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THE *H*-PROBLEM AND THE STRUCTURE OF *H*-GROUPS

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THE H_p-PROBLEM AND THE STRUCTURE OF H_p-GROUPS

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1. Introduction. Let G be a group, p a prime, and $H_p(G)$ the subgroup of G generated by the elements of G which do not have order p. In a research problem in the Bulletin of the American Mathematical Society, one of the authors posed the following problem: is it always true that $H_p(G) = 1$, $H_p(G) = G$, or $[G : H_p(G)] = p$? This problem is easily settled in the affirmative for $p = 2$, and a similar answer was recently given for $p = 3$ ([5]). In this paper (Section 2) we give an affirmative answer for the case that G is finite and not a p-group. Furthermore (Section 3) we are able to give a rather precise description of the structure of G in the most interesting case, when $[G : H_p(G)] = p$. This structure theorem depends heavily on the deep results of Hall and Higman ([4]) and Thompson ([6]) on finite groups. If $H\ (\neq 1)$ is a finite group and there exists a group G such that $H_p(G)$ is isomorphic to H, where $H_p(G) \neq G$, then we call H an H_p-group; it is seen that H_p-groups are natural generalizations of "Frobenius groups." By a Frobenius group we mean a finite group G possessing an automorphism σ of prime order p such that $x^\sigma = x$ if and only if $x = 1$. It is easy to show that this implies

$$x^{1+\sigma+\cdots+\sigma^{p-1}} = x(x^\sigma)\cdots(x^{\sigma^{p-1}}) = 1\ ,$$

for all x in G. This last equation characterizes H_p-groups,[1] and as a generalization of Thompson's result ([6]) that Frobenius groups are nilpotent, we show that H_p-groups are solvable, among other things.

Throughout the paper, if B is a group, A a subgroup of B, then $N_B(A)$ and $C_B(A)$ mean, respectively, the normalizer and centralizer of A in B. By $Z(A)$ we mean the center of A.

2. The H_p-problem. Let G be a group, and let $H = H_p(G)$. Suppose

(1) G is finite,

(2) G is not a p-group,

(3) the index of H in G is greater than p,

(4) G is a group of minimal order satisfying (1), (2), (3). Note that every element of G which is not in H has order p.

Let q be a prime dividing $[G : 1]$, $q \neq p$, and let Q be a Sylow q-group of G; then Q is also a Sylow q-group of H. Let $N = N_G(Q)$; then

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[1] Unless the group is a p-group; see Theorem 2.

by the Frattini argument (see [1], p. 117, for instance), $G = NH$. Thus $[G:1] = [NH:1] = [N:1][H:1]/[N \cap H:1]$.

First, let us suppose $N \neq G$. Then clearly $H_p(N) \subseteq H_p(G)$, so $H_p(N) \subseteq H \cap N$. Since $Q \subseteq H_p(N)$, it follows that $H_p(N) \neq 1$, so $[N:H_p(N)] \leq p$, and hence $[N:N \cap H] \leq p$. So $p^2 = [G:H] = [G:1]/[H:1] = [N:1]/[N \cap H:1] = [N:N \cap H] \leq p$. This is impossible, so we must have $N = G$, and thus Q is normal in G.

Now let Q_1 ($\neq 1$) be any subgroup of Q, normal in G, and consider G/Q_1. Clearly $H_p(G/Q_1) = 1$ or $H_p(G/Q_1)$ has index p in G/Q_1, unless G/Q_1 is a p-group. Indeed, it is obvious that $H_p(G/Q_1) \subseteq H/Q_1$. But $[G/Q_1:H/Q_1] = [G:H] = p^2$, so $[G/Q_1:H_p(G/Q_1)] \geq [G/Q:H/Q_1] = p_2$ implies $H_p(G/Q_1) = 1$. So G/Q_1 is a p-group.

LEMMA 1. *If $[G:H] = p^2$, then Q is an elementary abelian q-group, none of whose proper subgroups ($\neq 1$) is normal in G, Q is normal in G, and $G = PQ$, where P is a Sylow p-group of G.*

Proof. We have shown that Q is normal. If Q_1 above is taken to be the Frattini subgroup of Q, then Q_1 is normal in G, since it is characteristic in Q. Since $Q_1 \neq Q$, G/Q_1 cannot be a p-group, so we must have $Q_1 = 1$. Thus Q is elementary abelian. Since G/Q is a p-group, it is clear that $G = PQ$, and the rest of the lemma follows similarly.

In what follows, P is a Sylow p-group of G and $P_0 \subseteq P$ is a Sylow p-group of H; clearly $[P:P_0] = p^2$ and P_0 is normal in P, since $P_0 = P \cap H$.

If x ($\neq 1$) is in Q, while a is in G, not in H, and if $ax = xa$, then ax has order pq. But ax is not in H, since a is not in H, and thus ax has order p; hence $ax \neq xa$. If $P_0 = 1$, then P, of order p^2, is an automorphism group of $H = Q$ such that no non-identity element of P fixes any non-identity element of Q. But by ([2], pp. 334–335) this means that P is cyclic, whereas P is clearly elementary abelian in this case (for all its elements have order p). So $P_0 \neq 1$.

Since P_0 is normal in P, $P_0 \cap Z(P) \neq 1$ (see [3], p. 35, for instance). Let z be an element of $P_0 \cap Z(P)$, chosen to have order p, and let Z_0 be the subgroup (of order p) generated by z; note that z and Z_0 are contained in H. Let $K = Z_0Q$, and observe that $[K:1] = p[Q:1]$. Let a be an element of G, not in H, and $G_1 = \{a, K\}$ = the group generated by a and K. Then $Q \subseteq H_p(G_1) \subseteq H \cap G_1 \neq G_1$, so $[G_1:H_p(G_1)] = p$, by induction. Hence $Z_0 \subseteq K \subseteq H_p(G_1)$, so there must be an element y in K of order pq. Then y^p is in Q and y^q is in $x^{-1}Z_0x$, for some x in K, since Z_0 is a Sylow p-group of K. By adjusting our choice of P, we can assume that y^q is in Z_0; let $u = y^p$, $v = y^q$. Then $u \neq 1$, $v \neq 1$, u is in Q, v is in Z_0, and $uv = vu$. So if $Q_1 = \{u\}$, we have $Z_0 \subseteq C_G(Q_1)$. But then $x^{-1}Z_0x \subseteq C_G(x^{-1}Q_1x)$, and if x is in P, this implies $Z_0 \subseteq C_G(x^{-1}Q_1x)$, for all x in P. But, from Lemma 1, the subgroup generated by all

$x^{-1}Q_1x$, as x ranges over P, must be Q, and so $Z_0 \subseteq C_G(Q)$. Since Z_0 is in the center of P, it follows that Z_0 is normal in G, so we consider G/Z_0. One easily sees that $H_p(G/Z_0) \subseteq H/Z_0$, and $H_p(G/Z_0)$ equals neither 1 nor G/Z_0. Hence $p^2 = [G : H] = [G/Z_0 : H/Z_0] \leq [G/Z_0 : H_p(G/Z_0)] = p$, which is a contradiction. So:

THEOREM 1. *If $H_p(G) \neq 1$ or G, and if G is finite and not a p-group, then $[G : H_p(G)] = p$.*

If G is a p-group, or is infinite, the situation seems more inaccessible; as remarked earlier, Theorem 1 still holds if $p = 2$ or 3, no matter what G is. But the proof for $p = 3$ (see [5]) utilizes the Burnside theorem (for $p = 3$) and this strongly suggests that the infinite case at least is considerably harder.

3. Structure of H_p-groups. Let us suppose that G is a finite group, and that $H = H_p(G)$ has index p in G. Then we say that H is an H_p-group.

THEOREM 2. *If H is not a p-group, then H is an H_p-group if and only if H has an automorphism σ of order p such that*

$$x^{1+\sigma+\cdots+\sigma^{p-1}} = 1 ,$$

for all x in H.

Proof. If $H = H_p(G)$, let a be in G, a not in H, and define $x^\sigma = a^{-1}xa$, for x in H. Since $(ax)^p = 1$, while $(ax)^p = a^p(x)(x^\sigma) \cdots (x^{\sigma^{p-1}})$, the equation of the theorem follows immediately.

Conversely, if σ exists satisfying the hypotheses of the theorem, then let G be the holomorph of H by the automorphism group $\{\sigma\}$. It is easy to see that $H_p(G) \subseteq H$. Since $H_p(G) \neq 1$ (for H is not a p-group), it follows that $[G : H_p(G)] = p$, from Theorem 1, so $H_p(G) = H$.

Note that if $x^\sigma = x$, then the equation of Theorem 2 implies $x^p = 1$. So if p does not divide the order of the H_p-group H, then H is even a Frobenius group, and so is nilpotent ([6]).

THEOREM 3. *If H is an H_p-group, then $H = PK$, where P is a Sylow p-group of H, K is normal in H and is nilpotent, and $P \cap K = 1$. In particular, H is solvable.*

Proof. We can assume that $P \neq 1$, and that H is not a p-group. Inductively, suppose the theorem is true for all H_p-groups whose order is less than the order of H, and (using Theorem 2) let γ be an automorphism of H, of order p, such that

$$x^{1+\gamma+\cdots+\gamma^{p-1}} = 1 , \quad \text{all } x \text{ in } H.$$

If A is a γ-invariant subgroup of H, then A is an H_p-group or is a p-group, while if B is a γ-invariant normal subgroup of H, then H/B is an H_p-group or is a p-group.

Now let B be any γ-invariant subgroup of P, B normal in P, $B \neq 1$; let $N = N_H(B)$. If $N = H$, then H/B is an H_p-group, so $H/B = (P/B)(K_1/B)$, where K_1/B is normal in H/B and is nilpotent. So K_1 is normal in H and since K_1/B is γ-invariant in H/B, so is K_1 γ-invariant in H. So K_1 is an H_p-group. If $K_1 \neq H$, then $K_1 = BK$, where K is normal in K_1 and is nilpotent, and $K \cap B = 1$. But then K is characteristic in K_1, hence is normal in H; every Sylow q-group of H, $q \neq p$, is in K. So K is characteristic in H and clearly $H = PK$, $P \cap K = 1$.

If $K_1 = H$ for every such B, then $B = P$ is the only γ-invariant normal subgroup of P, other than 1. Hence in particular P is elementary abelian. Then H/P is an H_p-group, and even a Frobenius group, so is nilpotent. Furthermore (since H is then solvable), $H = PK$, where K is isomorphic to H/P. Let $K = Q_1 Q_2 \cdots Q_t$, where Q_i is a Sylow q_i-group of K (and of H) for distinct primes $q_1, q_2, \cdots, q_t$.

Now let G be the holomorph of H with the group $\{\gamma\}$. Then, by the Frattini argument, $N_G(Q_i) \cap H \neq N_G(Q_i)$, so by an appropriate choice of γ_i in G, γ_i not in H, we can assume that Q_i is γ_i-invariant. Thus PQ_i is γ_i-invariant and so it is an H_p-group (it is straightforward to check that any element of G, not in H, can play the role of γ).[2]

If $t > 1$, then PQ_i has order smaller than H, so Q_i is normal in PQ_i. Thus both P and K are contained in $N_H(Q_i)$, so Q_i is normal in H, hence K, which is the direct product of the Q_i, is normal in H, so we are done.

If $t = 1$, let $Q = Q_1$, and as above, choose γ in G, not in H, so that Q is γ-invariant. If $Q_0 \neq 1$ is a γ-invariant normal subgroup of Q, then PQ_0 is an H_p-group, smaller than $H = PQ$ if $Q_0 \neq Q$; thus P normalizes Q_0, so Q_0 is normal in H. Then by considering H/Q_0, we find that Q/Q_0 is normal, so Q is normal in H, and again we are done. Thus we can assume that Q is elementary abelian with only trivial γ-invariant normal subgroups.

Now we consider the holomorph G again. The maximal normal p-group of G is P, since $\{\gamma\}$ (as part of G) is not normalized modulo P by Q. Then G/P is a solvable (and in particular, p-solvable) group of automorphisms of the elementary abelian group P, and G/P has no normal p-group ($\neq 1$). Furthermore, this representation of G/P as a linear transformation group on P is faithful, since $C_H(P) \cap Q = 1$ (otherwise $C_H(P) \cap Q$ would be a non-trivial γ-invariant normal subgroup of Q). Thus we can utilize Theorem B of Hall and Higman ([4]); since Q is abelian, Theorem B asserts that γ, as a linear transformation of P, has the minimal

[2] In these references to the holomorph G, we are not making a distinction between an element as an automorphism of H and as an element of G; the automorphism is actually identified with an element of G which induces the prescribed automorphism in H.

polynomial $(x - 1)^p$. But in fact, γ has a minimal polynomial which divides $1 + x + \cdots + x^{p-1}$, since

$$b^{1+\gamma+\cdots+\gamma^{p-1}} = 1 ,$$

for all b in P. Thus we have a contradiction, and so Q is normal in H, and we are done.

Now we must consider the case that if B ($\neq 1$) is any γ-invariant subgroup of P, normal in P, then $N = N_H(B)$ is never equal to H. Hence N, being γ-invariant, is an H_p-group or is a p-group, so $N = P_1K_1$, where P_1 is a Sylow p-group of N, K_1 is normal in N and is nilpotent, and $K_1 \cap P_1 = 1$. Since B is normal in N, K_1 is contained in $C_N(B)$, and thus contained in $C_H(B)$, so $N_H(B)/C_H(B)$ is a p-group (i.e., is isomorphic to P_1/P_0, for some subgroup P_0 of P_1). But then, since this holds for all such B, Thompson's theorem ([6]) asserts that P has a normal complement K in H; i.e., $H = PK$, where $P \cap K = 1$ and K is normal in H. Since K consists exactly of the elements of H whose order is prime to p, K is characteristic. Thus K is an H_p-group (even a Frobenius group) and is nilpotent, so we are done.

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