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TRANSFORMATIONS ON TENSOR PRODUCT SPACES

MARVIN DAVID MARCUS AND BENJAMIN NELSON MOYLS

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1. Introduction. Let U and V be m - and n -dimensional vector spaces over an algebraically closed field F of characteristic 0. Then $U \otimes V$, the tensor product of U and V , is the dual space of the space of all bilinear functionals mapping the cartesian product of U and V into F . If $x \in U$, $y \in V$ and w is a bilinear functional, then $x \otimes y$ is defined by: $x \otimes y(w) = w(x, y)$. If e_1, \dots, e_m and f_1, \dots, f_n are bases for U and V , respectively, then the $e_i \otimes f_j$, $i = 1, \dots, m$, $j = 1, \dots, n$, form a basis for $U \otimes V$.

Let $M_{m,n}$ denote the vector space of $m \times n$ matrices over F . Then $U \otimes V$ is isomorphic to $M_{m,n}$ under the mapping ψ where $\psi(e_i \otimes f_j) = E_{i,j}$, and $E_{i,j}$ is the matrix with 1 in the (i, j) position and 0 elsewhere. An element $z \in U \otimes V$ is said to be of rank k if $z = \sum_{i=1}^k x_i \otimes y_i$, where x_1, \dots, x_k are linearly independent and so are y_1, \dots, y_k . If $R_k = \{z \in U \otimes V \mid \text{rank}(z) = k\}$, then $\psi(R_k)$ is the set of matrices of rank k , in $M_{m,n}$. In view of the isomorphism any linear map T of $U \otimes V$ into itself can be considered as a linear map of $M_{m,n}$ into itself.

In [2] and [3], Hua and Jacob obtained the structure of any mapping T that preserves the rank of every matrix in $M_{m,n}$ and whose inverse exists and has this property (coherence invariance). (In [3] F is replaced by a division ring, and T is shown to be semi-linear by appealing to the fundamental theorem of projective geometry.) In [4] we obtained the structure of T when $m = n$, T is linear and T preserves rank 1, 2 and n . Specifically, there exist non-singular matrices M and N such that $T(A) = MAN$ for all $A \in M_{nn}$, or $T(A) = MA'N$ for all A , where A' designates the transpose of A . Frobenius (cf. [1], p. 249) obtained this result when T is a linear map which preserves the determinant of every A . In [5] it was shown that this result can be obtained by requiring only that T be linear and preserve rank n . In the present paper we show that rank 1 suffices (Theorem 1), or rank 2 with the side condition that T maps no matrix of rank 4 or less into 0 (Theorem 2). Thus our hypothesis will be that T is linear and $T(R_i) \subseteq R_i$. We remark that T may be singular and still its kernel may have a zero intersection with R_i ; e.g., take $U = V$ and $T(x \otimes y) = x \otimes y + y \otimes x$.

2. Rank one preservers. Throughout this section T will be a linear transformation (l.t.) of $U \otimes V$ into $U \otimes V$ such that $T(R_i) \subseteq R_i$. Here

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U and V are m - and n -dimensional vector spaces over F . Let e_1, \dots, e_m and f_1, \dots, f_n be fixed bases for U and V , and set

$$(1) \quad T(e_i \otimes f_j) = u_{ij} \otimes v_{ij}, \quad i = 1, \dots, m; j = 1, \dots, n.$$

Note that no u_{ij} or v_{ij} can be zero. We shall show, in case $m \neq n$ that there exist vectors u_i and v_j such that $T(e_i \otimes f_j) = u_i \otimes v_j$, and hence that the l.t. T is a tensor product of transformations on U and V separately. In case $m = n$ it will be shown that a slight modification of T is a tensor product.

Denote by $L(x_1, \dots, x_r)$ the subspace spanned by the vectors x_1, \dots, x_r , and let $\rho(x_1, \dots, x_r)$ be the dimension of $L(x_1, \dots, x_r)$.

LEMMA 1. *Let $x_1, \dots, x_r, w_1, \dots, w_s$ be vectors in U , and let $y_1, \dots, y_r, z_1, \dots, z_s$ be vectors in V . Let*

$$(2) \quad \sum_{i=1}^r (x_i \otimes y_i) = \sum_{j=1}^s (w_j \otimes z_j).$$

If $\rho(x_1, \dots, x_r) = r$, then $y_i \in L(z_1, \dots, z_s)$, $i = 1, \dots, r$; and similarly if $\rho(y_1, \dots, y_r) = r$, then $x_i \in L(w_1, \dots, w_s)$, $i = 1, \dots, r$.

Proof. Suppose that $\rho(x_1, \dots, x_r) = r$. Let θ be a linear functional on U such that $\theta(x_1) = 1$, $\theta(x_i) = 0$, $i \neq 1$, and let α be an arbitrary linear functional on V . For $x \in U$, $y \in V$, define

$$(3) \quad g(x, y) = \theta(x)\alpha(y).$$

Applying (2) to g , we get

$$\alpha(y_1) = \sum_{i=1}^r \theta(x_i)\alpha(y_i) = \alpha\left(\sum_{j=1}^s \theta(w_j)z_j\right)$$

where each $\theta(w_j)$ is a scalar. Since α is arbitrary, y_1 , and similarly y_2, \dots, y_r , are contained in $L(z_1, \dots, z_s)$. The second part of the lemma is proved in the same way.

LEMMA 2. *If $T(R_1) \subseteq R_1$, and T satisfies (1), then for $i = 1, \dots, m$, either*

$$(4) \quad \rho(u_{i1}, \dots, u_{in}) = n \quad \text{and} \quad \rho(v_{i1}, \dots, v_{in}) = 1,$$

or

$$(5) \quad \rho(u_{i1}, \dots, u_{in}) = 1 \quad \text{and} \quad \rho(v_{i1}, \dots, v_{in}) = n.$$

Similarly, for $j = 1, \dots, n$, either

$$(6) \quad \rho(u_{1j}, \dots, u_{mj}) = m \quad \text{and} \quad \rho(v_{1j}, \dots, v_{mj}) = 1,$$

or

$$(7) \quad (u_{i_j}, \dots, u_{m_j}) = 1 \quad \text{and} \quad (v_{1_j}, \dots, v_{m_j}) = m .$$

Proof. Suppose that u_{i_α} and u_{i_β} are independent. Then

$$T(e_i \otimes (f_\alpha + f_\beta)) = (u_{i_\alpha} \otimes v_{i_\alpha}) + (u_{i_\beta} \otimes v_{i_\beta})$$

must be a tensor product $u \otimes v$. By Lemma 1, $v_{i_\alpha}, v_{i_\beta} \in L(v)$. Since all $v_{i_j} \neq 0$, $L(v_{i_\alpha}) = L(v_{i_\beta})$. For $\gamma \neq \alpha, \beta$, $L(v_{i_\gamma}) = L(v_{i_\alpha})$, since u_{i_γ} must be independent of at least one of $u_{i_\alpha}, u_{i_\beta}$. We have shown that if $\rho(u_{i_1}, \dots, u_{i_n}) \geq 2$, then $\rho(v_{i_1}, \dots, v_{i_n}) = 1$.

Suppose next that $\rho(u_{i_1}, \dots, u_{i_n}) = 1$, viz., $u_{i_\alpha} = c_\alpha u_{i_1}$, $c_\alpha \neq 0$, $\alpha = 1, \dots, n$. If

$$\rho(v_{i_1}, \dots, v_{i_n}) < n, \quad \text{let} \quad \sum_{\alpha=1}^n a_\alpha v_{i_\alpha} = 0$$

be a non-trivial dependence relation. Then

$$T\left(e_i \otimes \left(\sum_{\alpha=1}^n \frac{a_\alpha}{c_\alpha} f_\alpha\right)\right) = \sum_{\alpha=1}^n \left(c_\alpha u_{i_1} \otimes \frac{a_\alpha}{c_\alpha} v_{i_\alpha}\right) = u_{i_1} \otimes \left(\sum_{\alpha=1}^n a_\alpha v_{i_\alpha}\right) = 0,$$

which is impossible by the nature of T . Hence $\rho(u_{i_1}, \dots, u_{i_n}) = 1$ implies $\rho(v_{i_1}, \dots, v_{i_n}) = n$.

It follows by a similar argument that if $\rho(v_{i_1}, \dots, v_{i_n}) = 1$, then $\rho(u_{i_1}, \dots, u_{i_n}) = n$. Hence either (4) or (5) must hold. The second part of the lemma is proved similarly.

We remark that if $m < n$ (or $n < m$), then (4) (or (7)) cannot hold.

LEMMA 3. *Either (4) and (7) hold for all i, j ; or (5) and (6) hold for all i, j .*

Proof. We show first that either (4) or (5) holds uniformly in i . Suppose that for some i and k , $1 \leq i \leq k \leq m$, $\rho(u_{i_1}, \dots, u_{i_n}) = n$ while $\rho(u_{k_1}, \dots, u_{k_n}) = 1$. Then for some α , $1 \leq \alpha \leq n$, $\rho(u_{i_\alpha}, u_{k_\alpha}) = 2$. For $\beta \neq \alpha$ consider

$$\begin{aligned} \eta &= T[(e_i + e_k) \otimes (cf_\alpha + f_\beta)] \\ &= c(u_{i_\alpha} \otimes v_{i_\alpha}) + (u_{i_\beta} \otimes v_{i_\beta}) + c(u_{k_\alpha} \otimes v_{k_\alpha}) + (u_{k_\beta} \otimes v_{k_\beta}), \end{aligned}$$

where c is an arbitrary scalar.

By hypothesis and Lemma 2, $v_{i_\alpha} = av_{k_\alpha}$ and $v_{i_\beta} = b_1 v_{i_\alpha} = bv_{k_\alpha}$ for suitable non-zero scalars a and b , while $\rho(v_{k_\alpha}, v_{k_\beta}) = 2$. Thus $\eta = (acu_{i_\alpha} + bu_{i_\beta} + cu_{k_\alpha}) \otimes v_{k_\alpha} + (u_{k_\beta} \otimes v_{k_\beta})$, and by Lemma 1, $\rho(acu_{i_\alpha} + bu_{i_\beta} + cu_{k_\alpha}, u_{k_\beta}) = 1$ for all scalars c . Since $\rho(u_{k_\alpha}, u_{k_\beta}) = 1$, this implies that $\rho(cu_{i_\alpha} + u_{i_\beta}, u_{k_\beta}) = 1$ for all c . This is impossible, since $\rho(u_{i_\alpha}, u_{i_\beta}) = 2$. Thus either (4) is true for all i , or (5) is true for all i . A similar argument applies to (6) and (7).

If (4) and (6) hold for all i and j , then there exist non-zero scalars c_{ij} such that $v_{ij} = c_{ij}v_{11}$, $i = 1, \dots, m$, $j = 1, \dots, n$. For a_j, b scalars, consider

$$T\left[\left(\sum_{i=1}^m a_i e_i\right) \otimes (f_1 - b f_2)\right] = \left(\sum_{i=1}^m a_i c_{i1} u_{i1} - b \sum_{i=1}^m a_i c_{i2} u_{i2}\right) \otimes v_{11}.$$

Let z_1, \dots, z_m and w_1, \dots, w_m be the m -column vectors which are respectively the representations of u_{11}, \dots, u_{m1} and u_{12}, \dots, u_{m2} with respect to the basis e_1, \dots, e_m . Let C be the m -square matrix whose columns are $c_{11}z_1, \dots, c_{m1}z_m$ and let W be the m -square matrix whose columns are $c_{12}w_1, \dots, c_{m2}w_m$. Then with respect to the basis e_1, \dots, e_m the vector $\sum_{i=1}^m a_i c_{i1} u_{i1} - b \sum_{i=1}^m a_i c_{i2} u_{i2}$ has the representation $(C - bW)a$ where a is the column m -tuple (a_1, \dots, a_m) . Now C and W are non-singular since $\rho(u_{11}, \dots, u_{m1}) = \rho(u_{12}, \dots, u_{m2}) = m$, so choose b to be an eigenvalue of $W^{-1}C$ and choose a to be the corresponding eigenvector. Then $(C - bW)a = 0$ and hence there exist scalars a_1, \dots, a_m not all 0 and b such that

$$T\left(\sum_{i=1}^m a_i e_i \otimes (f_1 - b f_2)\right) = 0,$$

a contradiction since $T(R_1) \subseteq R_1$.

Hence (4) and (6) cannot hold for all i and j . Similarly both (5) and (7) cannot hold for all i and j . This completes the proof of the lemma.

In view of the remark preceding this lemma, (5) and (6) must hold when $m \neq n$.

THEOREM 1. *Let U and V be m - and n -dimensional vector spaces respectively. Let T be a linear transformation on $U \otimes V$ which maps elements of rank one into elements of rank one. Let T_1 be the l.t. of $V \otimes U$ into $U \otimes V$ which maps $y \otimes x$ onto $x \otimes y$. If $m = n$, let φ be any non-singular l.t. of U onto V . Then if $m \neq n$, there exist non-singular l.t.'s A and B on U and V , respectively, such that $T = A \otimes B$. If $m = n$, there exist non-singular A and B such that either $T = A \otimes B$ or $T = T_1(\varphi A \otimes \varphi^{-1}B)$.*

Proof. By (1) and Lemma 3, $T(e_i \otimes f_j) = u_{ij} \otimes v_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$, where either (5) and (6) hold or (4) and (7) hold. Suppose first that the former is the case; in particular, $\rho(u_{i1}, \dots, u_{in}) = 1$ for $i = 1, \dots, m$ and $\rho(v_{1j}, \dots, v_{mj}) = 1$ for $j = 1, \dots, n$. Then there exist non-zero scalars s_{ij}, t_{ij} such that $u_{ij} = s_{ij}u_{i1}$ and $v_{ij} = t_{ij}v_{1j}$. Thus

$$(8) \quad T(e_i \otimes f_j) = c_{ij}u_i \otimes v_j,$$

where $u_i = u_{i1}$, $v_j = v_{1j}$, and $c_{ij} = s_{ij}t_{ij}$. For $i = 2, \dots, n$,

$$T\left[(e_1 + e_i) \otimes \left(\sum_{j=1}^n f_j\right)\right] = u_1 \otimes \sum_{j=1}^n c_{1j}v_j + u_i \otimes \sum_{j=1}^n c_{ij}v_j$$

must be a direct product $x \otimes w$. By (6) and Lemma 1, $\sum_{j=1}^n c_{ij}v_j = d_i \sum_{j=1}^n c_{1j}v_j$ for some constant d_i . By (5), $c_{ij} = d_i c_{1j}$. Hence

$$(9) \quad T(e_i \otimes f_j) = x_i \otimes y_j,$$

where $x_i = d_i u_i$ and $y_j = c_{1j}v_j$. Since the $\{x_i\}$ and $\{y_j\}$ are each linearly independent sets, there non-singular linear transformations A and B such that $x_i = Ae_i$ and $y_j = Bf_j$. Then $T = A \otimes B$.

When $m = n$, (4) and (7) may hold; in particular,

$$\rho(v_{i1}, \dots, v_{in}) = 1 \text{ and } \rho(u_{1j}, \dots, u_{nj}) = 1 \text{ for } i, j = 1, \dots, n.$$

As in the preceding case, there exist linearly independent sets x_1, \dots, x_n and y_1, \dots, y_n such that

$$(10) \quad T(e_i \otimes f_j) = x_j \otimes y_i.$$

There exist non-singular transformations A and B of U and V , respectively, such that $Ae_i = \varphi^{-1}y_i$ and $Bf_j = \varphi x_j$, $i, j = 1, \dots, n$. Thus $T^{-1}T(e_i \otimes f_j) = \varphi Ae_i \otimes \varphi^{-1}Bf_j$. Q.E.D.

In matrix language we have the following.

COROLLARY. *Let T be a l.t. on the space M_{nn} of n -square matrices. If the set of rank one matrices is invariant under T , then there exist non-singular matrices A and B such that either $T(X) = AXB$ for all $X \in M_{nn}$ or $T(X) = AX'B$ for all $X \in M_{nn}$.*

3. Rank two preservers. In this section T will be a l.t. of $U \otimes V$ such that $T(R_2) \subseteq R_2$. We shall show that under certain conditions $T(R_1) \subseteq R_1$.

LEMMA 4. *If W is a subspace of $U \otimes V$ such that, for some integer r , $1 \leq r \leq \min(m, n)$,*

$$(11) \quad \dim W \geq mn - r \max(m, n) + 1,$$

then $W \cap \bigcup_{j=1}^r R_j \neq \phi$.

Proof. Suppose that $m = \max(m, n)$. The products $e_i \otimes f_j$, $i = 1, \dots, m$, $j = 1, \dots, r$, are linearly independent and span a space W_1 of dimension mr . Furthermore, $W_1 \subseteq \bigcup_{j=1}^r R_j$. Then $\dim(W_1 \cap W) = \dim W_1 + \dim W - \dim(W_1 \cup W) \geq mr + (mn - rm + 1) - mn = 1$. The result follows, since $W_1 \cap W \subseteq \bigcup_{j=1}^r R_j \cap W$.

LEMMA 5. *If $T(R_2) \subseteq T(R_2) \subseteq R_2$, then $T(R_1) \subseteq R_1 \cup R_2$.*

Proof. Suppose $x_1 \otimes y_1 \in R_1$, and choose $x_2 \otimes y_2 \in R_1$ such that $\rho(x_1, x_2) = \rho(y_1, y_2) = 2$. Then $\alpha = sT(x_1 \otimes y_1) + tT(x_2 \otimes y_2) \in R_2$ for all non-zero scalars s, t . Now suppose that $T(x_1 \otimes y_1) = \sum_{j=1}^p u_j \otimes v_j$, where $\rho(u_1, \dots, u_p) = \rho(v_1, \dots, v_p) = p$, and that $T(x_2 \otimes y_2) = \sum_{j=1}^q z_j \otimes w_j$, where $\rho(z_1, \dots, z_q) = \rho(w_1, \dots, w_q) = q$. Let u_{p+1}, \dots, u_m be a completion of u_1, \dots, u_p to a basis for U . It follows that

$$\sum_{j=1}^q z_j \otimes w_j = \sum_{j=1}^m u_j \otimes h_j$$

for some vectors $h_j \in V, j = 1, \dots, m$. Then

$$\begin{aligned} \alpha &= \sum_{j=1}^p u_j \otimes sv_j + \sum_{j=1}^p u_j \otimes th_j + \sum_{j=p+1}^m u_j \otimes th_j \\ &= \sum_{j=1}^p u_j \otimes (sv_j + th_j) + \sum_{j=p+1}^m u_j \otimes th_j. \end{aligned}$$

Since $\alpha \in R_2$, it follows by Lemma 1 that

$$\rho(sv_1 + th_1, \dots, sv_p + th_p) \leq 2 \text{ for } st \neq 0.$$

The vectors $sv_1 + th_1, \dots, sv_p + th_p$ are linearly independent when $s = 1$ and $t = 0$. By continuity, they remain independent for small values of t . Hence $p \leq 2$ and $T(x_1 \otimes y_1) \in R_1 \cup R_2$.

THEOREM 2. *If $T(R_2) \subseteq R_2$ and $0 \notin T(\mathbf{U}_{j=1}^i R_j)$, then $T(R_1) \subseteq R_1$.*

Proof. Suppose $x_1 \otimes y_1 \in R_1$ and $T(x_1 \otimes y_1) \notin R_1$. By Lemma 5, $T(x_1 \otimes y_1) \in R_2$, since $0 \notin T(R_1)$. Thus $T(x_1 \otimes y_1) = (u_1 \otimes v_1) + (u_2 \otimes v_2)$, where $\rho(u_1, u_2) = \rho(v_1, v_2) = 2$. Let x_1, \dots, x_m and y_1, \dots, y_n be bases for U and V respectively. Then for $st \neq 0$

$$(12) \quad sT(x_i \otimes y_1) + tT(x_i \otimes y_j) \in R_1 \cup R_2 \text{ for } i = 1, \dots, m, j = 1, \dots, n.$$

At this point it seems simpler to regard the images $T(x_i \otimes y_j)$ as elements of M_{mn} . It is clear that there is no loss in generality in taking $T(x_1 \otimes y_1) = E_{11} + E_{22}$.

Let i and j be fixed for this discussion, and let $A = T(x_i \otimes y_j)$. Let a_1, \dots, a_n be the m -dimensional vectors which are the columns of A , and let ε_k be the unit vector with 1 in the k th position. It follows from (12) that

$$(13) \quad \rho(s\varepsilon_1 + ta_1, s\varepsilon_2 + ta_2, ta_3, \dots, ta_n) = 2$$

for $st \neq 0$. The Grassmann products

$$(14) \quad (s\varepsilon_1 + ta_1) \wedge (s\varepsilon_2 + ta_2) \wedge ta_k, \quad 3 \leq k \leq n$$

must be zero for $st \neq 0$. In the expansion of (14) the coefficient of s^2t is 0; that is, $\varepsilon_1 \wedge \varepsilon_2 \wedge a_k = 0$.

Thus the matrix A has non-zero entries only in the first two rows and columns. It follows immediately that the dimension of the range of $T \leq 2(m+n) - 4$. Hence the dimension of the kernel of $T \geq mn - 2(m+n) + 4 > mn - 4 \max(m, n) + 1$.

By Lemma 4, there exists an element of $\mathbf{U}_{j=1}^4$ whose image is zero. This contradicts the hypothesis; hence $T(R_1) \subseteq R_1$.

We see then that the form of T satisfying Theorem 2 is given in the conclusions of Theorem 1.

REMARK. We feel that the hypothesis $0 \notin T(\mathbf{U}_{j=1}^4 R_j)$ of Theorem 2 should not be necessary, but we have not been able to prove the theorem without it. More generally, we conjecture that $T(R_k) \subseteq R_k$ for some fixed k , $1 \leq k \leq n$, should suffice to prove that T is essentially a tensor product.

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Vol. 9, No. 4

August, 1959

Frank Herbert Brownell, III, <i>A note on Kato's uniqueness criterion for Schrödinger operator self-adjoint extensions</i>	953
Edmond Darrell Cashwell and C. J. Everett, <i>The ring of number-theoretic functions</i>	975
Heinz Otto Cordes, <i>On continuation of boundary values for partial differential operators</i>	987
Philip C. Curtis, Jr., <i>n-parameter families and best approximation</i>	1013
Uri Fixman, <i>Problems in spectral operators</i>	1029
I. S. Gál, <i>Uniformizable spaces with a unique structure</i>	1053
John Mitchell Gary, <i>Higher dimensional cyclic elements</i>	1061
Richard P. Gosselin, <i>On Diophantine approximation and trigonometric polynomials</i>	1071
Gilbert Helmsberg, <i>Generating sets of elements in compact groups</i>	1083
Daniel R. Hughes and John Griggs Thompson, <i>The H-problem and the structure of H-groups</i>	1097
James Patrick Jans, <i>Projective injective modules</i>	1103
Samuel Karlin and James L. McGregor, <i>Coincidence properties of birth and death processes</i>	1109
Samuel Karlin and James L. McGregor, <i>Coincidence probabilities</i>	1141
J. L. Kelley, <i>Measures on Boolean algebras</i>	1165
John G. Kemeny, <i>Generalized random variables</i>	1179
Donald G. Malm, <i>Concerning the cohomology ring of a sphere bundle</i>	1191
Marvin David Marcus and Benjamin Nelson Moysls, <i>Transformations on tensor product spaces</i>	1215
Charles Alan McCarthy, <i>The nilpotent part of a spectral operator</i>	1223
Kotaro Oikawa, <i>On a criterion for the weakness of an ideal boundary component</i>	1233
Barrett O'Neill, <i>An algebraic criterion for immersion</i>	1239
Murray Harold Protter, <i>Vibration of a nonhomogeneous membrane</i>	1249
Victor Lenard Shapiro, <i>Intrinsic operators in three-space</i>	1257
Morgan Ward, <i>Tests for primality based on Sylvester's cyclotomic numbers</i>	1269
L. E. Ward, <i>A fixed point theorem for chained spaces</i>	1273
Alfred B. Willcox, <i>Šilov type C algebras over a connected locally compact abelian group</i>	1279
Jacob Feldman, <i>Correction to "Equivalence and perpendicularity of Gaussian processes"</i>	1295