n-PARAMETER FAMILIES AND BEST APPROXIMATION

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1. Introduction. Let \( f(x) \) be a real valued continuous function defined on a closed finite interval and let \( F \) be a class of approximating functions for \( f \). Suppose there exists a function \( g_0 \in F \) such that
\[
\| f - g_0 \| = \inf_{g \in F} \| f - g \| \quad \text{where} \quad \| f \| = \sup_{x \in [a, b]} |f(x)|.
\]
The problem of characterizing \( g_0 \) and giving conditions that it be unique is classical and has received attention from many authors. The well-known results for polynomials were generalized by Bernstein [2] to "Chebyshev" systems. Later Motzkin [10] and Tornheim [15] further extended these theorems to not necessarily linear families of continuous functions. The only essential requirement was that to any \( n \)-points in the plane with distinct abscissae lying in a finite interval \([a, b]\), there should be a unique function in the class \( F \) passing through the given points. Such a system \( F \) is called an \( n \)-parameter family. Constructive methods for determining the function from \( F \) of best approximation to \( f \), due to Remes [14] in the polynomial case, were extended to the above situation by Novodvorskii and Pinsker [13]. In this paper and in the paper of Motzkin two apparently additional requirements were placed on the system \( F \). One, a continuity condition, was shown by Tornheim to follow from the axioms of \( F \). The other, a condition on the multiplicity of the roots of \( f - g, f, g \in F \), also follows from the definitions as will be shown in § 2. In § 3 the characterization of \( g_0 \) is discussed. Methods for constructing \( g_0 \) are given in § 4. These are based on the maximization of a certain function of \( n + 1 \) variables. In § 5 it is shown that an \( n \)-parameter family has a unique function of best approximation to an arbitrary continuous function in the \( L_{p,N} \) norm if and only if \( F \) is the translate of a linear \( n \)-parameter family. The problem of the existence of \( n \)-parameter families on general compact spaces \( S \) is discussed in § 6. Under additional hypotheses on \( F \) it is shown that \( S \) must be homeomorphic to a subset of the circumference of the unit circle. If \( n \) is even this subset must be proper.

2. \( n \)-parameter families functions. Following Tornheim we define, for a fixed integer \( n \geq 1 \), an \( n \)-parameter family of functions \( F \) to be a class of real valued continuous functions on the finite interval \([a, b]\) such that for any real numbers
\[
x_1, \ldots, x_n, y_1, \ldots, y_n, \quad a \leq x_1 < x_2 < \cdots < x_n \leq b
\]
there exists a unique \( f \in F \) such that \( f(x_i) = y_i, i = 1, \ldots, n \). For convenience we will usually take \([a, b]\) to be the interval \([0, 1]\). We will include the possibility that 0 and 1 are identified. Then of course \( x_i \neq x_n \), and the functions of \( F \) are periodic of period 1. We call such a family a periodic \( n \)-parameter family. If we wish to consider specifically the case when 0 and 1 are not identified, we will refer to \( F \) as an ordinary \( n \)-parameter family. If \( F \) is a linear vector space of functions then we will call \( F \) a linear \( n \)-parameter family (e.g., polynomials of degree \( \leq n - 1 \)). The following continuity theorem of Tornheim [15] is a generalization of a result of Beckenbach [1] for \( n = 2 \).

**Theorem 1.** Let \( F \) be an \( n \)-parameter family on \([0, 1]\). For \( k = 1, 2, \ldots, \) let \( x_1^{(k)}, \ldots, x_n^{(k)}, y_1^{(k)}, \ldots, y_n^{(k)}, 0 \leq x_1^{(k)} < \cdots < x_n^{(k)} \leq 1 \) be given sequences of real numbers and let \( f_k \) be the unique function from \( F \) such that

\[
f_k(x_i^{(k)}) = y_i^{(k)}
\]

\( i = 1, \ldots, n \).

Suppose for each \( i \),

\[
limit_{k \to \infty} x_i^{(k)} = x_i, \lim_{k \to \infty} y_i^{(k)} = y_i \text{ and } 0 \leq x_1 < \cdots < x_n \leq 1.
\]

Let \( f \) be the unique function from \( F \) such that \( f(x_i) = y_i, i = 1, \ldots, n \). Then \( \lim_{k \to \infty} f_k = f \) uniformly on \([0, 1]\).

**Proof.** If 0 and 1 are not identified the proof is given in [15]. Therefore, let 0 and 1 be identified and the functions of \( F \) be periodic. Suppose \( f_k \) does not tend uniformly to \( f \). For some \( \varepsilon > 0 \), there exists a sequence \( \{u_k\} \subset [0, 1] \) such that for each \( k \), \( |f(u_k) - f_k(u_k)| \geq \varepsilon \). Since a subsequence of \( \{u_k\} \) converges, we may assume \( \{u_k\} \) does and let \( u = \lim_{k \to \infty} u_k \). By a suitable rotation of \([0, 1]\) we may assume \( u, x_1, \ldots, x_n \) all lie in the interior of an interval \([a, b]\), \( 0 < a < b < 1 \). But \( F \) forms an ordinary \( n \)-parameter family on \([a, b]\) and hence \( f_k \to f \) uniformly on \([a, b]\) which is a contradiction. This completes the proof.

We now verify that \( n \)-parameter families are unisolvent in the sense of Motzkin [10]. Let \( f, g \in F \) and let \( x \) be an interior point of \([0, 1]\). If \( x \) is a zero of \( f - g \) and if \( f - g \) does not change sign in a suitably small neighborhood about \( x \) then we will say the zero \( x \) has multiplicity 2, otherwise we say \( x \) has multiplicity 1. If 0 and 1 are not identified and either is a zero of \( f - g \), then the multiplicity is taken to be 1. We shall denote the sum of the multiplicities of the zeros of \( f - g \) within an interval \([a, b]\) by \( m_{a,b}(f, g) \). The following generalized con-

\[1\] If 0, 1 are identified we assume \( x_n^{(k)} < 1 \) and \( x_n < 1 \),
vexity notion is also useful. A continuous function \( h \) will be said to be convex to \( F \) if \( h \) intersects no function of \( F \) at more than \( n \) points. The following result extends Theorems 2 and 3 of [15].

**Theorem 2.** Let \( F \) be an \( n \)-parameter family on \([0, 1]\) and let \( h \) be convex to \( F \). Then for any \( f, g \in F, m_{0,1}(f, h) \leq n \) and \( m_{0,1}(f, g) \leq n - 1 \).

**Proof.** We assume first that 0 and 1 are not identified and that \( F \) is an ordinary \( n \)-parameter family. We verify the first statement by induction on \( n \). For \( n = 1 \) the result follows by [15] Theorem 2. Hence, let \( h \) be a continuous function convex to a \( k + 1 \) parameter family \( F \) and assume the conclusion holds for all \( k \)-parameter families. For \( f \in F \) let \( x_i, i = 1, \ldots, m, \) be the zeros of \( f - h \) ordered from left to right and assume \( m_{0,1}(f, h) > k + 1 \). Choose a point \( u \) such that \( x_i < u < x_{i+1} \). If \( F_1 = \{ g \in F | g(x_i) = h(x_i) \} \), then \( F_1 \) is a \( k \)-parameter family on \([u, 1]\). \( f \in F_1 \) and \( h \) is convex to \( F_1 \). By our inductive assumption \( m_{0,1}(f, h) \leq k \). Therefore \( x_i \) must be a zero of \( f - h \), and \( m_{0,1}(f, h) = k + 2 \). By the same reasoning we may assume \( x_m \) is a double zero of \( f - h \).

We now construct a set \( E \) of \( k \) points from \([0, 1]\) in the following manner. First choose an \( \varepsilon > 0 \) such that \( x_i + 2\varepsilon < x_{i+1} - 2\varepsilon, \) \( i = 1, \ldots, m - 1 \). If \( x \) is a single zero of \( f - h \) then let \( x \) belong to \( E \). If \( x \) is a double zero of \( f - h, \) \( x \neq x_i, x_m \) let \( x + \varepsilon, \) and \( x - \varepsilon \) belong to \( E \). We add the points \( x_i + \varepsilon, x_m - \varepsilon. \) Since \( m_{x_i+\varepsilon, x_m-\varepsilon}(f, h) = k - 2 \) it is clear that \( E \) contains exactly \( k \) points. Choose a point \( x', x_1 + \varepsilon < x' < x_2 - \varepsilon. \) Let \( f_n \) be the unique function in \( F \) such that

\[
\begin{align*}
  f_n(x) &= f(x), \ x \in E \\
  f_n(x') &= f(x') + \frac{1}{n} \text{sgn} [f(x') - h(x')]
\end{align*}
\]

Now \( f_n - f \) has \( k \) zeros which must all be simple by [15] Theorem 3. Within the interval \([x_1, x_m] \) \( f_n - h \) has exactly \( k \) simple zeros since \( f_n \) was chosen so that at the points \( x_i \pm 2\varepsilon, \) \( i = 2, \ldots, m - 1, x_i + 2\varepsilon, x_m - 2\varepsilon, f \) lies between \( f_n \) and \( h. \) Hence for \( 0 \leq x < x_1 \) and \( x_m < x \leq 1, f_n \) and \( h \) are on the same side of \( f \) (i.e., \( \text{sgn} [f_n(x) - f(x)] = \text{sgn} [h(x) - f(x)] \)). But by Theorem 1, \( f_n \) tends uniformly to \( f \) as \( n \to \infty. \) Hence for \( n \) sufficiently large \( f_n - h \) must have at least \( k + 2 \) zeros which is a contradiction.

The case when 0 and 1 are identified and \( F \) is periodic causes no difficulty. For if \( x_i, \ldots, x_m \) are the zeros of \( f - h, \) using a suitable rotation we may assume that there is an interval \([a, b] \), such that \( 0 < a < x_1 < \cdots < x_m < b < 1. \) \( F \) is an ordinary \( n \)-parameter family on \([a, b] \) and \( m_{0,1}(f, h) = m_{a,b}(f, h) \leq n. \)
The verification of the second assertion is very similar to the above, and we leave the details to the reader.

**COROLLARY.** There are no periodic \( n \)-parameter families when \( n \) is an even integer.

**Proof.** Suppose false. Let \( F \) be a periodic \( n \)-parameter family and \( n \) an even integer. Let \( f \in F \) and choose \( x_i = 1, \cdots, n \) such that \( 0 < x_1 < x_2 < \cdots < x_n < 1 \). Choose \( g \in T \) such that \( g(x_i) = f(x_i) \) \( i = 1, \cdots, n - 1 \), \( g(x_n) = f(x_n) + 1 \). By Theorem 2, \( f - g \) changes sign at each of the points \( x_i \), \( i = 1, \cdots, n - 1 \); and since \( f - g \) can have no other zeros within \([0, 1]\), \( g(1) > f(1) \). On the other hand \( g(0) < f(0) \) which is a contradiction, since \( f, g \) are periodic of period 1.

3. **Best approximation in the \( L_\infty \) norm.** If \( g \) is continuous on \([0, 1]\), \( g \notin F \), then \( \{g - f\} \) forms a new \( n \)-parameter family. Hence without loss of generality we may consider the characterization and construction of the function \( f \in F \) such that

\[
\|f\| = \inf_{f \in F} \|f\| = \delta
\]

We first adopt the following notation. If \( S \subset [0, 1] \)

\[
\delta_S = \inf_{f \in F} \sup_{t \in S} |f(t)|.
\]

Let \( T \) denote the class of vectors \( \mathbf{u} = (u_1, \cdots, u_{n+1}) \) satisfying the condition that \( 0 \leq u_1 < u_2 < \cdots u_{n+1} \leq 1 \). The statements and proofs of the results of this section are valid when \( F \) consists of continuous periodic functions on \([0, 1]\). We shall assume, however, that \( F \) is an ordinary \( n \)-parameter family and leave the details in the periodic case to the reader.

The following two lemmas are appropriate generalizations of results of de la Vallee Poussin [6] for polynomials. Where possible we refer the reader to [13] for proofs.

**LEMMA 1.** For any \( \mathbf{u} = (u_1, \cdots, u_{n+1}) \in T \) there exists a unique \( f \in F \) and unique real number \( \lambda \) such that \( f(u_i) = (-1)^i \lambda \cdot i = 1, \cdots, n + 1 \). Moreover \( |\lambda| = \delta_u \) and \( f \) is the only function in \( F \) with the property that \( \max_{i=1,\ldots,n+1} |f(u_i)| = \delta_u \). In addition suppose for \( k = 1, 2, \cdots \) that

\[
\mathbf{u}^{(k)} = (u_1^{(k)}, \cdots, u_{n+1}^{(k)}) \in T \text{ and } f_k(u_i^{(k)}) = (-1)^i \lambda^{(k)}.
\]

Then if \( \mathbf{u}^{(k)} \to \mathbf{u} \) and \( \mathbf{u} \in T \), it follows that \( f_k \to f \) uniformly on \([0, 1]\) and \( \lambda^{(k)} \to \lambda \).
Lemma 2. Let \( u \in T \) and a sequence of non-negative numbers \( \lambda_i \), \( i = 1, \ldots, n + 1 \) be given. If there exists an \( f \in F \) such that

\[
f(u_i) = (-1)^i \lambda_i \quad i = 1, \ldots, n + 1 \text{ or } f(u_i) = (-1)^{i+1} \lambda_i \quad i = 1, \ldots, n + 1
\]

then either \( \min \lambda_i < \delta_u < \max \lambda_i \) or \( \lambda_i = \delta_u \), \( i = 1, \ldots, n + 1 \).

Proof. Lemma 2 is a restatement of Lemma 1 of [13]. Everything in Lemma 1 except the facts that \( |\lambda| = \delta_u \) and the function \( f \) satisfying \( \max_{i=1,\ldots,n+1} |f(u_i)| = \delta_u \) is unique is proved explicitly in [13]. To prove the latter statements observe that if there is a \( g \in F \) satisfying \( |g(u_i)| < |\lambda| \) then \( f(u_i) - g(u_i) = (-1)^i \lambda_i \), \( i = 1, \ldots, n + 1 \) where either \( \lambda_i \geq 0 \), \( i = 1, 2, \ldots, n + 1 \) or \( \lambda_i \leq 0 \), \( i = 1, 2, \ldots, n + 1 \). In either case by [12], Lemma 1, \( f - g \) must have at least \( n \) zeros between \( u_i \) and \( u_{n+1} \) counting multiplicity which is a contradiction.

For \( u \in T \) we will usually denote the function \( f \) of Lemma 1 by \( f_u \).

Next we define a function \( \delta(u_1, \ldots, u_{n+1}) \) of \( n + 1 \) variables.

\[
\delta(u) = \delta(u_1, \ldots, u_{n+1}) = \delta_u \text{ if } u = (u_1, \ldots, u_{n+1}) \in T \text{ and } u = 0 \text{ otherwise}.
\]

If we restrict the points \( u_i \) to lie in some subset \( S \subset [0,1] \), then \( \delta(u_1, \ldots, u_{n+1}) \) will be denoted \( \delta_S(u_1, \ldots, u_{n+1}) \).

Lemma 3. \( \delta(u_1, \ldots, u_{n+1}) \) is continuous on \( R^{n+1} \)

Proof. Assume that \( \delta(u_1, \ldots, u_{n+1}) \) is not continuous at some point \( u = (u_1, \ldots, u_{n+1}) \). We may assume \( 0 \leq u_1 \leq u_2 \leq \cdots \leq u_{n+1} \leq 1 \), and by Lemma 1 we may assume that \( m(\leq n) \) of the points \( u_i \) are distinct. Consequently \( \delta(u_1, \ldots, u_{n+1}) = 0 \). Suppose there exists an \( \varepsilon > 0 \) and a sequence \( \{u_k\} \subset T \) such that \( u_k \to u \) and \( \delta_{u_k} \geq \varepsilon \). Let \( u_i^{(k)} \) be the \( i \)-th coordinate of \( u_k \). Choose \( n \) points \( u_i', 0 \leq u_i' < \cdots < u_n' \leq 1 \) such that \( m \) of the points \( u_i' \) coincide with the \( m \) distinct points \( u_i \). Let \( f_0 \) be the unique function in \( F \) such that \( f_0(u_i') = 0 \). Choose \( \eta \) such that for any \( i \) \( |u_i' - u_i| < \eta \) implies \( |f_0(u_i)| < \varepsilon/2 \). Choose \( k \) so large that all coordinates of \( u_k \) are within \( \eta \) neighborhoods of some coordinate of \( u_i' \). Then \( f_{u_k}^{(k)}(u_i^{(k)}) - f_0(u_i^{(k)}) = (-1)^i \lambda_i \) where \( \text{sgn } \lambda_i^{(k)} = \text{sgn } \lambda_i^{(k)} \), \( i = 1, \ldots, n \). As in the proof of Lemma 1 it follows that \( f_{u_k}^{(k)} - f_0 \) must have at least \( n \) zeros within \( [0,1] \) which is a contradiction.

Using the function \( \delta(u_1, \ldots, u_{n+1}) \) one can give a simple proof of the Theorem of Motzkin and Tornheim characterizing the function \( f \) which has minimum deviation from zero.

Theorem 3. There exists a unique \( \hat{f} \in F \) such that \( \|\hat{f}\| = \inf_{f \in F} \|f\| \).

\( \hat{f} \) is uniquely characterized by the fact that for some \( u = (u_1, \ldots, u_{n+1}) \in T \)
\[ \| \hat{f} \| = \delta_u. \text{ u will have this property if and only if } \delta(u, \cdots, u_{n+1}) \text{ is an absolute maximum, and then } \hat{f} = f_u. \]

**Proof.** Since \( \delta(u_1, \cdots, u_{n+1}) \) is a continuous function on a compact set, its maximum is attained for some \( u = (u_1, \cdots, u_{n+1}) \in T \). Assert that \( \| f_u \| > \delta_u \) if \( \| f_u \| > \delta_u \), then there is a point \( x' \) in \([0,1]\) for which \( |f_u(x')| = \| f_u \| \). We form a new vector \( u' \in T \) by replacing one coordinate \( u_i \) of \( u \) by \( x' \) in the following way. If \( u_i < x' < u_{i+1} \), and \( sgn \ f_u(u_i) = sgn \ f_u(x') \) then let \( u'_i = u_j \), \( j \neq i \), and \( u'_i = x' \). If \( sgn \ f_u(u_i) = (-1) \) \( sgn \ f_u(x') \) let \( u'_i = u_j \), \( j \neq i + 1 \) and \( u'_{i+1} = x' \). If \( x' < u_i(x' > u_{i+1}) \) and \( sgn \ f_u(u_i) = sgn \ f_u(x') \) \( (sgn \ f_u(u_{n+1}) = sgn \ f_u(x')) \) let \( u'_i = u_j, j \neq j = 1, 2, \cdots, n = 1 \) \( u_{n+1} = u'_{n+1} = x' \). Now either \( f_u(u'_i) = (-1) \) \( \lambda_i = 1, \cdots, n + 1 \) or \( f_u(u'_i) = (-1)^{i+1} \lambda_i = 1, \cdots, n + 1 \) where \( \lambda_i = \delta_u \) or \( \lambda_i = \| f_u \| \) respectively. Therefore by Lemma 2, \( \delta_u < \delta_u' < \| f_u \| \) which contradicts the maximality of \( \delta_u \).

It now follows immediately that \( \| f_u \| = \inf_{f \in F} \| f \| \) and that \( f_u \) is the only such function with this property. For if \( f_0 \in F \) and \( \| f_0 \| \leq \| f_u \| \) then \( \| f_0 \| \leq \delta_u \) which contradicts Lemma 1. Moreover the same argument shows that if there exists an \( f_0 \in F \) and a \( v \in T \) such that \( \| f_0 \| = \delta_v \) then \( \| f_0 \| = \inf_{f \in F} \| f \| \). It is clear that \( \delta(v_1, \cdots, v_{n+1}) \) must be an absolute maximum.

In the above theorem if \( \| f \| \) is replaced by \( \| f \|_S = \sup_{t \in S} |f(t)| \) where \( S \) is any closed set of \([0,1]\) containing at least \( n + 1 \) points, then the same conclusions hold. Here of course, the function \( \delta(u_1, \cdots, u_{n+1}) \) is replaced by \( \delta_S(u_1, \cdots, u_{n+1}) \) and the points \( u_k \) are assumed to be in \( S \). The following generalization of [11] Theorem 7.1 is therefore relevant.

**Theorem 4.** Let \( S_k, S \) be closed sets of \([0,1]\) such that for each \( k, S_k, \) contains at least \( n + 1 \) points; \( S \) contains infinitely many points, and \( S_k \subseteq S \). Let \( f_k, \hat{f}_k \) be functions from \( F \) which minimize \( \| f \|_S, \| f \|_S \) respectively. If for each \( \varepsilon > 0 \) there exists an integer \( k_0 \) such that for \( k > k_0 \) each point \( u \in S \) is at a distance less than \( \varepsilon \) from some point of \( S_k \), then \( \hat{f}_k \rightarrow f_0 \) uniformly on \([0,1]\).

**Proof.** We assume \( \delta_S > 0. \) \( S_k \subseteq S \) implies \( \delta_{S_k} \leq \delta_S \). Choose \( u = (u_1, \cdots, u_{n+1}) \in T, u \in S \) such that \( \delta_S(u_1, \cdots, u_{n+1}) \) is an absolute maximum. Let \( u_k = (u_1^{(k)}, \cdots, u_{n+1}^{(k)}) \in T, u_j^{(k)} \in S_k \) be chosen such that \( u_k \rightarrow u \). By Lemma 1, \( \delta_{u_k} \rightarrow \delta_u \) and since \( \delta_{u_k} < \delta_S, \delta_{u_k} \rightarrow \delta_S \) \( \delta_{u_k} = \delta_S \). Let \( v_k = (v_1^{(k)}, \cdots, v_{n+1}^{(k)}) \in T, v_i^{(k)} \in S_k \) be chosen so that for each \( k, \delta_{S_k}(v_1^{(k)}, \cdots, v_{n+1}^{(k)}) \) is an absolute maximum. Extract any convergent subsequence \( v_k \) with limit \( v \).
If \( v = (v_1, \ldots, v_{n+1}) \), then \( v_i \in S \) and \( \delta_v = \delta_S \). Also \( \hat{f}_{v_j} = f_{v_{v_j}} \) tends uniformly to \( f_v \), the function from \( F \) with minimum deviation on \( v \). But by the uniqueness of \( f_v, f_v = \hat{f}_0 \). The above argument shows that any subsequence of \( \{ \hat{f}_k \} \) contains a refinement which converges to \( \hat{f}_0 \). Hence \( \lim_{k \to \infty} \hat{f}_k = \hat{f}_0 \) uniformly on \([0, 1]\).

4. The estimation of \( \hat{f} \). In [13] Novodovorskii and Pinsker consider a direct method, due to Remes [14] in the polynomial case, for the estimation of \( \hat{f} \). However the following Lemma shows that \( \hat{f} \) is continuously dependent on estimates of the best approximation. Hence if \( u \) is a vector in \( T \) for which \( \delta(u) \) is an estimate of \( \inf_{f \in F} ||f|| \), then the solution of the equation \( f(u_i) = (-1)^i, i = 1, \ldots, n + 1 \) is the appropriate estimate of \( \hat{f} \).

**Lemma 4.** Let \( \{ \delta_n \} \) be a sequence of non-negative numbers converging to \( \delta = \inf_{f \in F} ||f|| \) from below. If \( u_n \) are vectors in \( T \) for which \( \delta(u_n) = \delta_n \), then \( \lim_{n \to \infty} f_{u_n} = \hat{f} \) uniformly on \([0, 1]\).

**Proof.** If the conclusion is false there exists a subsequence \( \{ u_{k_j} \} \) and a number \( \varepsilon > 0 \) such that \( ||\hat{f} - f_{u_{k_j}}|| \geq \varepsilon \). But \( \{ u_{k_j} \} \) may be further refined to obtain a convergent subsequence of vectors. Calling this \( \{ u_{k_j} \} \) and letting \( u_0 = \lim_{j \to \infty} u_{k_j} \) we have by Lemma 1 \( \delta(u_0) = \lim_{j \to \infty} \delta(u_{k_j}) \). By Theorem 3 \( f_{u_0} = \hat{f} \) which is a contradiction.

We shall consider two algorithms for estimating \( \delta \) and prove convergence of both.

Each of these algorithms can be used efficiently for actual numerical calculations. A detailed description of method 2 for polynomials on a finite point set can be found in [5]. Also for polynomials on an interval a maximization procedure has been announced by Bratton [3].

For both methods the following notation is convenient. For \( u = (u_1, \ldots, u_{n+1}) \in T \) define for \( j = 1, \ldots, n + 1 \).

\[
\delta_u^{(j)}(x) = \delta(u_1, \ldots, u_{j-1}, x, u_{j+1}, \ldots, u_{n+1}) \text{ if } u_{j-1} \leq x \leq u_{j+1} \\
= 0 \text{ otherwise}
\]

where we take \( u_0 = 0, u_{n+2} = 1 \). We now form \( \eta_u(x) = \max_{j=1,\ldots,n+1} \delta_u^{(j)}(x) \). From the continuity of \( \delta(u_1, \ldots, u_{n+1}) \) it follows that for each \( j \), \( \delta_u^{(j)}(x) \) is continuous, and hence \( \eta_u(x) \) is continuous. Therefore there exists a point \( x', 0 \leq x' \leq 1 \) and integer \( 1 \leq m \leq n + 1 \) such that

\[
\delta_u^{(m)}(x') = \max_{j=1,\ldots,n+1} ||\delta_u^{(j)}|| = ||\eta_u||.
\]

For a given vector \( u \) we define \( u' = (u_1', \ldots, u'_{n+1}) \) by setting \( u_j' = u_j, j \neq m \), \( u_m' = x' \).
Theorem 5. If vectors $u_k$ are defined inductively in the above fashion with $u_1 \in T$ chosen arbitrarily, then $\lim_{k \to \infty} \delta(u_k)$ exists and there exists $u_0 \in T$ such that $\delta(u_i) = \lim_{k \to \infty} \delta(u_k)$. Furthermore $\delta(u_i)$ is an absolute maximum of the function $\delta(u)$.

Proof. $\{\delta(u_k)\}$ is a monotonically increasing, bounded sequence hence convergent. If $\delta = \lim_{k \to \infty} \delta(u_k)$, then a suitable subsequence $\{u_{k_j}\}$ converges to $u_0$ and $\delta(u_0) = \delta$. We now assert $\gamma_{u_k}(x)$ converges uniformly to $\gamma_{u_0}(x)$. It suffices to assume $u_i \leq x \leq u_{i+1}$. Then
\[
|\gamma_{u_k}(x) - \gamma_{u_{k_j}}(x)| = |\max(\delta_{u_k}^i(x), \delta_{u_{k_j}}^{i+1}(x)) - \max(\delta_{u_0}^i(x), \delta_{u_{k_j}}^{i+1}(x))|
\leq |\delta_{u_0}^i(x) - \delta_{u_{k_j}}^i(x)| + |\delta_{u_k}^{i+1}(x) - \delta_{u_{k_j}}^{i+1}(x)|.
\]
Since $\delta(u)$ is a uniformly continuous function the latter expression tends to zero uniformly in $x$.
Hence
\[
|\gamma_{u_0} - \gamma_{u_{k_j}}| = \lim_{j \to \infty} |\gamma_{u_{k_j}}|.
\]
But
\[
|\gamma_{u_{k_j}}| = \delta(u_{k_j+1}) \leq \delta(u_{k_j+1}) \leq |\gamma_{u_{k_j+1}}|.
\]
Therefore $|\gamma_{u_0}| = \lim_{j \to \infty} \delta(u_{k_j}) = \delta(u_0)$. It now follows by the same argument as in the proof of Theorem 3 that $|f_{u_0}| = \delta(u_0)$ and by Theorem 3, $\delta(u_0)$ is a maximum.

For the second method of estimation of $f$ we alter slightly our definition of $\delta^i_u(x)$ and $\delta^{i+1}_u(x)$. We now define
\[
\delta^i_u(x) = \delta(x, u_2, \cdots, u_{n+1}) \text{ if } 0 \leq x \leq u_2.
\]
\[
\delta^i_u(x) = \delta(u_2, u_3, \cdots, u_{n+1}, x) \text{ if } u_{n+1} \leq x \leq 1
\]
\[
\delta^{i+1}_u(x) = \delta(u_1, \cdots, u_n, x) \text{ if } u_n \leq x \leq 1
\]
\[
\delta^{i+1}_u(x) = \delta(x, u_1, \cdots, u_n) \text{ if } 0 \leq x \leq u_1.
\]

The algorithm proceeds as follows. First let $\varepsilon > 0$ be chosen. Select an arbitrary vector $u \in T$. Maximize $\delta^i_u(x)$ over its domain of definition. Let $x'$ be a point for which $\delta^i_u(x)$ is a maximum. If $\delta^i_u(x') \geq (1 + \varepsilon)\delta(u)$, replace $u_2$ by $x'$ forming a new vector $u'$. If not, let $u' = u$. We now maximize $\delta^{i+1}_u(x)$ and continue inductively. Special attention is necessary for $\delta^{i+1}_u(x)$ and $\delta^i_u(x)$. If $x'$ is a point for which $\delta^{i+1}_u(x)$ is a maximum and $\delta^{i+1}_u(x) \geq (1 + \varepsilon)\delta(u)$, then $u'$ is formed in the following way. If $x' \geq u_n$ then $u_i = u_i, i = 1, \cdots, n$, $u_{n+1} = x'$; if $x' \leq u_i$ then $u_i = x' u_i = u_{i-1}, i = 2, \cdots, n+1$. In the latter case, the next function maximized is $\delta^i_u(x)$. If the first case occurs then $\delta^i_u(x)$ is maximized. Let $x''$ be a point for which $\delta^i_u(x)$.\]
is a maximum and \( \delta_{u_i}(x'') \geq (1 + \varepsilon)\delta(u') \). If \( x'' \leq u'_i \) then \( u''_i = x'' \) and \( u''_i = u'_i \) \( i = 2, 3, \ldots, n + 1 \). If \( x'' \geq u'_{n+1} \) then \( u''_i = u_{i+1} \) \( i = 1, \ldots, n \) and \( u''_{n+1} = x'' \). For the first case the next function maximized is \( \delta_{u_i}(x) \); the second case, \( \delta_{u_{i+1}}(x) \). If

\[
\delta_{u_i+1}(x') < (1 + \varepsilon)\delta(u) \quad (\delta_{u_i}(x'') < (1 + \varepsilon)\delta(u'))
\]

then we take \( u' = u \) (\( u'' = u' \)). When there have been \( n + 1 \) consecutive maximizations with no change in the vector \( u \), \( \varepsilon \) is now replaced by \( \varepsilon/2 \) and the process is repeated. We now continue inductively and pass to the limit as \( \varepsilon/2^k \to 0 \).

**Theorem 6.** The conclusions of Theorem 5 hold if the sequence \( \{u_k\} \) is chosen inductively in accordance with the above algorithm.

**Proof.** As before, \( \lim_{k \to \infty} \delta(u_k) = \delta \) exists. We choose a particular convergent subsequence \( \{u_{k_j}\} \) of \( \{u_k\} \). For each \( j \) let \( u_{k_j} \) be a vector of \( \{u_k\} \) such that for each \( i, \quad i = 1, \ldots, n + 1 \) and all appropriate \( x, \delta_{u_{k_j}}(x) < (1 + \varepsilon/2)i\delta(u_{k_j}) \). The algorithm guarantees that for each integer \( j \) such a vector \( u_{k_j} \) exists in the sequence \( \{u_k\} \). Since a refinement of this sequence is convergent, we assume \( \{u_{k_j}\} \) converges. Then if \( u_{k_j} \to u_0, \delta(u_0) = \delta \). Suppose \( \delta(u_0) \) is not a maximum of \( \delta(u) \), then \( ||f_{u_0}|| > \delta(u_0) \). Choose \( x' \) so that \( |f_{u_0}(x')| = \|f\| \), and form \( u' \) by replacing one point, the \( i \)th, say, of \( u_0 \) by \( x' \) in the manner of the proof of Theorem 3. Form \( u'_{k_j} \) by replacing the \( i \)th coordinate of \( u_{k_j} \) by \( x' \) Then \( u'_{k_j} \to u' \) and \( \delta(u'_{k_j}) \to \delta(u') \). Therefore for \( j \) sufficiently large, since \( \delta(u') > \delta \),

\[
\delta(u'_{k_j}) > \frac{\delta(u') + \delta}{2}
\]

On the other hand for each \( j \) there is a point \( x \) and an integer \( m \) such that

\[
\delta(u'_{k_j}) = \delta_{u_{k_j}}(x) \leq \left(1 + \frac{\varepsilon}{2^j}\right)\delta(u_{k_j}) \leq \left(1 + \frac{\varepsilon}{2^j}\right)\delta.
\]

For \( j \) sufficiently large this is a contradiction, therefore \( ||f_{u_0}|| = \delta(u_0) \) and \( \delta(u_0) \) is an absolute maximum.

**5. Approximation in \( L_{p,N} \) norm.** For \( N \geq n \) let \( x_1, \ldots, x_N \) be \( N \) distinct points of \([0,1]\). In place of the sup norm let \( \|f\| = \left\{\sum_{i=1}^N |f(x_i)|^p\right\}^{1/p} \) and assume \( p > 1 \). The fundamental problem to be considered here is to give necessary and sufficient conditions that the function \( \hat{f} \in F \) for which \( ||\hat{f}|| = \inf_{f \in F} ||f|| \) is unique. Now the image of \( F \) under the mapping \( f \to (f(x_1), \ldots, f(x_N)) \) is a closed set in \( N \) dimensional Euclidean
space. By a theorem of Motzkin [9] as generalized by Busemann [4],
to each point $x \in E_N$ there will exist a unique nearest point in a given
set $S \subset E_N$ with respect to a strictly convex metric if and only if $S$
is closed and convex. Hence $\hat{f}$ will be unique if and only if $F$ is convex,
but for $n$-parameter families we can say more.\footnote{For a discussion of related results see the article by Motzkin in the Symposium on
Numerical Approximation, University of Wisconsin Press, 1959.}

**Theorem 7.** An $n$-parameter family $F$ is convex if and only if $F'$
is the translate of a linear $n$-parameter family.

**Proof.** If $F$ is the translate of a linear $n$-parameter family, i.e.,
there exists a continuous $g$ on $[0,1]$ and a linear $n$-parameter family $F_0$
such that each $f \in F$ can be written uniquely as $f = g + f', f \in F_0$,
then $F$ is obviously convex. Conversely suppose $F$ is convex. Choose $n$
distinct points $x_1, \ldots, x_n$ in $[0,1]$. Let $f_0, f_1, \ldots, f_n$ be the unique functions
of $F$ such that $f_j(x_j) = 0$, $j = 1, \ldots, n$; $f_k(x_j) = \delta_{kj}$ for $k, j = 1, \ldots, n$
where $\delta_{kj}$ is the Kronecker delta. We assert that each $f \in F$ has a re-
presentation as

$$f = f_0 + \sum_{k=1}^{n} \lambda_k (f_k - f_0)$$

where $\lambda_k = f(x_k)$.

If such a representation exists it is obviously unique. Also the vector
space spanned by $f_1 - f_0, \ldots, f_n - f_0$, is obviously an $n$-parameter family
and the theorem is proved. To prove the assertion let

$$\begin{align*}
F'_k &= \{ f \in F' | f(x_{k+1}) = f(x_{k+2}) = \cdots = f(x_n) = 0 \} \\
F'_k' &= \{ f \in F | f(x_j) = 0 \; j \neq k \}.
\end{align*}$$

From the convexity of $F$, $F'_k'$ is a convex one parameter family on a suitably
small interval containing $x_k$. We assert $f \in F'_k'$ implies $f = f_0 + \lambda_k (f_k - f_0)$
where $\lambda_k = f(x_k)$. By convexity this is obviously true for $0 \leq \lambda_k \leq 1$.
For $\lambda_k > 1$ if $f \in F'_k', f(x_k) = \lambda_k$ then by convexity

$$f_k = \frac{1}{\lambda_k} f + \left(1 - \frac{1}{\lambda_k}\right) f_0$$

or $f = f_0 + \lambda_k (f_k - f_0)$. If $\lambda_k < 0$,

$$f_0 = \frac{1}{1 - \lambda_k} f + \frac{(-\lambda_k)}{1 - \lambda_k} f_k$$

or $f = f_0 + \lambda_k (f_k - f_0)$. To finish the proof we apply an induction. Assume $f \in F'_k'$ implies that $f = f_0 + \sum_{j=1}^{n} \lambda_j (x_j - x_0)$ where $f(x_j) = \lambda_j$ and
suppose \( g \in F_{k+1} \) and \( g(x_j) = \mu_j, j = 1, \ldots, k + 1 \). Then if \( g_1 = f_0 + \sum_{j=1}^k 2\mu_j(f_j - f_0), g_2 = f_0 + 2\mu_{k+1}(f_{k+1} - f_0) \) it follows that
\[
g' = \frac{g_1 + g_2}{2} \in F_{k+1}
\]
and \( g'(x_j) = \mu_j, j = 1, \ldots, k + 1 \). Therefore
\[
g = g' = f_0 + \sum_{j=1}^{k+1} \mu_j(f_j - f_0).
\]

6. The existence of \( n \)-parameter families on compact space. Let \( f_1, \ldots, f_n \), be \( n \) linearly independent real valued continuous functions defined on a compact set \( S \) in finite dimensional Euclidean space. Let \( V \) be the span of the functions \( f_1, \ldots, f_n \). In 1918 Haar [7] showed that to each continuous real valued function \( g \) defined on \( S \), there is a unique \( \hat{f} \in V \) satisfying \( \| \hat{f} - g \| = \inf_{\hat{f} \in V} \| f - g \| \) where \( \| f \| = \sup_{s \in S} |f(s)| \) if and only if no non-zero function in \( V \) vanished at more than \( n - 1 \) points of \( S \). Haar noted that the existence of such a set of functions \( V \) placed a severe restriction on the set \( S \). In 1956 Mairhuber [8] proved that if \( V \) satisfied the above condition of Haar then \( S \) is a homeomorphic image of a subset of the circumference of the unit circle. If \( n \) is even this subset must be proper. It is clear that \( V \) satisfies the condition of Haar if and only if \( V \) is a linear \( n \)-parameter family. The characterization of those compact Hausdorff spaces on which there exist \( n \)-parameter families \( F \) for \( n > 1 \) seems to be quite difficult. One can give a characterization if one imposes a rather strong local condition on \( F \). The result presented here includes the one of Mairhuber, and is proved by somewhat different means. The following fundamental lemma is perhaps of independent interest.

**Lemma 5.** Let \( S \) be a compact connected Hausdorff space with the property that for each point \( x \in S \) there exists a neighborhood \( U_x \) and continuous real valued functions \( f_1, f_2 \) defined on \( U_x \) such that for \( y, z \in U_x, y \neq z \)
\[
\begin{vmatrix}
f_1(y) & f_1(z) \\
f_2(y) & f_2(z)
\end{vmatrix} \neq 0.
\]

Then \( S \) may be embedded homeomorphically into the circumference \( C \) of the unit circle.

**Proof.** Without loss of generality we assume \( U_x \) is a closed, therefore compact neighborhood of \( x \). \( f_1, f_2 \) never vanish simultaneously on \( U_x \) and therefore \( f_1/f_2 \) defines a continuous mapping of \( U_x \) into the
compactified real line. (1) guarantees that the mapping is one to one and \( \phi_x(u) = \arctan(f_1/f_2)(u) \) gives a homeomorphism of \( U_x \) into \( C \).

We next verify that \( S \) is locally connected. To do this it suffices to show that for each \( x \in S \) there exists a connected neighborhood which can be mapped homeomorphically into \( C \). In fact if \( \phi_x \) is the homeomorphism for a point \( x \in S \) constructed above, and if \( C_x = \phi_x(U_x) \), it is enough to show that there exists a connected neighborhood \( V_x \) in \( C \) of \( \lambda_x := \phi_x(x) \). For then \( \phi^{-1}_x(V_x) \) is a connected neighborhood of \( x \) contained in \( U_x \). But \( C_x \) is a compact subset of \( C \). Therefore let \( I_x \) be the component of \( \lambda_x \) in \( C_x \). \( I_x \) is a compact connected subset of \( C \). \( I_x \) is then either an interval or all of \( C \). If \( I_x \) is the latter we are through. Also if \( I_x \) is an interval and \( \lambda_x \) an interior point (relative to \( C \)) then \( \phi^{-1}_x(I_x) \) is the required neighborhood. Hence assume that \( \lambda_x \) is an end point of \( I_x \). This will include that degenerate case when \( I_x \) is just one point.

We may also assume that there does not exist a suitably small connected neighborhood \( N \) of \( \lambda_x \) in \( C \) such that \( N \cap C_x \subset I_x \). For then \( \phi^{-1}_x(N \cap N_x) \) is an appropriate neighborhood of \( x \). Therefore it now must follow that for any connected neighborhood \( N \) of \( \lambda_x \) in \( C \) there exists \( \lambda_1, \lambda_2 \) in the interior of \( N \) such that \( \lambda_1, \lambda_2 \notin C_x \) and \( (\lambda_1, \lambda_2) \cap C_x \neq \phi \). If we let \( F = \phi^{-1}_x[(\lambda_1, \lambda_2) \cap C_x] \) and \( G = \phi^{-1}_x[C_x \sim (\lambda_1, \lambda_2)] \) then \( F \cup (S \sim U_x) \) and \( G \) separate \( S \) which is a contradiction.

We note that \( S \) is certainly a separable metric since a finite number of homeomorphic images of subsets of \( C \) cover \( S \). Hence by [16] Theorem 5.1, \( S \) is arc wise connected.

We now assert \( S \) is homeomorphic to a subset of \( C \). Let \( U_1, \ldots, U_n \) be a finite collection of connected neighborhoods covering \( S \) each of which is homeomorphic to a subset of \( C \). By a suitable rearrangement we may assume that \( U_1 \cap U_1 \neq \phi \) and \( U_1 \not\subset U_1 \). Let \( x_1 \in U_1 \sim U_2, x_2 \in U_2 \sim U_1 \). Let \( x \in U_1 \cup U_2 \) connecting \( x_1, x, x_2 \). This must be all of \( U_1 \cup U_2 \), for if \( y \in U_1 \cup U_2 \) and \( y \notin A \), then \( y \) may be connected to any point in \( A \) by an arc in \( U_1 \cup U_2 \). If \( y \) is connected to \( A \) at an end point of \( A \), this is an enlargement of \( A \) which contradicts maximality. If \( y \) is connected to \( A \) at a point other than an end point, then no neighborhood of this point is homeomorphic to a subset of \( C \). This also is a contradiction. If \( U_1 \cup U_2 \) is not all of \( S \) then \( U_1 \cup U_2 \) is homeomorphic to an arc, and by induction the homeomorphism may be extended to all of \( S \).

**Theorem 8.** For \( n > 1 \) let \( F \) be an \( n \)-parameter family of functions defined on a compact Hausdorff space \( S \). Suppose in addition that to each point \( x \in S \) there exists a neighborhood \( N_x \) and functions \( f_1, f_2 \in F \) such that
for \( y, z \in \mathbb{N}, y \neq z \). Then there exists a homeomorphism of \( S \) into the circumference of the unit circle. If \( n \) is even the image of \( S \) must be a proper subset of \( C \).

Proof. First we note that \( S \) cannot have a proper subset \( W \) homeomorphic to \( C \). If \( n \) is even this follows directly from the Corollary to Theorem 2. If \( n \) is odd, choose \( x \in S \sim W \) and let \( F' = \{ f \in F | f(x) = 0 \} \); then \( F' \) is an \( n - 1 \) parameter family defined on \( W \). Since \( n - 1 \) is even this is a contradiction. We may therefore assume that if \( n \) is even \( S \) is not homeomorphic to \( C \).

If \( I \) is a component of \( S \) then by Lemma 5 there exists a homeomorphism \( \phi \) of \( I \) onto the closed interval \([0, 1]\) considered as a subset of \( C \). We assert that if \( I \) is not all of \( S \), then \( \phi \) can be extended to an open and closed set \( U \supseteq I \). \( U \) and its complement then separate \( S \). If \( I \) is itself open in \( S \) then we take \( U = I \). If not, let \( x = \phi^{-1}(0), y = \phi^{-1}(1) \). Let \( N_x, N_y \) be compact neighborhoods of \( x \) and \( y \) respectively and let \( \phi_x, \phi_y \) be homeomorphisms of \( N_x \) and \( N_y \) respectively into \( C \). We may assume \( \phi_x(x) = 0, \phi_y(y) = 1 \) and

\[
\phi_x(N_x \cap I) \subset [0, 1] \text{ and } \phi_y(N_y \cap I) \subset [0, 1].
\]

If we define \( \phi' \) by

\[
\phi'(z) = \phi(z) \quad \text{if } z \in I \\
= \phi_x(z) \quad \text{if } z \in N_x \sim I \\
= \phi_y(z) \quad \text{if } z \in N_y \sim I
\]

then \( \phi' \) is a homeomorphism of \( N_x \cup N_y \cup I = N \) into \( C \). Also int. \( N \supseteq I \). Now \([0, 1] = \phi'(I) \) is the maximal connected subset of \( \phi'(N) \) containing \( \phi'(I) \). Therefore there exist sequences \( \{\lambda_n\}, \{\mu_n\} \) of real numbers tending monotonically to 0 from below, and monotonically to 1 from above, respectively such that \( \{\lambda_n\} \cap \phi'(N) = \phi \) and \( \{\mu_n\} \cap \phi'(N) = \phi \). Choose \( n \) large enough that \( \phi'^{-1}[\lambda_n, 0] \subset \text{interior of } N_x \) and \( \phi'^{-1}[1, \mu_n] \subset \text{interior of } N_y \). Clearly \( J_n = \phi'^{-1}[\lambda_n, \mu_n] \) is a closed set containing \( I \). \( J_n \) is open in the interior of \( N \). Hence \( J_n \) is open in \( S \).

Let \( T \) be the class of open sets \( O \) of \( S \) which can be mapped homeomorphically into \( C \). We partially order \( T \) in the following way. If \( O_1, O_2 \in T \) then \( O_1 \subseteq O_2 \) if \( O_1 \subset O_2 \) and if there exist homeomorphisms \( \phi_1, \phi_2 \) of \( O_1, O_2 \) respectively into \( C \) such that \( \phi_2 \) agrees with \( \phi_1 \) on \( O_1 \). By Zorn's lemma there exists a maximal element \( O \) of \( T \). We assert \( O = S \). If not, let \( x \in S \sim O \). Then there exists an open and closed set \( U \ni x \) and mapping \( \phi \) such that \( \phi \) maps \( U \) homeomorphically into \( C \).
$O \cap U$ and $O \sim U$ are separated open sets of $S$. Hence if $\phi'$ is any homeomorphism of $O$ into $C$ such $\phi'(O) \cap \phi(U) = \phi$. $\phi''$ defined by $\phi''(x) \equiv \phi(x)$, $x \in O \cap U$, $\phi''(x) \equiv \phi'(x)$, $x \in O \sim U$ is also a homeomorphism of $O$ into $C$. $\phi''$ has an obvious extension to $U \cup O$ which contradicts the maximality of $O$.

**Corollary.** If $F$ is a linear $n$-parameter family ($n > 1$) defined on the compact Hausdorff space $S$, then $S$ is homeomorphic to a subset of $C$. If $n$ is even the subset must be proper.

**Proof.** We assume $S$ contains more than $n$ points. For a given $x \in S$ choose $n - 2$ distinct points $x_1, \cdots, x_{n-2}$ of $S$ outside a suitably small compact neighborhood $N_x$ of $x$. If $F_x = \{f \in F | f(x_i) = 0, i = 1, \cdots, n - 2\}$ then $F_x$ is a linear $2$-parameter family defined on $N_x$. Therefore, for any two linearly independent functions $f_1, f_2$ in $F_x$,

$$\begin{vmatrix} f_1(y) & f_1(z) \\ f_2(y) & f_2(z) \end{vmatrix} \neq 0 \text{ for } y, z \in N_x, y \neq z.$$  

We now apply the theorem.

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