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**$n$ -PARAMETER FAMILIES AND BEST APPROXIMATION**

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**1. Introduction.** Let  $f(x)$  be a real valued continuous function defined on a closed finite interval and let  $F$  be a class of approximating functions for  $f$ . Suppose there exists a function  $g_0 \in F$  such that  $\|f - g_0\| = \inf_{g \in F} \|f - g\|$  where  $\|f\| \equiv \sup_{x \in [a, b]} |f(x)|$ . The problem of characterizing  $g_0$  and giving conditions that it be unique is classical and has received attention from many authors. The well-known results for polynomials were generalized by Bernstein [2] to "Chebyshev" systems. Later Motzkin [10] and Tornheim [15] further extended these theorems to not necessarily linear families of continuous functions. The only essential requirement was that to any  $n$ -points in the plane with distinct abscissae lying in a finite interval  $[a, b]$ , there should be a unique function in the class  $F$  passing through the given points. Such a system  $F$  is called an  $n$ -parameter family. Constructive methods for determining the function from  $F$  of best approximation to  $f$ , due to Remes [14] in the polynomial case, were extended to the above situation by Novodvorskii and Pinsker [13]. In this paper and in the paper of Motzkin two apparently additional requirements were placed on the system  $F$ . One, a continuity condition, was shown by Tornheim to follow from the axioms of  $F$ . The other, a condition on the multiplicity of the roots of  $f - g, f, g \in F$ , also follows from the definitions as will be shown in § 2. In § 3 the characterization of  $g_0$  is discussed. Methods for constructing  $g_0$  are given in § 4. These are based on the maximization of a certain function of  $n + 1$  variables. In § 5 it is shown that an  $n$ -parameter family has a unique function of best approximation to an arbitrary continuous function in the  $L_{p, N}$  norm if and only if  $F$  is the translate of a linear  $n$ -parameter family. The problem of the existence of  $n$ -parameter families on general compact spaces  $S$  is discussed in § 6. Under additional hypotheses on  $F$  it is shown that  $S$  must be homeomorphic to a subset of the circumference of the unit circle. If  $n$  is even this subset must be proper.

**2.  $n$ -parameter families functions.** Following Tornheim we define, for a fixed integer  $n \geq 1$ , an  $n$ -parameter family of functions  $F$  to be a class of real valued continuous functions on the finite interval  $[a, b]$  such that for any real numbers

$$x_1, \dots, x_n, y_1, \dots, y_n, a \leq x_1 < x_2 < \dots < x_n \leq b$$

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there exists a unique  $f \in F$  such that  $f(x_i) = y_i$   $i = 1, \dots, n$ . For convenience we will usually take  $[a, b]$  to be the interval  $[0, 1]$ . We will include the possibility that 0 and 1 are identified. Then of course  $x_1 \neq x_n$ , and the functions of  $F$  are periodic of period 1. We call such a family a periodic  $n$ -parameter family. If we wish to consider specifically the case when 0 and 1 are not identified, we will refer to  $F$  as an ordinary  $n$ -parameter family. If  $F$  is a linear vector space of functions then we will call  $F$  a linear  $n$ -parameter family (e.g., polynomials of degree  $\leq n - 1$ ). The following continuity theorem of Tornheim [15] is a generalization of a result of Beckenbach [1] for  $n = 2$ .

**THEOREM 1.** *Let  $F$  be an  $n$ -parameter family on  $[0, 1]$ . For*

$$k = 1, 2, \dots, \text{ let } x_1^{(k)}, \dots, x_n^{(k)}, y_1^{(k)}, \dots, y_n^{(k)}, 0 \leq x_1^{(k)} < \dots < x_n^{(k)} \leq 1$$

*be given sequences of real numbers and let  $f_k$  be the unique function from  $F$  such that*

$$f_k(x_i^{(k)}) = y_i^{(k)} \quad i = 1, \dots, n.$$

*Suppose for each*

$$i, \lim_{k \rightarrow \infty} x_i^{(k)} = x_i, \lim_{k \rightarrow \infty} y_i^{(k)} = y_i \text{ and } 0 \leq x_1 < \dots < x_n \leq 1.^1$$

*Let  $f$  be the unique function from  $F$  such that  $f(x_i) = y_i$   $i = 1, \dots, n$ . Then  $\lim_{k \rightarrow \infty} f_k = f$  uniformly on  $[0, 1]$ .*

*Proof.* If 0 and 1 are not identified the proof is given in [15]. Therefore, let 0 and 1 be identified and the functions of  $F$  be periodic. Suppose  $f_k$  does not tend uniformly to  $f$ . For some  $\varepsilon > 0$ , there exists a sequence  $\{u_k\} \subset [0, 1]$  such that for each  $k$ ,  $|f(u_k) - f_k(u_k)| \geq \varepsilon$ . Since a subsequence of  $\{u_k\}$  converges, we may assume  $\{u_k\}$  does and let  $u = \lim_{k \rightarrow \infty} u_k$ . By a suitable rotation of  $[0, 1]$  we may assume  $u, x_1, \dots, x_n$  all lie in the interior of an interval  $[a, b]$ ,  $0 < a < b < 1$ . But  $F$  forms an ordinary  $n$ -parameter family on  $[a, b]$  and hence  $f_k \rightarrow f$  uniformly on  $[a, b]$  which is a contradiction. This completes the proof.

We now verify that  $n$ -parameter families are unisolvent in the sense of Motzkin [10]. Let  $f, g \in F$  and let  $x$  be an interior point of  $[0, 1]$ . If  $x$  is a zero of  $f - g$  and if  $f - g$  does not change sign in a suitably small neighborhood about  $x$  then we will say the zero  $x$  has multiplicity 2, otherwise we say  $x$  has multiplicity 1. If 0 and 1 are not identified and either is a zero of  $f - g$ , then the multiplicity is taken to be 1. We shall denote the sum of the multiplicities of the zeros of  $f - g$  within an interval  $[a, b]$  by  $m_{a,b}(f, g)$ . The following generalized con-

<sup>1</sup> If 0, 1 are identified we assume  $x_n^{(k)} < 1$  and  $x_n < 1$ ,

vexity notion is also useful. A continuous function  $h$  will be said to be convex to  $F$  if  $h$  intersects no function of  $F$  at more than  $n$  points. The following result extends Theorems 2 and 3 of [15].

**THEOREM 2.** *Let  $F$  be an  $n$ -parameter family on  $[0, 1]$  and let  $h$  be convex to  $F$ . Then for any  $f, g \in F$ ,  $m_{0,1}(f, h) \leq n$  and  $m_{0,1}(f, g) \leq n - 1$ .*

*Proof.* We assume first that 0 and 1 are not identified and that  $F$  is an ordinary  $n$ -parameter family. We verify the first statement by induction on  $n$ . For  $n = 1$  the result follows by [15] Theorem 2. Hence, let  $h$  be a continuous function convex to a  $k + 1$  parameter family  $F$  and assume the conclusion holds for all  $k$ -parameter families. For  $f \in F$  let  $x_i, i = 1, \dots, m$ , be the zeros of  $f - h$  ordered from left to right and assume  $m_{0,1}(f, h) > k + 1$ . Choose a point  $u$  such that  $x_1 < u < x_2$ . If  $F_1 = \{g \in F \mid g(x_1) = h(x_1)\}$ , then  $F_1$  is a  $k$ -parameter family on  $[u, 1]$ .  $f \in F_1$  and  $h$  is convex to  $F_1$ . By our inductive assumption  $m_{u,1}(f, h) \leq k$ . Therefore  $x_1$  must be a zero of  $f - h$ , and  $m_{0,1}(f, h) = k + 2$ . By the same reasoning we may assume  $x_m$  is a double zero of  $f - h$ .

We now construct a set  $E$  of  $k$  points from  $[0, 1]$  in the following manner. First choose an  $\varepsilon > 0$  such that  $x_i + 2\varepsilon < x_{i+1} - 2\varepsilon, i = 1, \dots, m - 1$ . If  $x$  is a single zero of  $f - h$  then let  $x$  belong to  $E$ . If  $x$  is a double zero of  $f - h, x \neq x_1, x_m$  let  $x + \varepsilon$ , and  $x - \varepsilon$  belong to  $E$ . We add the points  $x_1 + \varepsilon, x_m - \varepsilon$ . Since  $m_{x_1+\varepsilon, x_m-\varepsilon}(f, h) = k - 2$  it is clear that  $E$  contains exactly  $k$  points. Choose a point  $x', x_1 + \varepsilon < x' < x_2 - \varepsilon$ . Let  $f_n$  be the unique function in  $F$  such that

$$\begin{aligned} f_n(x) &= f(x), x \in E \\ f_n(x') &= f(x') + \frac{1}{n} \operatorname{sgn} [f(x') - h(x')] \end{aligned}$$

Now  $f_n - f$  has  $k$  zeros which must all be simple by [15] Theorem 3. Within the interval  $[x_1, x_m]$   $f_n - h$  has exactly  $k$  simple zeros since  $f_n$  was chosen so that at the points  $x_i \pm 2\varepsilon, i = 2, \dots, m - 1, x_1 + 2\varepsilon, x_m - 2\varepsilon, f$  lies between  $f_n$  and  $h$ . Hence for  $0 \leq x < x_1$  and  $x_m < x \leq 1, f_n$  and  $h$  are on the same side of  $f$  (i.e.,  $\operatorname{sgn} [f_n(x) - f(x)] = \operatorname{sgn} [h(x) - f(x)]$ ). But by Theorem 1,  $f_n$  tends uniformly to  $f$  as  $n \rightarrow \infty$ . Hence for  $n$  sufficiently large  $f_n - h$  must have at least  $k + 2$  zeros which is a contradiction.

The case when 0 and 1 are identified and  $F$  is periodic causes no difficulty. For if  $x_1, \dots, x_m$  are the zeros of  $f - h$ , using a suitable rotation we may assume that there is an interval  $[a, b]$ , such that  $0 < a < x_1 < \dots < x_m < b < 1$ .  $F$  is an ordinary  $n$ -parameter family on  $[a, b]$  and  $m_{0,1}(f, h) = m_{a,b}(f, h) \leq n$ .

The verification of the second assertion is very similar to the above, and we leave the details to the reader.

**COROLLARY.** *There are no periodic  $n$ -parameter families when  $n$  is an even integer.*

*Proof.* Suppose false. Let  $F$  be a periodic  $n$ -parameter family and  $n$  an even integer. Let  $f \in F$  and choose  $x_i$   $i = 1, \dots, n$  such that  $0 < x_1 < x_2 < \dots < x_n < 1$ . Choose  $g \in T$  such that  $g(x_i) = f(x_i)$   $i = 1, \dots, n-1$ ,  $g(x_n) = f(x_n) + 1$ . By Theorem 2,  $f - g$  changes sign at each of the points  $x_i$ ,  $i = 1, \dots, n-1$ ; and since  $f - g$  can have no other zeros within  $[0, 1]$ ,  $g(1) > f(1)$ . On the other hand  $g(0) < f(0)$  which is a contradiction, since  $f, g$  are periodic of period 1.

**3. Best approximation in the  $L_\infty$  norm.** If  $g$  is continuous on  $[0, 1]$ ,  $g \notin F$ , then  $\{g - f\}$  forms a new  $n$ -parameter family. Hence without loss of generality we may consider the characterization and construction of the function  $\hat{f} \in F$  such that

$$\|\hat{f}\| = \inf_{f \in F} \|f\| \equiv \delta$$

We first adopt the following notation. If  $S \subset [0, 1]$

$$\delta_S = \inf_{f \in F} \sup_{t \in S} |f(t)|.$$

Let  $T$  denote the class of vectors  $\mathbf{u} = (u_1, \dots, u_{n+1})$  satisfying the condition that  $0 \leq u_1 < u_2 < \dots < u_{n+1} \leq 1$ . The statements and proofs of the results of this section are valid when  $F$  consists of continuous periodic functions on  $[0, 1]$ . We shall assume, however, that  $F$  is an ordinary  $n$ -parameter family and leave the details in the periodic case to the reader.

The following two lemmas are appropriate generalizations of results of de la Vallee Poussin [6] for polynomials. Where possible we refer the reader to [13] for proofs.

**LEMMA 1.** *For any  $\mathbf{u} = (u_1, \dots, u_{n+1}) \in T$  there exists a unique  $f \in F$  and unique real number  $\lambda$  such that  $f(u_i) = (-1)^i \lambda \cdot i = 1, \dots, n+1$ . Moreover  $|\lambda| = \delta_{\mathbf{u}}$  and  $f$  is the only function in  $F$  with the property that  $\max_{i=1, \dots, n+1} |f(u_i)| = \delta_{\mathbf{u}}$ . In addition suppose for  $k = 1, 2, \dots$  that*

$$\mathbf{u}^{(k)} = (u_1^{(k)}, \dots, u_{n+1}^{(k)}) \in T \text{ and } f_k(u_i^{(k)}) = (-1)^i \lambda^{(k)}.$$

*Then if  $\mathbf{u}^{(k)} \rightarrow \mathbf{u}$  and  $\mathbf{u} \in T$ , it follows that  $f_k \rightarrow f$  uniformly on  $[0, 1]$  and  $\lambda^{(k)} \rightarrow \lambda$ .*

LEMMA 2. Let  $\mathbf{u} \in T$  and a sequence of non-negative numbers  $\lambda_i$   $i = 1, \dots, n+1$  be given. If there exists an  $f \in F$  such that

$$f(u_i) = (-1)^i \lambda_i \quad i = 1, \dots, n+1 \text{ or } f(u_i) = (-1)^{i+1} \lambda_i \quad i = 1, \dots, n+1$$

then either  $\min \lambda_i < \delta_{\mathbf{u}} < \max \lambda_i$  or  $\lambda_i = \delta_{\mathbf{u}} \quad i = 1, \dots, n+1$ .

*Proof.* Lemma 2 is a restatement of Lemma 1 of [13]. Everything in Lemma 1 except the facts that  $|\lambda| = \delta_{\mathbf{u}}$  and the function  $f$  satisfying  $\max_{i=1, \dots, n+1} |f(u_i)| = \delta_{\mathbf{u}}$  is unique is proved explicitly in [13]. To prove the latter statements observe that if there is a  $g \in F$  satisfying  $|g(u_i)| < |\lambda|$  then  $f(u_i) - g(u_i) = (-1)^i \lambda_i \quad i = 1, \dots, n+1$  where either  $\lambda_i \geq 0$ ,  $i = 1, 2, \dots, n+1$  or  $\lambda_i \leq 0 \quad i = 1, 2, \dots, n+1$ . In either case by [12], Lemma 1,  $f - g$  must have at least  $n$  zeros between  $u_1$  and  $u_{n+1}$  counting multiplicity which is a contradiction.

For  $\mathbf{u} \in T$  we will usually denote the function  $f$  of Lemma 1 by  $f_{\mathbf{u}}$ . Next we define a function  $\delta(u_1, \dots, u_{n+1})$  of  $n+1$  variables.

$$\begin{aligned} \delta(\mathbf{u}) \equiv \delta(u_1, \dots, u_{n+1}) &= \delta_{\mathbf{u}} \text{ if } \mathbf{u} = (u_1, \dots, u_{n+1}) \in T \\ &= 0 \text{ otherwise.} \end{aligned}$$

If we restrict the points  $u_i$  to lie in some subset  $S \subset [0, 1]$ , then  $\delta(u_1, \dots, u_{n+1})$  will be denoted  $\delta_S(u_1, \dots, u_{n+1})$ .

LEMMA 3.  $\delta(u_1, \dots, u_{n+1})$  is continuous on  $R^{n+1}$

*Proof.* Assume that  $\delta(u_1, \dots, u_{n+1})$  is not continuous at some point  $\mathbf{u} = (u_1, \dots, u_{n+1})$ . We may assume  $0 \leq u_1 \leq u_2 \leq \dots \leq u_{n+1} \leq 1$ , and by Lemma 1 we may assume that  $m$  ( $\leq n$ ) of the points  $u_i$  are distinct. Consequently  $\delta(u_1, \dots, u_{n+1}) = 0$ . Suppose there exists an  $\varepsilon > 0$  and a sequence  $\{\mathbf{u}_k\} \subset T$  such that  $\mathbf{u}_k \rightarrow \mathbf{u}$  and  $\delta_{\mathbf{u}_k} \geq \varepsilon$ . Let  $u_i^{(k)}$  be the  $i$ th coordinate of  $\mathbf{u}_k$ . Choose  $n$  points  $u'_i$ ,  $0 \leq u'_i < \dots < u'_n \leq 1$  such that  $m$  of the points  $u'_i$  coincide with the  $m$  distinct points  $u_i$ . Let  $f_0$  be the unique function in  $F$  such that  $f_0(u'_i) = 0$ . Choose  $\eta$  such that for any  $i$   $|u'_i - u_i| < \eta$  implies  $|f_0(u'_i)| < \varepsilon/2$ . Choose  $k$  so large that all coordinates of  $\mathbf{u}_k$  are within  $\eta$  neighborhoods of some coordinate of  $\mathbf{u}'$ . Then  $f_{\mathbf{u}_k}(u_i^{(k)}) - f_0(u_i^{(k)}) = (-1)^i \lambda_i$  where  $\text{sgn } \lambda_i^{(k)} = \text{sgn } \lambda_{i+i}^{(k)} \quad i = 1, \dots, n$ . As in the proof of Lemma 1 it follows that  $f_{\mathbf{u}_k} - f_0$  must have at least  $n$  zeros within  $[0, 1]$  which is a contradiction.

Using the function  $\delta(u_1, \dots, u_{n+1})$  one can give a simple proof of the Theorem of Motzkin and Tornheim characterizing the function  $\hat{f}$  which has minimum deviation from zero.

THEOREM 3. There exists a unique  $\hat{f} \in F$  such that  $\|\hat{f}\| = \inf_{f \in F} \|f\|$ .  $\hat{f}$  is uniquely characterized by the fact that for some  $\mathbf{u} = (u_1, \dots, u_{n+1}) \in T$

$\|\hat{f}\| = \delta_u$ .  $\mathbf{u}$  will have this property if and only if  $\delta(u_1, \dots, u_{n+1})$  is an absolute maximum, and then  $\hat{f} = f_u$ .

*Proof.* Since  $\delta(u_1, \dots, u_{n+1})$  is a continuous function on a compact set, its maximum is attained for some  $\mathbf{u} = (u_1, \dots, u_{n+1}) \in T$ . Assert  $\|f_u\| = \delta_u$ . If  $\|f_u\| > \delta_u$ , then there is a point  $x'$  in  $[0, 1]$  for which  $|f_u(x')| = \|f_u\|$ . We form a new vector  $\mathbf{u}' \in T$  by replacing one coordinate  $u_i$  of  $\mathbf{u}$  by  $x'$  in the following way. If  $u_i < x' < u_{i+1}$   $i = 1, \dots, n$  and  $\text{sgn } f_u(u_i) = \text{sgn } f_u(x')$  then let  $u'_j = u_j$ ,  $j \neq i$ , and  $u'_i = x'$ . If  $\text{sgn } f_u(u_i) = (-1) \text{sgn } f_u(x')$  let  $u'_j = u_j$   $j \neq i+1$  and  $u'_{i+1} = x'$ . If  $x' < u_1$  ( $x' > u_{n+1}$ ) and  $\text{sgn } f_u(u_1) = \text{sgn } f_u(x')$  ( $\text{sgn } f_u(u_{n+1}) = \text{sgn } f_u(x')$ ) let  $u'_j = u_j$   $j \neq 1$  ( $j \neq n+1$ ) and  $u'_1 = x'$  ( $u'_{n+1} = x'$ ). If  $\text{sgn } f_u(u_1) = (-1) \text{sgn } f_u(x')$  ( $\text{sgn } f_u(u_{n+1}) = (-1) \text{sgn } f_u(x')$ ) then let  $u'_1 = x'$ ,  $u'_j = u_{j-1}$   $j = 2, \dots, n+1$  ( $u'_j = u_{j+1}$ ,  $j = 1, \dots, n$ ,  $u'_{n+1} = x'$ ). Now either  $f_u(u'_i) = (-1)^i \lambda_i$   $i = 1, \dots, n+1$  or  $f_u(u'_i) = (-1)^{i+1} \lambda_i$   $i = 1, \dots, n+1$  where  $\lambda_i = \delta_u$  or  $\lambda_i = \|f_u\|$ . Therefore by Lemma 2,  $\delta_u < \delta_{u'} < \|f_u\|$  which contradicts the maximality of  $\delta_u$ .

It now follows immediately that  $\|f_u\| = \inf_{f \in F} \|f\|$  and that  $f_u$  is the only such function with this property. For if  $f_0 \in F$  and  $\|f_0\| \leq \|f_u\|$  then  $\|f_0\| \leq \delta_u$  which contradicts Lemma 1. Moreover the same argument shows that if there exists an  $f_0 \in F$  and a  $\mathbf{v} \in T$  such that  $\|f_0\| = \delta_v$  then  $\|f_0\| = \inf_{f \in F} \|f\|$ . It is clear that  $\delta(v_1, \dots, v_{n+1})$  must be an absolute maximum.

In the above theorem if  $\|f\|$  is replaced by  $\|f\|_S = \sup_{t \in S} |f(t)|$  where  $S$  is any closed set of  $[0, 1]$  containing at least  $n+1$  points, then the same conclusions hold. Here of course, the function  $\delta(u_1, \dots, u_{n+1})$  is replaced by  $\delta_S(u_1, \dots, u_{n+1})$  and the points  $u_k$  are assumed to be in  $S$ . The following generalization of [11] Theorem 7.1 is therefore relevant.

**THEOREM 4.** Let  $S_k$ ,  $S$  be closed sets of  $[0, 1]$  such that for each  $k$ ,  $S_k$  contains at least  $n+1$  points;  $S$  contains infinitely many points, and  $S_k \subset S$ . Let  $\hat{f}_k, \hat{f}_0$  be functions from  $F$  which minimize  $\|f\|_{S_k}, \|f\|_S$  respectively. If for each  $\varepsilon > 0$  there exists an integer  $k_0$  such that for  $k > k_0$  each point  $u \in S$  is at a distance less than  $\varepsilon$  from some point of  $S_k$ , then  $\hat{f}_k \rightarrow \hat{f}_0$  uniformly on  $[0, 1]$ .

*Proof.* We assume  $\delta_S > 0$ .  $S_k \subset S$  implies  $\delta_{S_k} \leq \delta_S$ . Choose  $\mathbf{u} = (u_1, \dots, u_{n+1}) \in T$ ,  $u_i \in S$  such that  $\delta_S(u_1, \dots, u_{n+1})$  is an absolute maximum. Let  $\mathbf{u}_k = (u_1^{(k)}, \dots, u_{n+1}^{(k)}) \in T$ ,  $u_j^{(k)} \in S_k$  be chosen such that  $\mathbf{u}_k \rightarrow \mathbf{u}$ . By Lemma 1,  $\delta_{\mathbf{u}_k} \rightarrow \delta_{\mathbf{u}}$  and since  $\delta_{\mathbf{u}_k} \leq \delta_{S_k}$ ,  $\delta_{S_k} \rightarrow \delta_{\mathbf{u}} = \delta_S$ . Let  $\mathbf{v}_k = (v_1^{(k)}, \dots, v_{n+1}^{(k)}) \in T$ ,  $v_i^{(k)} \in S_k$  be chosen so that for each  $k$ ,  $\delta_{S_k}(v_1^{(k)}, \dots, v_{n+1}^{(k)})$  is an absolute maximum. Extract any convergent subsequence  $\mathbf{v}_{k_j}$  with limit  $\mathbf{v}$ .

If  $\mathbf{v} = (v_1, \dots, v_{n+1})$ , then  $v_i \in S$  and  $\delta_v = \delta_s$ . Also  $\hat{f}_{k_j} = f_{v_{k_j}}$  tends uniformly to  $f_v$ , the function from  $F$  with minimum deviation on  $\mathbf{v}$ . But by the uniqueness of  $f_v$ ,  $f_v = \hat{f}_0$ . The above argument shows that any subsequence of  $\{\hat{f}_k\}$  contains a refinement which converges to  $\hat{f}_0$ . Hence  $\lim_{k \rightarrow \infty} \hat{f}_k = \hat{f}_0$  uniformly on  $[0, 1]$ .

**4. The estimation of  $f$ .** In [13] Novodvorskii and Pinsker consider a direct method, due to Remes [14] in the polynomial case, for the estimation of  $\hat{f}$ . However the following Lemma shows that  $\hat{f}$  is continuously dependent on estimates of the best approximation. Hence if  $\mathbf{u}$  is a vector in  $T$  for which  $\delta(\mathbf{u})$  is an estimate of  $\inf_{h \in F} \|f\|$ , then the solution of the equation  $f(u_i) = (-1)^i \lambda$   $i = 1, \dots, n+1$  is the appropriate estimate of  $\hat{f}$ .

**LEMMA 4.** *Let  $\{\delta_n\}$  be a sequence of non-negative numbers converging to  $\delta = \inf_{f \in F} \|f\|$  from below. If  $\mathbf{u}_n$  are vectors in  $T$  for which  $\delta(\mathbf{u}_n) = \delta_n$ , then  $\lim_{n \rightarrow \infty} f_{\mathbf{u}_n} = \hat{f}$  uniformly on  $[0, 1]$ .*

*Proof.* If the conclusion is false there exists a subsequence  $\{\mathbf{u}_{k_j}\}$  and a number  $\varepsilon > 0$  such that  $\|\hat{f} - f_{\mathbf{u}_{k_j}}\| \geq \varepsilon$ . But  $\{\mathbf{u}_{k_j}\}$  may be further refined to obtain a convergent subsequence of vectors. Calling this  $\{\mathbf{u}_{k_j}\}$  and letting  $\mathbf{u}_0 = \lim_{j \rightarrow \infty} \mathbf{u}_{k_j}$  we have by Lemma 1  $\delta(\mathbf{u}_0) = \lim_{j \rightarrow \infty} \delta(\mathbf{u}_{k_j})$ . By Theorem 3  $f_{\mathbf{u}_0} = \hat{f}$  which is a contradiction.

We shall consider two algorithms for estimating  $\delta$  and prove convergence of both.

Each of these algorithms can be used efficiently for actual numerical calculations. A detailed description of method 2 for polynomials on a finite point set can be found in [5]. Also for polynomials on an interval a maximization procedure has been announced by Bratton [3].

For both methods the following notation is convenient. For  $\mathbf{u} = (u_1, \dots, u_{n+1}) \in T$  define for  $j = 1, \dots, n+1$ .

$$\begin{aligned} \delta_{\mathbf{u}}^{(j)}(x) &= \delta(u_1, \dots, u_{j-1}, x, u_{j+1}, \dots, u_{n+1}) \text{ if } u_{j-1} \leq x \leq u_{j+1} \\ &= 0 \text{ otherwise} \end{aligned}$$

where we take  $u_0 = 0, u_{n+2} = 1$ . We now form  $\eta_{\mathbf{u}}(x) \equiv \max_{j=1, \dots, n+1} \delta_{\mathbf{u}}^{(j)}(x)$ . From the continuity of  $\delta(u_1, \dots, u_{n+1})$  it follows that for each  $j$ ,  $\delta_{\mathbf{u}}^{(j)}(x)$  is continuous, and hence  $\eta_{\mathbf{u}}(x)$  is continuous. Therefore there exists a point  $x', 0 \leq x' \leq 1$  and integer  $1 \leq m \leq n+1$  such that

$$\delta_{\mathbf{u}}^m(x') = \max_{j=1, \dots, n+1} \|\delta_{\mathbf{u}}^{(j)}\| = \|\eta_{\mathbf{u}}\|.$$

For a given vector  $\mathbf{u}$  we define  $\mathbf{u}' = (u'_1, \dots, u'_{n+1})$  by setting  $u'_j = u_j, j \neq m, u'_m = x'$ .



**THEOREM 5.** *If vectors  $\mathbf{u}_k$  are defined inductively in the above fashion with  $\mathbf{u}_1 \in T$  chosen arbitrarily, then  $\lim_{k \rightarrow \infty} \delta(\mathbf{u}_k)$  exists and there exists  $\mathbf{u}_0 \in T$  such that  $\delta(\mathbf{u}_0) = \lim_{k \rightarrow \infty} \delta(\mathbf{u}_k)$ . Furthermore  $\delta(\mathbf{u}_0)$  is an absolute maximum of the function  $\delta(\mathbf{u})$ .*

*Proof.*  $\{\delta(\mathbf{u}_k)\}$  is a monotonically increasing, bounded sequence hence convergent. If  $\delta = \lim_{k \rightarrow \infty} \delta(\mathbf{u}_k)$ , then a suitable subsequence  $\{\mathbf{u}_{k_j}\}$ , converges to  $\mathbf{u}_0$  and  $\delta(\mathbf{u}_0) = \delta$ . We now assert  $\eta_{\mathbf{u}_{k_j}}(x)$  converges uniformly to  $\eta_{\mathbf{u}_0}(x)$ . It suffices to assume  $u_i \leq x \leq u_{i+1}$ . Then

$$\begin{aligned} |\eta_{\mathbf{u}_0}(x) - \eta_{\mathbf{u}_{k_j}}(x)| &= |\max(\delta_{\mathbf{u}_0}^i(x), \delta_{\mathbf{u}_0}^{i+1}(x)) - \max(\delta_{\mathbf{u}_{k_j}}^i(x), \delta_{\mathbf{u}_{k_j}}^{i+1}(x))| \\ &\leq |\delta_{\mathbf{u}_0}^i(x) - \delta_{\mathbf{u}_{k_j}}^i(x)| + |\delta_{\mathbf{u}_0}^{i+1}(x) - \delta_{\mathbf{u}_{k_j}}^{i+1}(x)|. \end{aligned}$$

Since  $\delta(\mathbf{u})$  is a uniformly continuous function the latter expression tends to zero uniformly in  $x$ .

Hence

$$\|\eta_{\mathbf{u}_0}\| = \lim_{j \rightarrow \infty} \|\eta_{\mathbf{u}_{k_j}}\|.$$

But

$$\|\eta_{\mathbf{u}_{k_j}}\| = \delta(\mathbf{u}_{k_{j+1}}) \leq \delta(\mathbf{u}_{k_{j+1}}) \leq \|\eta_{\mathbf{u}_{k_{j+1}}}\|$$

Therefore  $\|\eta_{\mathbf{u}_0}\| = \lim_{j \rightarrow \infty} \delta(\mathbf{u}_{k_j}) = \delta(\mathbf{u}_0)$ . It now follows by the same argument as in the proof of Theorem 3 that  $\|f_{\mathbf{u}_0}\| = \delta(\mathbf{u}_0)$  and by Theorem 3,  $\delta(\mathbf{u}_0)$  is a maximum.

For the second method of estimation of  $f$  we alter slightly our definition of  $\delta_{\mathbf{u}}^i(x)$  and  $\delta_{\mathbf{u}}^{n+1}(x)$ . We now define

$$\begin{aligned} \delta_{\mathbf{u}}^1(x) &= \delta(x, u_2, \dots, u_{n+1}) \text{ if } 0 \leq x \leq u_2. \\ &= \delta(u_2, u_3, \dots, u_{n+1}, x) \text{ if } u_{n+1} \leq x \leq 1 \end{aligned}$$

$$\begin{aligned} \delta_{\mathbf{u}}^{n+1}(x) &= \delta(u_1, \dots, u_n, x) \text{ if } u_n \leq x \leq 1 \\ &= \delta(x, u_1, \dots, u_n) \text{ if } 0 \leq x \leq u_1. \end{aligned}$$

The algorithm proceeds as follows. First let  $\varepsilon > 0$  be chosen. Select an arbitrary vector  $\mathbf{u} \in T$ . Maximize  $\delta_{\mathbf{u}}^2(x)$  over its domain of definition. Let  $x'$  be a point for which  $\delta_{\mathbf{u}}^2(x)$  is a maximum. If  $\delta_{\mathbf{u}}^2(x') \geq (1 + \varepsilon)\delta(\mathbf{u})$ , replace  $u_2$  by  $x'$  forming a new vector  $\mathbf{u}'$ . If not, let  $\mathbf{u}' = \mathbf{u}$ . We now maximize  $\delta_{\mathbf{u}'}^3(x)$  and continue inductively. Special attention is necessary for  $\delta_{\mathbf{u}}^{n+1}(x)$  and  $\delta_{\mathbf{u}}^1(x)$ . If  $x'$  is a point for which  $\delta_{\mathbf{u}}^{n+1}(x)$  is a maximum and  $\delta_{\mathbf{u}}^{n+1}(x) \geq (1 + \varepsilon)\delta(\mathbf{u})$ , then  $\mathbf{u}'$  is formed in the following way. If  $x' \geq u_n$  then  $u'_i = u_i$ ,  $i = 1, \dots, n$ ,  $u'_{n+1} = x'$ ; if  $x' \leq u_1$  then  $u'_1 = x'$ ,  $u'_i = u_{i-1}$ ,  $i = 2, \dots, n+1$ . In the latter case, the next function maximized is  $\delta_{\mathbf{u}'}^2(x)$ . If the first case occurs then  $\delta_{\mathbf{u}'}^1(x)$  is maximized. Let  $x''$  be a point for which  $\delta_{\mathbf{u}'}^1(x)$

is a maximum and  $\delta_{u'}^1(x'') \geq (1 + \varepsilon)\delta(u')$ . If  $x'' \leq u'_2$  then  $u'_1 = x''$  and  $u'_i = u_i$   $i = 2, 3, \dots, n + 1$ . If  $x'' \geq u'_{n+1}$  then  $u'_i = u_{i+1}$   $i = 1, \dots, n$  and  $u'_{n+1} = x''$ . For the first case the next function maximized is  $\delta_{u''}^2(x)$ ; the second case,  $\delta_{u''}^{(1)}(x)$ . If

$$\delta_u^{n+1}(x') < (1 + \varepsilon)\delta(u) \quad (\delta_{u'}^1(x'') < (1 + \varepsilon)\delta(u'))$$

then we take  $u' = u$  ( $u'' = u'$ ). When there have been  $n + 1$  consecutive maximizations with no change in the vector  $u$ ,  $\varepsilon$  is now replaced by  $\varepsilon/2$  and the process is repeated. We now continue inductively and pass to the limit as  $\varepsilon/2^k \rightarrow 0$ .

**THEOREM 6.** *The conclusions of Theorem 5 hold if the sequence  $\{u_k\}$  is chosen inductively in accordance with the above algorithm.*

*Proof.* As before,  $\lim_{k \rightarrow \infty} \delta(u_k) = \delta$  exists. We choose a particular convergent subsequence  $\{u_{k_j}\}$  of  $\{u_k\}$ . For each  $j$  let  $u_{k_j}$  be a vector of  $\{u_k\}$  such that for each  $i$ ,  $i = 1, \dots, n + 1$  and all appropriate  $x$ ,  $\delta_{u_{k_j}}^i(x) < (1 + \varepsilon/2^j)\delta(u_{k_j})$ . The algorithm guarantees that for each integer  $j$  such a vector  $u_{k_j}$  exists in the sequence  $\{u_k\}$ . Since a refinement of this sequence is convergent, we assume  $\{u_{k_j}\}$  converges. Then if  $u_{k_j} \rightarrow u_0$ ,  $\delta(u_0) = \delta$ . Suppose  $\delta(u_0)$  is not a maximum of  $\delta(u)$ , then  $\|f_{u_0}\| > \delta(u_0)$ . Choose  $x'$  so that  $|f_u(x')| = \|f\|$ , and form  $u'$  by replacing one point, the  $i$ th say, of  $u_0$  by  $x'$  in the manner of the proof of Theorem 3. Form  $u'_{k_j}$  by replacing the  $i$ th coordinate of  $u_{k_j}$  by  $x'$ . Then  $u'_{k_j} \rightarrow u'$  and  $\delta(u'_{k_j}) \rightarrow \delta(u')$ . Therefore for  $j$  sufficiently large, since  $\delta(u') > \delta$ ,

$$\delta(u'_{k_j}) > \frac{\delta(u') + \delta}{2}$$

On the other hand for each  $j$  there is a point  $x$  and an integer  $m$  such that

$$\delta(u'_{k_j}) = \delta_{u_{k_j}}^m(x) \leq \left(1 + \frac{\varepsilon}{2^j}\right) \delta(u_{k_j}) \leq \left(1 + \frac{\varepsilon}{2^j}\right) \delta.$$

For  $j$  sufficiently large this is a contradiction, therefore  $\|f_{u_0}\| = \delta(u_0)$  and  $\delta(u_0)$  is an absolute maximum.

**5. Approximation in  $L_{p,N}$  norm.** For  $N \geq n$  let  $x_1, \dots, x_N$  be  $N$  distinct points of  $[0, 1]$ . In place of the sup norm let  $\|f\| = \{\sum_{i=1}^N |f(x_i)|^p\}^{1/p}$  and assume  $p > 1$ . The fundamental problem to be considered here is to give necessary and sufficient conditions that the function  $\hat{f} \in F$  for which  $\|\hat{f}\| = \inf_{f \in F} \|\hat{f}\|$  is unique. Now the image of  $F$  under the mapping  $f \rightarrow (f(x_1), \dots, f(x_N))$  is a closed set in  $N$  dimensional Euclidean

space. By a theorem of Motzkin [9] as generalized by Busemann [4], to each point  $x \in E_N$  there will exist a unique nearest point in a given set  $S \subset E_N$  with respect to a strictly convex metric if and only if  $S$  is closed and convex. Hence  $\hat{f}$  will be unique if and only if  $F$  is convex, but for  $n$ -parameter families we can say more.<sup>2</sup>

**THEOREM 7.** *An  $n$ -parameter family  $F$  is convex if and only if  $F$  is the translate of a linear  $n$ -parameter family.*

*Proof.* If  $F$  is the translate of a linear  $n$ -parameter family, i.e., there exists a continuous  $g$  on  $[0, 1]$  and a linear  $n$ -parameter family  $F_0$  such that each  $f \in F$  can be written uniquely as  $f = g + f'$ ,  $f' \in F_0$ , then  $F$  is obviously convex. Conversely suppose  $F$  is convex. Choose  $n$  distinct points  $x_1, \dots, x_n$  in  $[0, 1]$ . Let  $f_0, f_1, \dots, f_n$  be the unique functions of  $F$  such that  $f_0(x_j) = 0$ ,  $j = 1, \dots, n$ ;  $f_k(x_j) = \delta_{kj}$  for  $k, j = 1, \dots, n$  where  $\delta_{kj}$  is the Kronecker delta. We assert that each  $f \in F$  has a representation as

$$f = f_0 + \sum_{k=1}^n \lambda_k (f_k - f_0) \text{ where } \lambda_k = f(x_k).$$

If such a representation exists it is obviously unique. Also the vector space spanned by  $f_1 - f_0, \dots, f_n - f_0$ , is obviously an  $n$ -parameter family and the theorem is proved. To prove the assertion let

$$F_k = \{f \in F \mid f(x_{k+1}) = f(x_{k+2}) = \dots = f(x_n) = 0\}$$

$$F'_k = \{f \in F \mid f(x_j) = 0 \text{ } j \neq k\}.$$

From the convexity of  $F$ ,  $F'_k$  is a convex one parameter family on a suitably small interval containing  $x_k$ . We assert  $f \in F'_k$  implies  $f = f_0 + \lambda_k (f_k - f_0)$  where  $\lambda_k = f(x_k)$ . By convexity this is obviously true for  $0 \leq \lambda_k \leq 1$ . For  $\lambda_k > 1$  if  $f \in F'_k$ ,  $f(x_k) = \lambda_k$  then by convexity

$$f_k = \frac{1}{\lambda_k} f + \left(1 - \frac{1}{\lambda_k}\right) f_0$$

or  $f = f_0 + \lambda_k (f_k - f_0)$ . If  $\lambda_k < 0$ ,

$$f_0 = \frac{1}{1 - \lambda_k} f + \frac{(-\lambda_k)}{1 - \lambda_k} f_k$$

or  $f = f_0 + \lambda_k (f_k - f_0)$ . To finish the proof we apply an induction. Assume  $f \in F'_k$  implies that  $f = f_0 + \sum_{j=1}^k \lambda_j (f_j - f_0)$  where  $f(x_j) = \lambda_j$  and

<sup>2</sup> For a discussion of related results see the article by Motzkin in the Symposium on Numerical Approximation, University of Wisconsin Press, 1959.

suppose  $g \in F_{k+1}$  and  $g(x_j) = \mu_j, j = 1, \dots, k+1$ . Then if  $g_1 = f_0 + \sum_{j=1}^k 2\mu_j(f_j - f_0), g_2 = f_0 + 2\mu_{k+1}(f_{k+1} - f_0)$  it follows that

$$g' = \frac{g_1 + g_2}{2} \in F_{k+1}$$

and  $g'(x_j) = \mu_j, j = 1, \dots, k+1$ . Therefore

$$g = g' = f_0 + \sum_{j=1}^{k+1} \mu_j(f_j - f_0).$$

**6. The existence of  $n$ -parameter families on compact space.** Let  $f_1, \dots, f_n$ , be  $n$  linearly independent real valued continuous functions defined on a compact set  $S$  in finite dimensional Euclidean space. Let  $V$  be the span of the functions  $f_1, \dots, f_n$ . In 1918 Haar [7] showed that to each continuous real valued function  $g$  defined on  $S$ , there is a unique  $\hat{f} \in V$  satisfying  $\|\hat{f} - g\| = \inf_{f \in V} \|f - g\|$  where  $\|f\| = \sup_{s \in S} |f(s)|$  if and only if no non-zero function in  $V$  vanished at more than  $n-1$  points of  $S$ . Haar noted that the existence of such a set of functions  $V$  placed a severe restriction on the set  $S$ . In 1956 Mairhuber [8] proved that if  $V$  satisfied the above condition of Haar then  $S$  is a homeomorphic image of a subset of the circumference of the unit circle. If  $n$  is even this subset must be proper. It is clear that  $V$  satisfies the condition of Haar if and only if  $V$  is a linear  $n$ -parameter family. The characterization of those compact Hausdorff spaces on which there exist  $n$ -parameter families  $F$  for  $n > 1$  seems to be quite difficult. One can give a characterization if one imposes a rather strong local condition on  $F$ . The result presented here includes the one of Mairhuber, and is proved by somewhat different means. The following fundamental lemma is perhaps of independent interest.

**LEMMA 5.** *Let  $S$  be a compact connected Hausdorff space with the property that for each point  $x \in S$  there exists a neighborhood  $U_x$  and continuous real valued functions  $f_1, f_2$  defined on  $U_x$  such that for  $y, z \in U_x, y \neq z$*

$$(1) \quad \begin{vmatrix} f_1(y) & f_1(z) \\ f_2(y) & f_2(z) \end{vmatrix} \neq 0.$$

*Then  $S$  may be embedded homeomorphically into the circumference  $C$  of the unit circle.*

*Proof.* Without loss of generality we assume  $U_x$  is a closed, therefore compact neighborhood of  $x$ .  $f_1, f_2$  never vanish simultaneously on  $U_x$  and therefore  $f_1/f_2$  defines a continuous mapping of  $U_x$  into the

compactified real line. (1) guarantees that the mapping is one to one and  $\phi_x(u) = \text{Arctan}(f_1/f_2)(u)$  gives a homeomorphism of  $U_x$  into  $C$ .

We next verify that  $S$  is locally connected. To do this it suffices to show that for each  $x \in S$  there exists a connected neighborhood which can be mapped homeomorphically into  $C$ . In fact if  $\phi_x$  is the homeomorphism for a point  $x \in S$  constructed above, and if  $C_x = \phi_x(U_x)$ , it is enough to show that there exists a connected neighborhood  $V_x$  in  $C_x$  of  $\lambda_x \equiv \phi_x(x)$ . For then  $\phi_x^{-1}(V_x)$  is a connected neighborhood of  $x$  contained in  $U_x$ . But  $C_x$  is a compact subset of  $C$ . Therefore let  $I_x$  be the component of  $\lambda_x$  in  $C_x$ .  $I_x$  is a compact connected subset of  $C$ .  $I_x$  is then either an interval or all of  $C$ . If  $I_x$  is the latter we are through. Also if  $I_x$  is an interval and  $\lambda_x$  an interior point (relative to  $C$ ) then  $\phi_x^{-1}(I_x)$  is the required neighborhood. Hence assume that  $\lambda_x$  is an end point of  $I_x$ . This will include that degenerate case when  $I_x$  is just one point. We may also assume that there does not exist a suitably small connected neighborhood  $N$  of  $\lambda_x$  in  $C$  such that  $N \cap C_x \subset I_x$ . For then  $\phi_x^{-1}(N \cap N_x)$  is an appropriate neighborhood of  $x$ . Therefore it now must follow that for any connected neighborhood  $N$  of  $\lambda_x$  in  $C$  there exists  $\lambda_1, \lambda_2$  in the interior of  $N$  such that  $\lambda_1, \lambda_2 \notin C_x$  and  $(\lambda_1, \lambda_2) \cap C_x \neq \emptyset$ . If we let  $F = \phi_x^{-1}[(\lambda_1, \lambda_2) \cap C_x]$  and  $G = \phi_x^{-1}[C_x \sim (\lambda_1, \lambda_2)]$  then  $F \cup (S \sim U_x)$  and  $G$  separate  $S$  which is a contradiction.

We note that  $S$  is certainly a separable metric since a finite number of homeomorphic images of subsets of  $C$  cover  $S$ . Hence by [16] Theorem 5.1,  $S$  is arc wise connected.

We now assert  $S$  is homeomorphic to a subset of  $C$ . Let  $U_1, \dots, U_n$  be a finite collection of connected neighborhoods covering  $S$  each of which is homeomorphic to a subset of  $C$ . By a suitable rearrangement we may assume that  $U_2 \cap U_1 \neq \emptyset$  and  $U_2 \not\subset U_1$ . Let  $x_1 \in U_1 \sim U_2$ ,  $x_2 \in U_2 \sim U_1$ ,  $x \in U_1 \cap U_2$ . Let  $A$  be the maximal subset of  $U_1 \cup U_2$  connecting  $x_1, x, x_2$ . This must be all of  $U_1 \cup U_2$ , for if  $y \in U_1 \cup U_2$  and  $y \notin A$ , then  $y$  may be connected to any point in  $A$  by an arc in  $U_1 \cup U_2$ . If  $y$  is connected to  $A$  at an end point of  $A$ , this is an enlargement of  $A$  which contradicts maximality. If  $y$  is connected to  $A$  at a point other than an end point, then no neighborhood of this point is homeomorphic to a subset of  $C$ . This also is a contradiction. If  $U_1 \cup U_2$  is not all of  $S$  then  $U_1 \cup U_2$  is homeomorphic to an arc, and by induction the homeomorphism may be extended to all of  $S$ .

**THEOREM 8.** *For  $n > 1$  let  $F$  be an  $n$ -parameter family of functions defined on a compact Hausdorff space  $S$ . Suppose in addition that to each point  $x \in S$  there exists a neighborhood  $N_x$  and functions  $f_1, f_2 \in F$  such that*

$$\begin{vmatrix} f_1(y) & f_1(z) \\ f_2(y) & f_2(z) \end{vmatrix} \neq 0$$

for  $y, z \in N_x, y \neq z$ . Then there exists a homeomorphism of  $S$  into the circumference of the unit circle. If  $n$  is even the image of  $S$  must be a proper subset of  $C$ .

*Proof.* First we note that  $S$  cannot have a proper subset  $W$  homeomorphic to  $C$ . If  $n$  is even this follows directly from the Corollary to Theorem 2. If  $n$  is odd, choose  $x \in S \sim W$  and let  $F' = \{f \in F \mid f(x) = 0\}$ ; then  $F'$  is an  $n - 1$  parameter family defined on  $W$ . Since  $n - 1$  is even this is a contradiction. We may therefore assume that if  $n$  is even  $S$  is not homeomorphic to  $C$ .

If  $I$  is a component of  $S$  then by Lemma 5 there exists a homeomorphism  $\phi$  of  $I$  onto the closed interval  $[0, 1]$  considered as a subset of  $C$ . We assert that if  $I$  is not all of  $S$ , then  $\phi$  can be extended to an open and closed set  $U \supset I$ .  $U$  and its complement then separate  $S$ . If  $I$  is itself open in  $S$  then we take  $U = I$ . If not, let  $x = \phi^{-1}(0), y = \phi^{-1}(1)$ . Let  $N_x, N_y$  be compact neighborhoods of  $x$  and  $y$  respectively and let  $\phi_x, \phi_y$  be homeomorphisms of  $N_x$  and  $N_y$  respectively into  $C$ . We may assume  $\phi_x(x) = 0, \phi_y(y) = 1$  and

$$\phi_x(N_x \cap I) \subset [0, 1] \text{ and } \phi_y[N_y \cap I] \subset [0, 1].$$

If we define  $\phi'$  by

$$\begin{aligned} \phi'(z) &= \phi(z) && \text{if } z \in I \\ &= \phi_x(z) && \text{if } z \in N_x \sim I \\ &= \phi_y(z) && \text{if } z \in N_y \sim I \end{aligned}$$

then  $\phi'$  is a homeomorphism of  $N_x \cup N_y \cup I \equiv N$  into  $C$ . Also  $\text{int. } N \supset I$ . Now  $[0, 1] = \phi'(I)$  is the maximal connected subset of  $\phi'(N)$  containing  $\phi'(I)$ . Therefore there exist sequences  $\{\lambda_n\}, \{\mu_n\}$  of real numbers tending monotonically to 0 from below, and monotonically to 1 from above, respectively such that  $\{\lambda_n\} \cap \phi'(N) = \emptyset$  and  $\{\mu_n\} \cap \phi'(N) = \emptyset$ . Choose  $n$  large enough that  $\phi'^{-1}[\lambda_n, 0] \subset \text{interior of } N_x$  and  $\phi'^{-1}[1, \mu_n] \subset \text{interior of } N_y$ . Clearly  $J_n = \phi'^{-1}[\lambda_n, \mu_n]$  is a closed set containing  $I$ .  $J_n$  is open in the interior of  $N$ . Hence  $J_n$  is open in  $S$ .

Let  $T$  be the class of open sets  $O$  of  $S$  which can be mapped homeomorphically into  $C$ . We partially order  $T$  in the following way. If  $O_1, O_2 \in T$  then  $O_1 \leq O_2$  if  $O_1 \subset O_2$  and if there exist homeomorphisms  $\phi_1, \phi_2$  of  $O_1, O_2$  respectively into  $C$  such that  $\phi_2$  agrees with  $\phi_1$  on  $O_1$ . By Zorn's lemma there exists a maximal element  $O$  of  $T$ . We assert  $O = S$ . If not, let  $x \in S \sim O$ . Then there exists an open and closed set  $U \ni x$  and mapping  $\phi$  such that  $\phi$  maps  $U$  homeomorphically into  $C$ .

$O \cap U$  and  $O \sim U$  are separated open sets of  $S$ . Hence if  $\phi'$  is any homeomorphism of  $O$  into  $C$  such  $\phi'(O) \cap \phi(U) = \phi$ .  $\phi''$  defined by  $\phi''(x) \equiv \phi(x)$ ,  $x \in O \cap U$ ,  $\phi''(x) \equiv \phi'(x)$ ,  $x \in O \sim U$  is also a homeomorphism of  $O$  into  $C$ .  $\phi''$  has an obvious extension to  $U \cup O$  which contradicts the maximality of  $O$ .

**COROLLARY.** *If  $F$  is a linear  $n$ -parameter family ( $n > 1$ ) defined on the compact Hausdorff space  $S$ , then  $S$  is homeomorphic to a subset of  $C$ . If  $n$  is even the subset must be proper.*

*Proof.* We assume  $S$  contains more than  $n$  points. For a given  $x \in S$  choose  $n - 2$  distinct points  $x_1, \dots, x_{n-2}$  of  $S$  outside a suitably small compact neighborhood  $N_x$  of  $x$ . If  $F_x = \{f \in F \mid f(x_i) = 0, i = 1, \dots, n - 2\}$  then  $F_x$  is a linear 2-parameter family defined on  $N_x$ . Therefore, for any two linearly independent functions  $f_1, f_2$  in  $F_x$ ,

$$\left| \begin{array}{cc} f_1(y) & f_1(z) \\ f_2(y) & f_2(z) \end{array} \right| \neq 0 \text{ for } y, z \in N_x, y \neq z.$$

We now apply the theorem.

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