{D}_1$  transforms onto  $\mathcal{D}_1$ ) a sequence  $u_k \in \mathcal{D}_1$  such that  $\|L(\hat{u} - \hat{u}_k)\|_2 \rightarrow 0$ , and thus simultaneously

$$\int_{R_n} |\mathbf{y}|^{2\nu_p} |\hat{u}(\mathbf{y}) - \hat{u}_k(\mathbf{y})|^2 d\mu_n(\mathbf{y}) \rightarrow 0$$

for the finite set of  $\nu_p$  as well as  $\|u - u_k\|_2 = \|\hat{u} - \hat{u}_k\|_2 \rightarrow 0$ . Applying this result to  $u_0 \in L_2(R_n)$  with the finite set  $\{1, \nu_1\}$  of  $\nu$ 's, since  $|\mathbf{y}|\hat{u}_0(\mathbf{y})$  and  $|\mathbf{y}|^{\nu_1}\hat{u}_0(\mathbf{y})$  were shown to be in  $L_2(R_n)$ , there thus exists a sequence of  $u_k \in \mathcal{D}_1$  such that simultaneously

$$\|u_0 - u_k\|_2 = \|\hat{u} - \hat{u}_k\|_2 \rightarrow 0, \quad \int_{R_n} |\mathbf{y}|^2 |\hat{u}_0(\mathbf{y}) - \hat{u}_k(\mathbf{y})|^2 d\mu_n(\mathbf{y}) \rightarrow 0,$$

and

$$\int_{R_n} |\mathbf{y}|^{2\nu_1} |\hat{u}_0(\mathbf{y}) - \hat{u}_k(\mathbf{y})|^2 d\mu_n(\mathbf{y}) \rightarrow 0$$

with

$$\nu_1 = \frac{n}{2} + \frac{\rho}{n + \rho} > \frac{n}{2}.$$

From the second limit statement just proved, and from  $|\mathbf{y}|\hat{u}_0(\mathbf{y}) \in L_2(R_n)$ , we see that  $\mathcal{V}_{\text{gen}}u_0$  exists as defined and that, since  $i\mathbf{y}\hat{u}_k(\mathbf{y})$  clearly has its components the  $L_2$  transforms of the  $C_n$  vector valued function  $\mathcal{V}u_k(\mathbf{x})$ ,

$$\int_{R_n} |\mathcal{V}_{\text{gen}}u_0(\mathbf{x}) - \mathcal{V}u_k(\mathbf{x})|^2 d\mu_n(\mathbf{x}) = \int_{R_n} |\mathbf{y}|^2 |\hat{u}_0(\mathbf{y}) - \hat{u}_k(\mathbf{y})|^2 d\mu_n(\mathbf{y}) \rightarrow 0.$$

Next for  $u \in L_2(R_n)$  having  $|\mathbf{y}|^{\nu_1}\hat{u}(\mathbf{y}) \in L_2(R_n)$ ,

$$\begin{aligned} (36) \quad \|\hat{u}\|_1 &\leq M \left[ \int_{R_n} (1 + |\mathbf{y}|^{2\nu_1}) |\hat{u}(\mathbf{y})|^2 d\mu_n(\mathbf{y}) \right]^{1/2} \\ &\leq M \|\hat{u}\|_2 + M \left[ \int_{R_n} |\mathbf{y}|^{2\nu_1} |\hat{u}(\mathbf{y})|^2 d\mu_n(\mathbf{y}) \right]^{1/2}, \\ M &= \left[ \sigma_n \int_0^\infty \frac{t^{n-1}}{1 + t^{2\nu_1}} dt \right]^{1/2} < +\infty, \end{aligned}$$

using the Schwarz inequality and  $2\nu_1 > n$ . Thus from  $|\mathbf{y}|^{\nu_1}\hat{u}_0(\mathbf{y}) \in L_2(R_n)$  we conclude  $\|\hat{u}_0\|_1 < +\infty$  and  $\hat{u}_0 \in L_1(R_n)$ , and likewise  $\|\hat{u}_0 - \hat{u}_k\|_1 \rightarrow 0$  follows from

$$\int_{R_n} |\mathbf{y}|^{2\nu_1} |\hat{u}_0(\mathbf{y}) - \hat{u}_k(\mathbf{y})|^2 d\mu_n(\mathbf{y}) \rightarrow 0$$

shown above. Thus we have

$$\|u_0 - u_k\|_\infty \leq (2\pi)^{-n/2} \|\hat{u}_0 - \hat{u}_k\|_1 \rightarrow 0$$

from the  $L_1$  sense of (4) agreeing here with the  $L_2$  sense as usual. Hence finally

$$(37) \quad \int_{R_n} |V(\mathbf{x})| |u_0(\mathbf{x}) - u_k(\mathbf{x})|^2 d\mu_n(\mathbf{x}) \leq \|u_0 - u_k\|_\infty^2 \int_{\{\mathbf{x} \mid |\mathbf{x}| \leq b\}} |V(\mathbf{x})| d\mu_n(\mathbf{x}) + A \|u_0 - u_k\|_2^2$$

by Condition I with the right side  $\rightarrow 0$  as  $k \rightarrow +\infty$ . Thus the proof of Lemma T. 12 is complete.

We now are ready to give our variational characterization of the spectrum  $\Sigma$  of  $H_2$ , assuming only Condition I. Define

$$h_1 = \lim_{r \rightarrow \infty} \left( \operatorname{ess\,inf}_{|\mathbf{x}| \geq r} V(\mathbf{x}) \right),$$

and by Theorem T. 7 we know that  $\Sigma \cap (-\infty, h)$  for  $h < h_1$  consists of a finite set of  $\lambda$  which are each in the point spectrum of  $H_2$  with finite multiplicity. Thus there is uniquely defined a finite or countable set  $\{\lambda_p\} = \Sigma \cap (-\infty, h_1)$ ,  $\lambda_p \leq \lambda_{p+1}$ , and the  $\lambda_p = \lambda$  repeat according to the multiplicity of each  $\lambda$  in the point spectrum of  $H_2$ . In the statement following,  $u \perp S$  means  $(u, w) = 0$  for all  $w \in S$ .

**THEOREM T. 13.** *Let  $V$  satisfy Condition I and let  $\{\lambda_p\}$ , possibly empty, be defined as above. Then each such  $\lambda_p$  satisfies*

$$(38) \quad \lambda_p = \sup_{\substack{S \subseteq L_2(R_n), \\ \operatorname{card} S < p}} \left\{ \inf_{\substack{u \in \mathcal{D}_1 \\ \|u\|=1, u \perp S}} \int_{R_n} (|\nabla u(\mathbf{x})|^2 + V(\mathbf{x})|u(\mathbf{x})|^2) d\mu_n(\mathbf{x}) \right\},$$

and such  $\lambda_p$  exists for any integer  $p \geq 1$  for which the right side of (38) is  $< h_1$ . Moreover, in this statement  $\mathcal{D}_1$  may be replaced by  $\mathcal{D}_0$ , the set of all  $u \in L_2(R_n)$  which possess continuous second partials everywhere and such that  $u(\mathbf{x})$  together with all its partial derivatives of order  $\leq 2$  is  $O([1 + |\mathbf{x}|^m] \exp(-\frac{1}{2}|\mathbf{x}|^2))$  over  $\mathbf{x} \in R_n$  for some integer  $m > 0$  depending on  $u$ .

For integer  $p \geq 1$  define  $\tau_p(\mathcal{D}_1)$  as the right side of (38), and similarly  $\tau_p(\mathcal{D}_0)$  with  $\mathcal{D}_1$  replaced by  $\mathcal{D}_0$ .  $\mathcal{D}_0 \supseteq \mathcal{D}_1$  clearly makes  $\tau_p(\mathcal{D}_0) \leq \tau_p(\mathcal{D}_1)$ . Thus to prove theorem T. 13 we need only show first that any existing  $\lambda_p$  has  $\lambda_p \geq \tau_p(\mathcal{D}_1)$ , and secondly that  $\tau_p(\mathcal{D}_0) < h_1$  has  $\lambda_p$  existing with  $\tau_p(\mathcal{D}_0) \geq \lambda_p$ .

Now for each  $\lambda_p$  we may choose  $\varphi_p \in \mathcal{D}_2$ , the domain of  $H_2$ , such that  $H_2\varphi_p = \lambda_p\varphi_p$  and  $(\varphi_p, \varphi_{p'}) = \delta_{p,p'}$ , since  $H_2$  is self-adjoint. Thus using T. 10 and multiplying (32) by  $\varphi_{p'}(\mathbf{y})$  and integrating over  $R_n$  we have, since  $(\hat{\varphi}_p, \hat{\varphi}_{p'}) = (\varphi_p, \varphi_{p'}) = \delta_{p,p'}$ ,

$$(39) \quad \lambda_p \delta_{p,p'} = \int_{R_n} \{ |\mathbf{y}|^2 \hat{\phi}_p(\mathbf{y}) \overline{\hat{\phi}_{p'}(\mathbf{y})} + \psi_p(\mathbf{y}) \overline{\hat{\phi}_{p'}(\mathbf{y})} \} d\mu_n(\mathbf{y}),$$

the integral of each term in (39) existing finite in the Lebesgue sense. This finiteness is clear if  $n + \rho \geq 4$ , since then Condition II holds and  $\varphi_p \in \mathcal{D}_2 = \mathcal{D}$ ,  $|\mathbf{y}| \varphi_p(\mathbf{y}) \in L_2(R_n)$  and  $\psi_p \in L_2(R_n)$  by T. 2. Otherwise  $2 \leq n + \rho < 4$ , and T. 12 yields  $|\mathbf{y}| \hat{\phi}_p(\mathbf{y}) \in L_2(R_n)$  and  $\hat{\phi}_p \in L_1(R_n) \cap L_2(R_n)$ ; hence  $\psi_p = \hat{f}_p + \hat{g}_p$  with  $\hat{g}_p \in L_2(R_n)$  and  $\hat{f}_p \in L_\infty(R_n)$  from  $f_p \in L_1(R_n)$  also makes the second term integral be finite as well as the first. Also Parseval's equality applied to the terms on the right side of (39) yields

$$(40) \quad \lambda_p \delta_{p,p'} = \int_{R_n} \{ (\nabla_{\text{gen}} \varphi_p(\mathbf{x}) \cdot \nabla_{\text{gen}} \overline{\varphi_{p'}(\mathbf{x})}) + V(\mathbf{x}) \varphi_p(\mathbf{x}) \overline{\varphi_{p'}(\mathbf{x})} \} d\mu_n(\mathbf{x}),$$

provided that in addition we show

$$\int_{R_n} \hat{f}_p(\mathbf{y}) \overline{\hat{\phi}_{p'}(\mathbf{y})} d\mu_n(\mathbf{y}) = \int_{R_n} f_p(\mathbf{x}) \overline{\varphi_{p'}(\mathbf{x})} d\mu_n(\mathbf{x})$$

in the case  $2 \leq n + \rho < 4$ , where as usual  $f_p(\mathbf{x}) = V(\mathbf{x}) \varphi_p(\mathbf{x}) \chi_b(\mathbf{x})$  as in T. 10. Replacing  $V$  by the truncate  $V_q$  defined for (16) and defining  ${}_q f_p = V_q \varphi_p \chi_b$ , then  ${}_q f_p \in L_2(R_n)$  and  $({}_q f_p, \hat{\phi}_{p'}) = ({}_q f_p, \varphi_{p'})$  follows by Parseval's equality. Clearly Condition I,  $\varphi_p \in L_\infty(R_n)$  by T. 12, and dominated convergence over  $\{\mathbf{x} \mid |\mathbf{x}| \leq b\}$  yields  $\|f_p - {}_q f_p\|_1 \rightarrow 0$  as  $q \rightarrow +\infty$ , and hence also  $\|\hat{f}_p - {}_q \hat{f}_p\|_\infty \rightarrow 0$ . Thus  $\varphi_{p'} \in L_\infty(R_n)$  and  $\hat{\phi}_{p'} \in L_1(R_n)$  by T. 12 in our case  $2 \leq n + \rho < 4$  gives the desired result

$$(41) \quad \int_{R_n} \hat{f}_p(\mathbf{y}) \overline{\hat{\phi}_{p'}(\mathbf{y})} d\mu_n(\mathbf{y}) = \lim_{q \rightarrow \infty} ({}_q \hat{f}_p, \hat{\phi}_{p'}) = \lim_{q \rightarrow \infty} ({}_q f_p, \varphi_{p'}) \\ = \int_{R_n} f_p(\mathbf{x}) \overline{\varphi_{p'}(\mathbf{x})} d\mu_n(\mathbf{x}),$$

and (40) is completely proved.

Now from (40), for  $u = \sum_{j=1}^p c_j \varphi_j$  we have

$$(42) \quad \int_{R_n} \{ |\nabla_{\text{gen}} u(\mathbf{x})|^2 + V(\mathbf{x}) |u(\mathbf{x})|^2 \} d\mu_n(\mathbf{x}) = \sum_{j=1}^p \lambda_j |c_j|^2 \\ \leq \lambda_p \left[ \sum_{j=1}^p |c_j|^2 \right] = \lambda_p \|u\|^2.$$

Next by T. 11 and T. 12, since  $\mathcal{D}_2 = \mathcal{D}$  if  $n + \rho \geq 4$ , for each  $\varphi_j \in \mathcal{D}_2$ ,  $1 \leq j \leq p$ , we can choose a sequence  ${}_k \varphi_j \in \mathcal{D}_1$  having  $\|\varphi_j - {}_k \varphi_j\|_2 \rightarrow 0$ ,

$$\int_{R_n} |V(\mathbf{x})| |\varphi_j(\mathbf{x}) - {}_k \varphi_j(\mathbf{x})|^2 d\mu_n(\mathbf{x}) \rightarrow 0,$$

and

$$\int_{R_n} |\nabla_{\text{gen}} \mathcal{P}_j(\mathbf{x}) - \nabla_k \mathcal{P}_j(\mathbf{x})|^2 d\mu_n(\mathbf{x}) \rightarrow 0$$

as  $k \rightarrow \infty$ , and also satisfying  $\|\mathcal{P}_j - {}_k\mathcal{P}_j\|_2 < 1/(3p)$  for all  $k$ . This last requirement assures that  $|({}_k\mathcal{P}_j, {}_k\mathcal{P}_{j'}) - \delta_{j,j'}| \leq \theta/p$  for some fixed  $\theta < 1$  (actually  $\theta = \frac{8}{9}$  here), and hence the set  $\{{}_k\mathcal{P}_j\}$  over  $1 \leq j \leq p$  is linearly independent and thus spans a  $p$  dimensional manifold  $\mathcal{M}_k$  of  $\mathcal{D}_1$ . Thus given  $S \subseteq L_2(R_n)$  with  $\text{card } S < p$ , the orthogonal projection of  $S$  into the subspace  $\mathcal{M}_k$  spans at most a  $p - 1$  dimensional manifold, and hence there exists  $u_k \in \mathcal{M}_k$ ,  $\|u_k\| = 1$ ,  $u_k \perp S$ . Also

$$u_k = \sum_{j=1}^p {}_k c_j {}_k \mathcal{P}_j$$

has

$$\begin{aligned} 1 = \|u_k\|^2 &= \sum_{j,j'=1}^p {}_k c_j {}_k \bar{c}_{j'} ({}_k \mathcal{P}_j, {}_k \mathcal{P}_{j'}) \geq \sum_{j=1}^p |{}_k c_j|^2 \\ &\quad - \frac{\theta}{p} \left( \sum_{j=1}^p |{}_k c_j| \right)^2 \geq (1 - \theta) \sum_{j=1}^p |{}_k c_j|^2 \end{aligned}$$

by the Schwarz inequality,

$$\sum_{j=1}^p |{}_k c_j|^2 \leq (1 - \theta)^{-1} < +\infty,$$

and hence by taking subsequences we can assume  ${}_k c_j \rightarrow {}_0 c_j$  for some complex  ${}_0 c_j$  as  $k \rightarrow +\infty$  for each  $j$ ,  $1 \leq j \leq p$ . Thus  $u_0 = \sum_{j=1}^p {}_0 c_j \mathcal{P}_j$  has  $u_k \rightarrow u_0$  in each of the three quadratic form norms for which  ${}_k \mathcal{P}_j \rightarrow \mathcal{P}_j$  above, using the Minkowski inequality. Hence (42) for  $u_0$  has the left side to be equal the limit as  $k \rightarrow +\infty$  of the same expression with  $u_k$  replacing  $u_0$ . Since  $u_k \in \mathcal{M}_k \subseteq \mathcal{D}_1$ ,  $\|u_k\| = 1$ , and  $u_k \perp S$ , we thus see that  $\tau_p(\mathcal{D}_1) \leq \lambda_p$  holds for existing  $\lambda_p < h_1$ , which completes the first part of our proof.

In order to complete the proof Theorem T. 13, we must show  $\tau_p(\mathcal{D}_0) < h_1$  has  $\tau_p(\mathcal{D}_0) \geq \lambda_p$  with  $\lambda_p$  existing. Consider fixed  $u_0 \in \mathcal{D}_0$ . The truncate  $V_q$ , defined as for (16), with  $q > A$  satisfies Condition II clearly, and thus defines the self-adjoint  ${}_q H$  with domain  $\mathcal{D} \supseteq \mathcal{D}_0$  as in T. 1, and  ${}_q H \supseteq {}_q H_0$  defined on  $\mathcal{D}_0$  by (2) with  $V_q$ . Hence by integrating by parts, and using the exponential bounds in the definition of  $\mathcal{D}_0$ ,  ${}_q E$  being the spectral measure for  ${}_q H$ ,

$$\begin{aligned} (43) \quad &\int_{R_n} \{|\nabla u_0(\mathbf{x})|^2 + V_q(\mathbf{x})|u_0(\mathbf{x})|^2\} d\mu_n(\mathbf{x}) = ({}_q H u_0, u_0) \\ &= \int_{-\infty}^{\infty} \lambda d({}_q E(\lambda) u_0, u_0) \\ &= \left\{ \sum_{q \wedge j < h} \tilde{\lambda}_j {}_q \lambda_j ({}_q E(\{\tilde{\lambda}_j\}) u_0, u_0) \right\} + \int_{\lambda \geq h} \lambda d({}_q E(\lambda) u_0, u_0) \\ &\geq \left\{ \sum_{q \wedge j < h} \tilde{\lambda}_j {}_q \lambda_j ({}_q E(\{\tilde{\lambda}_j\}) u_0, u_0) \right\} + h \left\{ \|u_0\|^2 - \sum_{q \wedge j < h} ({}_q E(\{\tilde{\lambda}_j\}) u_0, u_0) \right\} \end{aligned}$$

for any  $h < h_1$ , the sum  $\sum_{\tilde{\lambda}_j < h}$  being finite then by T. 7 and here being defined to give one term for each distinct  $\lambda \in \Sigma_q$ .

Now taking  $q \rightarrow +\infty$  in (43), by Condition I and dominated convergence the limit of the left side is obtained by replacing  $V_q$  by  $V$ . On the right side  $|\overline{V - V_q}|_\omega \rightarrow 0$  by (16) under Condition I, and hence  $\|G_\omega - {}_qG_\omega\| \rightarrow 0$  ([1], 3.20, p. 561). Defining  $F_\omega$  as the spectral measure of  $G_\omega$  and  $f_\omega(\lambda) = 1/(\lambda + \omega^2)$ , we have ([1], Theorem 4.5, p. 567)  $E(B) = F_\omega(f_\omega(B))$  for Borel subsets  $B$  of the spectrum  $\Sigma$  of  $H_2$ ; also the usual loop integral formula

$$F_\omega([a, c]) = \frac{1}{2\pi i} \int (zI - G_\omega)^{-1} dz$$

holds in the weak sense, where  $C$  is a rectangular curve in the complex plane with sides parallel to the axes whose interior region intersects the real axis in  $(a, c)$ , provided both "a" and "c" are at a positive distance from  $f_\omega(\Sigma)$ . Thus  $\|G_\omega - {}_qG_\omega\| \rightarrow 0$  implies  $\|E(B) - {}_qE(B)\| \rightarrow 0$  for any closed interval  $B \subseteq (-\infty, h_1)$  whose endpoints are not in  $\{\lambda_p\}$ . Hence  ${}_q\lambda_j \rightarrow \lambda_j$  for  $\lambda_j$  existing, and (43) becomes

$$\begin{aligned} (44) \quad & \int_{R_n} \{|\nabla u(x)|^2 + V(x)|u(x)|^2\} d\mu_n(x) \\ & \geq \left\{ \sum_{\tilde{\lambda}_j < h} \lambda_j (E(\{\lambda_j\})u, u) \right\} + h \left\{ \|u\|^2 - \sum_{\tilde{\lambda}_j < h} (E(\{\lambda_j\})u, u) \right\} \\ & = \left\{ \sum_{\lambda_j < h} \lambda_j |(u, \varphi_j)|^2 \right\} + h \left\{ \|u\|^2 - \sum_{\lambda_j < h} |(u, \varphi_j)|^2 \right\} \end{aligned}$$

for  $u \in \mathcal{D}_0$  and  $h < h_1$ , the sum  $\sum_{\lambda_j < h}$  meaning as usual one term for each index  $j$  satisfying  $\lambda_j < h$ .

Now assume  $\tau_p(\mathcal{D}_0) < h_1$  for some integer  $p \geq 1$ , set  $h' = \frac{1}{2}[h_1 + \tau_p(\mathcal{D}_0)]$ , and thus  $\tau_p(\mathcal{D}_0) < h' < h_1$ . Now consider the particular  $S = \{\varphi_j | \lambda_j < h' \text{ exists and } j < p\} \subseteq L_2(R_n)$ , for which  $(\text{card } S) < p$  clearly. Thus (44) with  $\|u\| = 1$ ,  $(u, \varphi_j) = 0$  for  $\varphi_j \in S$ , and  $h = h'$  would give  $\tau_p(\mathcal{D}_0) \geq h'$  if either  $\lambda_p$  did not exist or else  $\lambda_p \geq h'$ , yielding the contradiction  $h' \leq \tau_p(\mathcal{D}_0) < h'$ . Thus  $\lambda_p < h' < h_1$  must exist, and (44) with  $\|u\| = 1$ ,  $(u, \varphi_j) = 0$  for  $j < p$ , and  $h = \lambda_p$  gives  $\tau_p(\mathcal{D}_0) \geq \lambda_p$  as desired. Thus the proof of Theorem T. 13 is complete.

### REFERENCES

1. F. H. Brownell, *Spectrum of the static potential Schrödinger equation over  $E_n$* , Annals of Math., **54** (1951), 554-594.
2. T. Kato, *Fundamental properties of Hamiltonian operators of Schrödinger type*, Trans. Amer. Math. Soc., **70** (1951), 195-211.
3. B. v. S. Nagy, *Spectraldarstellung linearer Transformationen des Hilbertschen Raumes*, Ergebnisse Math., **5** (1942).

4. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford U. Press, London, 1937.
5. F. Stummel, *Singulare elliptische Differentialoperatoren in Hilbertschen Raumen*, Math. Annalen, **132** (1956), 150-176.
6. (added in proof) N. Nilsson, *Essential self-adjointness and the spectral resolution of Hamiltonian operators*, Kungl. Fysiogr. Sällsk. i Lund Förh., **29** (1959), no. 1, 1-19.

UNIVERSITY OF WASHINGTON



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DAVID GILBARG

Stanford University  
Stanford, California

R. A. BEAUMONT

University of Washington  
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California  
Los Angeles 7, California

L. J. PAIGE

University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH

C. E. BURGESS

E. HEWITT

A. HORN

V. GANAPATHY IYER

R. D. JAMES

M. S. KNEBELMAN

L. NACHBIN

I. NIVEN

T. G. OSTROM

H. L. ROYDEN

M. M. SCHIFFER

E. G. STRAUS

G. SZEKERES

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA

OREGON STATE COLLEGE

UNIVERSITY OF OREGON

OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE COLLEGE

UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY

CALIFORNIA RESEARCH CORPORATION

HUGHES AIRCRAFT COMPANY

SPACE TECHNOLOGY LABORATORIES

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.



Frank Herbert Brownell, III, <i>A note on Kato's uniqueness criterion for Schrödinger operator self-adjoint extensions</i> .....	953
Edmond Darrell Cashwell and C. J. Everett, <i>The ring of number-theoretic functions</i> .....	975
Heinz Otto Cordes, <i>On continuation of boundary values for partial differential operators</i> .....	987
Philip C. Curtis, Jr., <i>n-parameter families and best approximation</i> .....	1013
Uri Fixman, <i>Problems in spectral operators</i> .....	1029
I. S. Gál, <i>Uniformizable spaces with a unique structure</i> .....	1053
John Mitchell Gary, <i>Higher dimensional cyclic elements</i> .....	1061
Richard P. Gosselin, <i>On Diophantine approximation and trigonometric polynomials</i> .....	1071
Gilbert Helmsberg, <i>Generating sets of elements in compact groups</i> .....	1083
Daniel R. Hughes and John Griggs Thompson, <i>The H-problem and the structure of H-groups</i> .....	1097
James Patrick Jans, <i>Projective injective modules</i> .....	1103
Samuel Karlin and James L. McGregor, <i>Coincidence properties of birth and death processes</i> .....	1109
Samuel Karlin and James L. McGregor, <i>Coincidence probabilities</i> .....	1141
J. L. Kelley, <i>Measures on Boolean algebras</i> .....	1165
John G. Kemeny, <i>Generalized random variables</i> .....	1179
Donald G. Malm, <i>Concerning the cohomology ring of a sphere bundle</i> .....	1191
Marvin David Marcus and Benjamin Nelson Moyls, <i>Transformations on tensor product spaces</i> .....	1215
Charles Alan McCarthy, <i>The nilpotent part of a spectral operator</i> .....	1223
Kotaro Oikawa, <i>On a criterion for the weakness of an ideal boundary component</i> .....	1233
Barrett O'Neill, <i>An algebraic criterion for immersion</i> .....	1239
Murray Harold Protter, <i>Vibration of a nonhomogeneous membrane</i> .....	1249
Victor Lenard Shapiro, <i>Intrinsic operators in three-space</i> .....	1257
Morgan Ward, <i>Tests for primality based on Sylvester's cyclotomic numbers</i> .....	1269
L. E. Ward, <i>A fixed point theorem for chained spaces</i> .....	1273
Alfred B. Willcox, <i>Šilov type C algebras over a connected locally compact abelian group</i> .....	1279
Jacob Feldman, <i>Correction to "Equivalence and perpendicularity of Gaussian processes"</i> .....	1295