Pacific Journal of Mathematics

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Vol. 9, No. 4 August 1959

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1. Introduction. Let U and V be m- and n-dimensional vector spaces over an algebraically closed field F of characteristic 0. Then $U \otimes V$, the tensor product of U and V, is the dual space of the space of all bilinear functionals mapping the cartesian product of U and V into F. If $x \in U$, $y \in V$ and w is a bilinear functional, then $x \otimes y$ is defined by: $x \otimes y(w) = w(x, y)$. If $e_1, \cdots, e_m$ and $f_1, \cdots, f_n$ are bases for U and V, respectively, then the $e_i \otimes f_j$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, form a basis for $U \otimes V$.

Let $M_{m,n}$ denote the vector space of $m \times n$ matrices over F. Then $U \otimes V$ is isomorphic to $M_{m,n}$ under the mapping ψ where $\psi(e_i \otimes f_j) = E_{ij}$, and E_{ij} is the matrix with 1 in the (i, j) position and 0 elsewhere. An element $z \in U \otimes V$ is said to be of rank k if $z = \sum_{i=1}^{k} x_i \otimes y_i$, where $x_1, \cdots, x_k$ are linearly independent and so are $y_1, \cdots, y_k$. If $R_k = \{z \in U \otimes V \mid \operatorname{rank}(z) = k\}$, then $\psi(R_k)$ is the set of matrices of rank k, in $M_{m,n}$. In view of the isomorphism any linear map T of $U \otimes V$ into itself can be considered as a linear map of $M_{m,n}$ into itself.

In [2] and [3], Hua and Jacob obtained the structure of any mapping T that preserves the rank of every matrix in $M_{m,n}$ and whose inverse exists and has this property (coherence invariance). (In [3] F is replaced by a division ring, and T is shown to be semi-linear by appealing to the fundamental theorem of projective geometry.) In [4] we obtained the structure of T when $m = n$, T is linear and T preserves rank 1, 2 and n. Specifically, there exist non-singular matrices M and N such that $T(A) = MAN$ for all $A \in M_{nn}$, or $T(A) = MA'N$ for all A, where A' designates the transpose of A. Frobenius (cf. [1], p. 249) obtained this result when T is a a linear map which preserves the determinant of every A. In [5] it was shown that this result can be obtained by requiring only that T be linear and preserve rank n. In the present paper we show that rank 1 suffices (Theorem 1), or rank 2 with the side condition that T maps no matrix of rank 4 or less into 0 (Theorem 2). Thus our hypothesis will be that T is linear and $T(R_1) \subseteq R_1$. We remark that T may be singular and still its kernel may have a zero intersection with R_1; e.g., take $U = V$ and $T(x \otimes y) = x \otimes y + y \otimes x$.

2. Rank one preservers. Throughout this section T will be a linear transformation (l.t.) of $U \otimes V$ into $U \otimes V$ such that $T(R_1) \subseteq R_1$. Here

Received March 2, 1959. The work of the first author was sponsored by U.S. National Science Foundation Grant G. 5416.

U and V are m- and n-dimensional vector spaces over F. Let $e_1, \cdots, e_m$ aud $f_1 \cdots, f_n$ be fixed bases for U and V, and set

$$T(e_i \otimes f_j) = u_{ij} \otimes v_{ij}\,, \qquad i = 1, \cdots, m;\ j = 1, \cdots, n. \tag{1}$$

Note that no u_{ij} or v_{ij} can be zero. We shall show, in case $m \neq n$ that there exist vectors u_i and v_j such that $T(e_i \otimes f_j) = u_i \otimes v_j$, and hence that the l.t. T is a tensor product of transformations on U and V separately. In case $m = n$ it will be shown that a slight modification of T is a tensor product.

Denote by $L(x_1, \cdots, x_t)$ the subspace spanned by the vectors $x_1, \cdots, x_t$, and let $\rho(x_1, \cdots, x_t)$ be the dimension of $L(x_1, \cdots, x_t)$.

LEMMA 1. *Let $x_1, \cdots, x_r, w_1, \cdots, w_s$ be vectors in U, and let $y_1, \cdots, y_r, z_1, \cdots, z_s$ be vectors in V. Let*

$$\sum_{i=1}^{r} (x_i \otimes y_i) = \sum_{j=1}^{s} (w_j \otimes z_j)\,. \tag{2}$$

If $\rho(x_1, \cdots, x_r) = r$, then $y_i \in L(z_1, \cdots, z_s)$, $i = 1, \cdots, r$; and similarly if $\rho(y_1, \cdots, y_r) = r$, then $x_i \in L(w_1, \cdots, w_s)$, $i = 1, \cdots, r$.

Proof. Suppose that $\rho(x_1, \cdots, x_r) = r$. Let θ be a linear functional on U such that $\theta(x_1) = 1$, $\theta(x_i) = 0$, $i \neq 1$, and let α be an arbitrary linear functional on V. For $x \in U$, $y \in V$, define

$$g(x, y) = \theta(x)\alpha(y)\,. \tag{3}$$

Applying (2) to g, we get

$$\alpha(y_1) = \sum_{i=1}^{s} \theta(w_j)\alpha(z_j) = \alpha\Big(\sum_{j=1}^{s} \theta(w_j) z_j\Big)$$

where each $\theta(w_j)$ is a scalar. Since α is arbitrary, y_1, and similarly $y_2, \cdots, y_r$, are contained in $L(z_1, \cdots, z_s)$. The second part of the lemma is proved in the same way.

LEMMA 2. *If $T(R_1) \subseteq R_1$, and T satisfies* (1), *then for $i = 1, \cdots, m$, either*

$$\rho(u_{i1}, \cdots, u_{in}) = n \quad \textit{and} \quad \rho(v_{i1}, \cdots, v_{in}) = 1\,, \tag{4}$$

or

$$\rho(u_{i1}, \cdots, u_{in}) = 1 \quad \textit{and} \quad \rho(v_{i1}, \cdots, v_{in}) = n\,. \tag{5}$$

Similarly, for $j = 1, \cdots, n$, either

$$\rho(u_{1j}, \cdots, u_{mj}) = m \quad \textit{and} \quad \rho(v_{1j}, \cdots, v_{mj}) = 1\,, \tag{6}$$

or

$$(7) \qquad (u_{1j}, \cdots, u_{mj}) = 1 \quad \textit{and} \quad (v_{1j}, \cdots, v_{mj}) = m\,.$$

Proof. Suppose that $u_{i\alpha}$ and $u_{i\beta}$ are independent. Then

$$T(e_i \otimes (f_\alpha + f_\beta)) = (u_{i\alpha} \otimes v_{i\alpha}) + (u_{i\beta} \otimes v_{i\beta})$$

must be a tensor product $u \otimes v$. By Lemma 1, $v_{i\alpha}$, $v_{i\beta} \in L(v)$. Since all $v_{ij} \neq 0$, $L(v_{i\alpha}) = L(v_{i\beta})$. For $\gamma \neq \alpha, \beta$, $L(v_{i\gamma}) = L(v_{i\alpha})$, since $u_{i\gamma}$ must be independent of at least one of $u_{i\alpha}$, $u_{i\beta}$. We have shown that if $\rho(u_{i1}, \cdots, u_{in}) \geq 2$, then $\rho(v_{i1}, \cdots, v_{in}) = 1$.

Suppose next that $\rho(u_{i1}, \cdots, u_{in}) = 1$, viz., $u_{i\alpha} = c_\alpha u_{i1}$, $c_\alpha \neq 0$, $\alpha = 1, \cdots, n$. If

$$\rho(v_{i1}, \cdots, v_{in}) < n\,, \quad \text{let} \quad \sum_{\alpha=1}^{n} a_\alpha v_{i\alpha} = 0$$

be a non-trivial dependence relation. Then

$$T\left(e_i \otimes \left(\sum_{\alpha=1}^{n} \frac{a_\alpha}{c_\alpha} f_\alpha\right)\right) = \sum_{\alpha=1}^{n} \left(c_\alpha u_{i1} \otimes \frac{a_\alpha}{c_\alpha} v_{i\alpha}\right) = u_{i1} \otimes \left(\sum_{\alpha=1}^{n} a_\alpha v_{i\alpha}\right) = 0\,,$$

which is impossible by the nature of T. Hence $\rho(u_{i1}, \cdots, u_{in}) = 1$ implies $\rho(v_{i1}, \cdots, v_{in}) = n$.

It follows by a similar argument that if $\rho(v_{i1}, \cdots, v_{in}) = 1$, then $\rho(u_{i1}, \cdots, u_{in}) = n$. Hence either (4) or (5) must hold. The second part of the lemma is proved similarly.

We remark that if $m < n$ (or $n < m$), then (4) (or (7)) cannot hold.

LEMMA 3. *Either* (4) *and* (7) *hold for all* i, j; *or* (5) *and* (6) *hold for all* i, j.

Proof. We show first that either (4) or (5) holds uniformly in i. Suppose that for some i and k, $1 \leq i \leq k \leq m$, $\rho(u_{i1}, \cdots, u_{in}) = n$ while $\rho(u_{k1}, \cdots, u_{kn}) = 1$. Then for some α, $1 \leq \alpha \leq n$, $\rho(u_{i\alpha}, u_{k\alpha}) = 2$. For $\beta \neq \alpha$ consider

$$\begin{aligned}\eta &= T[(e_i + e_k) \otimes (cf_\alpha + f_\beta)] \\ &= c(u_{i\alpha} \otimes v_{i\alpha}) + (u_{i\beta} \otimes v_{i\beta}) + c(u_{k\alpha} \otimes v_{k\alpha}) + (u_{k\beta} \otimes v_{k\beta})\,,\end{aligned}$$

where c is an arbitrary scalar.

By hypothesis and Lemma 2, $v_{i\alpha} = av_{k\alpha}$ and $v_{i\beta} = b_1 v_{i\alpha} = bv_{k\alpha}$ for suitable non-zero scalars a and b, while $\rho(v_{k\alpha}, v_{k\beta}) = 2$. Thus $\eta = (acu_{i\alpha} + bu_{i\beta} + cu_{k\alpha}) \otimes v_{k\alpha} + (u_{k\beta} \otimes v_{k\beta})$, and by Lemma 1, $\rho(acu_{i\alpha} + bu_{i\beta} + cu_{k\alpha}, u_{k\beta}) = 1$ for all scalars c. Since $\rho(u_{k\alpha}, u_{k\beta}) = 1$, this implies that $\rho(cu_{i\alpha} + u_{i\beta}, u_{k\beta}) = 1$ for all c. This is impossible, since $\rho(u_{i\alpha}, u_{i\beta}) = 2$. Thus either (4) is true for all i, or (5) is true for all i. A similar argument applies to (6) and (7).

If (4) and (6) hold for all i and j, then there exist non-zero scalars c_{ij} such that $v_{ij} = c_{ij}v_{11}$, $i = 1, \cdots, m$, $j = 1. \cdots, n$. For a_j, b scalars, consider

$$T\left[\left(\sum_{i=1}^{m} a_i e_i\right) \otimes (f_1 - bf_2)\right] = \left(\sum_{i=1}^{m} a_i c_{i1} u_{i1} - b \sum_{i=1}^{m} a_i c_{i2} u_{i2}\right) \otimes v_{11} .$$

Let $z_1, \cdots, z_m$ and $w_1, \cdots, w_m$ be the m-column vectors which are respectively the representations of $u_{11}, \cdots, u_{m1}$ and $u_{12}, \cdots, u_{m2}$ with respect to the basis $e_1, \cdots, e_m$. Let C be the m-square matrix whose columns are $c_{11}z_1, \cdots, c_{m1}z_m$ and let W be the m-square matrix whose columns are $c_{12}w_1, \cdots, c_{m2}w_m$. Then with respect to the basis $e_1, \cdots, e_m$ the vector $\sum_{i=1}^{m} a_i c_{i1} u_{i1} - b \sum_{i=1}^{m} a_i c_{i2} u_{i2}$ has the representation $(C - bW)a$ where a is the column m-tuple $(a_1, \cdots, a_m)$. Now C and W are non-singular since $\rho(u_{11}, \cdots, u_{m1}) = \rho(u_{12}, \cdots, u_{m2}) = m$, so choose b to be an eigenvalue of $W^{-1}C$ and choose a to be the corresponding eigenvector. Then $(C - bW)a = 0$ and hence there exist scalars $a_1, \cdots, a_m$ not all 0 and b such that

$$T\left(\sum_{i=1}^{m} a_i e_i \otimes (f_1 - bf_2)\right) = 0 ,$$

a contradiction since $T(R_1) \subseteq R_1$.

Hence (4) and (6) cannot hold for all i and j. Similarly both (5) and (7) cannot hold for all i and j. This completes the proof of the lemma.

In view of the remark preceding this lemma, (5) and (6) must hold when $m \neq n$.

THEOREM 1. *Let U and V be m- and n-dimensional vector spaces respectively. Let T be a linear transformation on $U \otimes V$ which maps elements of rank one into elements of rank one. Let T_1 be the* l.t. *of $V \otimes U$ into $U \otimes V$ which maps $y \otimes x$ onto $x \otimes y$. If $m = n$, let φ be any non-singular* l.t. *of U onto V. Then if $m \neq n$, there exist non-singular* l.t.'s *A and B on U and V, respectively, such that $T = A \otimes B$. If $m = n$, there exist non-singular A and B such that either $T = A \otimes B$ or $T = T_1(\varphi A \otimes \varphi^{-1} B)$.*

Proof. By (1) and Lemma 3, $T(e_i \otimes f_j) = u_{ij} \otimes v_{ij}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, where either (5) and (6) hold or (4) and (7) hold. Suppose first that the former is the case; in particular, $\rho(u_{i1}, \cdots, u_{in}) = 1$ for $i = 1, \cdots, m$ and $\rho(v_{1j}, \cdots, v_{mj}) = 1$ for $j = 1, \cdots, n$. Then there exist non-zero scalars s_{ij}, t_{ij} such that $u_{ij} = s_{ij}u_{i1}$ and $v_{ij} = t_{ij}v_{1j}$. Thus

$$T(e_i \otimes f_j) = c_{ij} u_i \otimes v_j , \tag{8}$$

where $u_i = u_{i1}$, $v_j = v_{1j}$, and $c_{ij} = s_{ij}t_{ij}$. For $i = 2, \cdots, n$,

$$T\left[(e_1 + e_i) \otimes \left(\sum_{j=1}^{n} f_j\right)\right] = u_1 \otimes \sum_{j=1}^{n} c_{1j}v_j + u_i \otimes \sum_{j=1}^{n} c_{ij}v_j$$

must be a direct product $x \otimes w$. By (6) and Lemma 1, $\sum_{j=1}^{n} c_{ij}v_j = d_i \sum_{j=1}^{n} c_{1j}v_j$ for some constant d_i. By (5), $c_{ij} = d_i c_{1j}$. Hence

$$T(e_i \otimes f_j) = x_i \otimes y_j \,, \tag{9}$$

where $x_i = d_i u_i$ and $y_j = c_{1j}v_j$. Since the $\{x_i\}$ and $\{y_j\}$ are each linearly independent sets, there non-singular linear transformations A and B such that $x_i = Ae_i$ and $y_j = Bf_j$. Then $T = A \otimes B$.

When $m = n$, (4) and (7) may hold; in particular,

$$\rho(v_{i1}, \cdots, v_{in}) = 1 \text{ and } \rho(u_{1j}, \cdots, u_{nj}) = 1 \quad \text{for} \quad i, j = 1, \cdots, n.$$

As in the preceding case, there exist linearly independent sets $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ such that

$$T(e_i \otimes f_j) = x_j \otimes y_i \,. \tag{10}$$

There exist non-singular transformations A and B of U and V, respectively, such that $Ae_i = \varphi^{-1}y_i$ and $Bf_j = \varphi x_j$, $i, j = 1, \cdots, n$. Thus $T_1^{-1}T(e_i \otimes f_j) = \varphi Ae_i \otimes \varphi^{-1}Bf_j$. Q.E.D.

In matrix language we have the following.

COROLLARY. *Let T be a* l.t. *on the space M_{nn} of n-square matrices. If the set of rank one matrices is invariant under T, then there exist non-singular matrices A and B such that either $T(X) = AXB$ for all $X \in M_{nn}$ or $T(X) = AX'B$ for all $X \in M_{nn}$.*

3. **Rank two preservers.** In this section T will be a l.t. of $U \otimes V$ such that $T(R_2) \subseteq R_2$. We shall show that under certain conditions $T(R_1) \subseteq R_1$.

LEMMA 4. *If W is a subspace of $U \otimes V$ such that, for some integer r, $1 \leq r \leq \min(m, n)$,*

$$\dim W \geq mn - r \max(m, n) + 1 \,, \tag{11}$$

then $W \cap \bigcup_{j=1}^{r} R_j \neq \phi$.

Proof. Suppose that $m = \max(m, n)$. The products $e_i \otimes f_j$, $i = 1, \cdots, m$, $j = 1, \cdots, r$, are linearly independent and span a space W_1 of dimension mr. Furthermore, $W_1 \subseteq \bigcup_{j=1}^{r} R_j$. Then $\dim(W_1 \cap W) = \dim W_1 + \dim W - \dim(W_1 \cup W) \geq mr + (mn - rm + 1) - mn = 1$. The result follows, since $W_1 \cap W \subseteq \bigcup_{j=1}^{r} R_j \cap W$.

LEMMA 5. *If $T(R_2) \subseteq T(R_2) \subseteq R_2$, then $T(R_1) \subseteq R_1 \cup R_2$.*

Proof. Suppose $x_1 \otimes y_1 \in R_1$, and choose $x_2 \otimes y_2 \in R_1$ such that $\rho(x_1, x_2) = \rho(y_1, y_2) = 2$. Then $\alpha = sT(x_1 \otimes y_1) + tT(x_2 \otimes y_2) \in R_2$ for all non-zero scalars s, t. Now suppose that $T(x_1 \otimes y_1) = \sum_{j=1}^{p} u_j \otimes v_j$, where $\rho(u_1, \cdots, u_p) = \rho(v_1, \cdots, v_p) = p$, and that $T(x_2 \otimes y_2) = \sum_{j=1}^{q} z_j \otimes w_j$, where $\rho(z_1, \cdots, z_q) = \rho(w_1, \cdots, w_q) = q$. Let $u_{p+1}, \cdots, u_m$ be a completion of $u_1, \cdots, u_p$ to a basis for U. It follows that

$$\sum_{j=1}^{q} z_j \otimes w_j = \sum_{j=1}^{m} u_j \otimes h_j$$

for some vectors $h_j \in V$, $j = 1, \cdots, m$. Then

$$\begin{aligned}\alpha &= \sum_{j=1}^{p} u_j \otimes sv_j + \sum_{j=1}^{p} u_j \otimes th_j + \sum_{j=p+1}^{m} u_j \otimes th_j \\ &= \sum_{j=1}^{p} u_j \otimes (sv_j + th_j) + \sum_{j=p+1}^{m} u_j \otimes th_j .\end{aligned}$$

Since $\alpha \in R_2$, it follows by Lemma 1 that

$$\rho(sv_1 + th_1, \cdots, sv_p + th_p) \leq 2 \quad \text{for} \quad st \neq 0 .$$

The vectors $sv_1 + th_1, \cdots, sv_p + th_p$ are linearly independent when $s = 1$ and $t = 0$. By continuity, they remain independent for small values of t. Hence $p \leq 2$ and $T(x_1 \otimes y_1) \in R_1 \cup R_2$.

THEOREM 2. *If* $T(R_2) \subseteq R_2$ *and* $0 \notin T(\bigcup_{j=1}^{4} R_j)$, *then* $T(R_1) \subseteq R_1$.

Proof. Suppose $x_1 \otimes y_1 \in R_1$ and $T(x_1 \otimes y_1) \notin R_1$. By Lemma 5, $T(x_1 \otimes y_1) \in R_2$, since $0 \notin T(R_1)$. Thus $T(x_1 \otimes y_1) = (u_1 \otimes v_1) + (u_2 \otimes v_2)$, where $\rho(u_1, u_2) = \rho(v_1, v_2) = 2$. Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ be bases for U and V respectively. Then for $st \neq 0$

$$sT(x_1 \otimes y_1) + tT(x_i \otimes y_j) \in R_1 \cup R_2 \quad \text{for } i = 1, \cdots, m,\ j = 1, \cdots, n. \tag{12}$$

At this point it seems simpler to regard the images $T(x_i \otimes y_j)$ as elements of M_{mn}. It is clear that there is no loss in generality in taking $T(x_1 \otimes y_1) = E_{11} + E_{22}$.

Let i and j be fixed for this discussion, and let $A = T(x_i \otimes y_j)$. Let $a_1, \cdots, a_n$ be the m-dimensional vectors which are the columns of A, and let ε_k be the unit vector with 1 in the kth position. It follows from (12) that

$$\rho(s\varepsilon_1 + ta_1, s\varepsilon_2 + ta_2, ta_3, \cdots, ta_n) = 2 \tag{13}$$

for $st \neq 0$. The Grassmann products

$$(s\varepsilon_1 + ta_1) \wedge (s\varepsilon_2 + ta_2) \wedge ta_k , \quad 3 \leq k \leq n \tag{14}$$

must be zero for $st \neq 0$. In the expansion of (14) the coefficient of s^2t is 0; that is, $\varepsilon_1 \wedge \varepsilon_2 \wedge a_k = 0$.

Thus the matrix A has non-zero entries only in the first two rows and columns. It follows immediately that the dimension of the range of $T \leq 2(m+n) - 4$. Hence the dimension of the kernel of $T \geq mn - 2(m+n) + 4 > mn - 4\max(m, n) + 1$.

By Lemma 4, there exists an element of $\bigcup_{j=1}^{4}$ whose image is zero. This contradicts the hypothesis; hence $T(R_1) \subseteq R_1$.

We see then that the form of T satisfying Theorem 2 is given in the conclusions of Theorem 1.

REMARK. We feel that the hypothesis $0 \notin T(\bigcup_{j=1}^{4} R_j)$ of Theorem 2 should not be necessary, but we have not been able to prove the theorem without it. More generally, we conjecture that $T(R_k) \subseteq R_k$ for some fixed k, $1 \leq k \leq n$, should suffice to prove that T is essentially a tensor product.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 9, No. 4 August, 1959