ON A CRITERION FOR THE WEAKNESS OF AN IDEAL BOUNDARY COMPONENT

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1. Exhaustion. Let $F$ be an open Riemann surface. An exhaustion $\{F_n\}$ of $F$ is an increasing (i.e., $F_n \subset F_{n+1}$) sequence of subregions with compact closures such that $\bigcup_{n=1}^{\infty} F_n = F$. We assume that $\partial F_n$ consists of a finite number of closed analytic curves and that each component of $F - F_n$ is noncompact. This is the most common definition used in the theory of open Riemann surfaces. Sometimes, however, we shall add the restriction that each component of $\partial F_n$ is a dividing cycle; if this is the case we shall call the exhaustion canonical.

2. Weak boundary component. Let $\gamma$ be an ideal boundary component of $F$, and let $\{F'_n\}$ be a canonical exhaustion of $F$. Then there exists a component $\gamma_n$ of $\partial F_n$ which separates $\gamma$ from $F'_n$. Let $n_0$ be a fixed number and consider the component $G_n$ of $F_n - F_{n_0}$ ($n > n_0$) such that $\gamma_n \subset \partial G_n$. There exists a harmonic function $s_n(p)$ on $G_n$ which satisfies the following conditions:

(i) $s_n = 0$ on $\gamma_{n_0}$ and $\int_{\gamma_{n_0}}^{} *ds_n = 2\pi$, ($\gamma_{n_0} = \partial F_{n_0} \cap \partial G_n$)

(ii) $s_n = \log r_n = \text{const.}$ on $\gamma_n$,

(iii) $s_n = \text{const.}$ on each component $\beta_{n\gamma}$ of $\partial G_n - \gamma_n - \gamma_{n_0}$ and $\int_{\beta_{n\gamma}}^{} *ds_n = 0$.

The condition $\lim_{n \to \infty} r_n = \infty$ depends neither on $n_0$ nor on the exhaustion. If it is satisfied, $\gamma$ is said to be weak.

Weak boundary components were introduced for plane regions by Grötzch [1] in connection with the so-called Kreisnormierungsproblem. He called them vollkommen punktförmig. They were generalized for open Riemann surfaces by Sario [6] and discussed also by Savage [7] and Jurchescu [2]. The above definition was given by Jurchescu [2].

A noncompact subregion $N$ whose relative boundary $\partial N$ consists of a finite number of closed analytic curves is called a neighborhood of $\gamma$ if $\gamma$ is an ideal boundary component of $N$ as well. Let $\{c\}$ be the family of all cycles $c$ (i.e., unions of finite numbers of closed curves) which are in $N$ and separate $\gamma$ from $\partial N$. Jurchescu [2] showed that $\lambda\{c\} = 0$ if and only if $\gamma$ is weak, where $\lambda\{c\}$ is the extremal length of the family $\{c\}$.

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3. Savage's criterion. Let \( \{F_n\} \) be an arbitrary exhaustion. Let \( E_n \) be the smallest union of components of \( F_n - \overline{F}_{n-1} \) such that \( \gamma_{n-1} = \partial E_n \cap \partial F_{n-1} \) is a cycle which separates \( \gamma \) from \( F_{n-1} \) \( (n = 2, 3, \ldots) \). Evidently \( \gamma_n \subset \partial E_n \). If \( \{F_n\} \) is canonical, \( E_n \) is connected and \( \gamma_n \) is a closed analytic curve.

There exists a harmonic function \( u_n(p) \) on \( E_n \) such that

(i) \( u_n = 0 \) on \( \gamma_{n-1} \) and \( \int_{\gamma_{n-1}} *d u_n = 2\pi \),

(ii) \( u_n = \log \mu_n = \text{const.} \) on \( \partial E_n - \gamma_{n-1} = \partial E_n \cap \partial F_n \).

The quantity \( \log \mu_n \) is called the modulus of \( E_n \) (cf. Sario [4,5], who called \( \mu_n \) the modulus). It is expressed in terms of extremal length as follows:

\[
\log \mu_n = \frac{2\pi}{\lambda \{\gamma_n\}}
\]

where \( \{\gamma\}_n \) is the family of cycles in \( E_n \) homologous to \( \gamma_{n-1} \).

Since \( \sum_{n=1}^{\infty} 1/\lambda \{\gamma\}_n \leq 1/\lambda \{\gamma\} \), we get the following criterion:

**Theorem 1** (Savage [7]). *If there exists an exhaustion such that \( \prod_{n=1}^{\infty} \mu_n = \infty \), then \( \gamma \) is weak.*

The purpose of the present note is to discuss the converse of this theorem.

4. Jurchescu's criterion. Suppose the exhaustion \( \{F_n\} \) is canonical. There exists a harmonic function \( U_n(p) \) on \( E_n \) such that

(i) \( U_n = 0 \) on \( \gamma_{n-1} \) and \( \int_{\gamma_{n-1}} *d U_n = 2\pi \),

(ii) \( U_n = \log M_n = \text{const.} \) on \( \gamma_n \),

(iii) \( U_n = \text{const.} \) on each component \( \beta_{n\nu} \) of \( \partial E_n - \gamma_n - \gamma_{n-1} \) and \( \int_{\beta_{n\nu}} *d U_n = 0 \).

Jurchescu's paper [2] contains implicitly the following result:

**Theorem 2** (Jurchescu). *A boundary component \( \gamma \) is weak if and only if there exists a canonical exhaustion such that \( \prod_{n=2}^{\infty} M_n = \infty \).*

**Proof.** *Sufficiency:* Let \( \{\gamma\}'_n \) be the family of cycles in \( E_n \) separating \( \gamma_n \) from \( \gamma_{n-1} \). It is not difficult to see that \( \log M_n = 2\pi/\lambda \{\gamma\}'_n \). Since \( \sum_{n=2}^{\infty} 1/\lambda \{\gamma\}'_n \leq 1/\lambda \{\gamma\} \), we conclude that \( \sum_{n=2}^{\infty} \log M_n = \infty \) implies \( \lambda \{\gamma\} = 0 \).

*Necessity:* Consider a canonical exhaustion \( \{F_n^0\} \). The desired exhaustion \( \{F_n\} \) is obtained by taking its subsequence as follows:

\( F_1 = F_1^0 \). To define \( F_2 \), consider the quantity \( r_n \) introduced in No. 2 with respect to \( F_n^0 - \overline{F}_n^0 \) \( (n = 2, 3, \ldots) \). Take \( n_2 \) so large that \( r_{n_2} \geq 2 \),
and put $F_2 = F_{n_2}^0$. Evidently $M_2 = r_{n_2}$. Similarly, $F_3 = F_{n_3}^0$ is defined by considering $F_{n_3}^0 - F_{n_2}^0 (n = n_2 + 1, n_2 + 2, \cdots)$ and by taking $n_3 > n_2$ so large that $r_{n_3} \geq 2$ where $r_n$ is the quantity $r_1$ introduced in No. 2 with respect to $F_n - F_{n-1}^0$. We have $M_3 = r_{n_3}$. On continuing this process, we obtain a canonical exhaustion such that $\sum_{n=2}^{\infty} \log M_n \geq \sum_{n=2}^{\infty} \log 2 = \infty$. The idea of this proof was first used by Noshiro [3].

5. The converse of Savage’s criterion. We shall now show that Savage’s criterion in Theorem 1 is also necessary.

**Theorem 3.** If $\gamma$ is weak, then there exists an exhaustion such that $\prod_{n=2}^{\infty} \mu_n = \infty$. It is not necessarily canonical.

**Proof.** By Theorem 2 there exists a canonical exhaustion \( \{F_n^0\} \) such that $\prod_{n=2}^{\infty} M_n^0 = \infty$. From this we construct a canonical exhaustion $\{F_n^*\}$ as follows:

\[ F_1^* = F_1^0. \]

To construct $F_2^*$, let $\partial E_2^0 - \gamma_2^0 - \gamma_2^0 = \beta_{21} \cup \beta_{22} \cup \cdots \cup \beta_{2k_2}$ be the decomposition into components, and let $H_3^*$ be the component of $F_3^0 - F_2^0$ such that $\partial H_3^* \cap \overline{F_2^0} = \beta_{2v} (v = 1, 2, \cdots, k_2)$. $F_2^*$ is the union of $F_1^*, E_2^0 \cup \gamma_2^0$, all the other components of $F_3^0 - F_2^0$, and $\bigcup_{v=1}^{k_2} H_v^*$. In this way, $F_n^*$ is defined as the union of $F_{n-1}^*, E_n^0 \cup \gamma_{n-1}^0$, every component of $F_{n+1}^0 - F_n^0 (m \geq n)$ which is adjacent to $F_{n-1}^*$, and $\bigcup_{v=1}^{k_n} H_v^*$. By construction, $E_n^* = E_n^0 \cup \bigcup_{v=1}^{k_n} H_v^*$.

The desired exhaustion $\{F_n^*\}$ is obtained by taking a refinement of $\{F_n^*\}$ as follows: Consider $E_n^0$ and the function $U_n^0$ for the exhaustion $\{F_n^0\}$. Let $\partial E_n^0 - \gamma_n^0 - \gamma_{n-1}^0 = \beta_{n1} \cup \beta_{n2} \cup \cdots \cup \beta_{nk_n}$ be the decomposition into components and let $U_n^0 = \alpha_v$ on $\beta_{nv} (v = 1, 2, \cdots, k_n)$. We may assume, without loss of generality, that the $\alpha_v$’s are different by pairs. We suppose that

\[ 0 = a_0 < a_1 < \cdots < a_{k_n} < a_{k_n + 1} \equiv \log M_n^0. \]

Take $a_v' (a_{v-1} < a_v' < a_v; \nu = 1, 2, \cdots, k_n, a_{k_n + 1} \equiv \log M_n^0)$ and $a_v'' (a_v < a_v'' < a_{v+1}; \nu = 1, 2, \cdots, k_n, a_0'' = 0)$ so close to $\alpha_v$ that

\[ (1) \quad \sum_{v=1}^{k_n+1} (a_v' - a_{v-1}') \geq \log M_n^0 - 2^{-\nu}. \]

Consider the sets

\[ D_n^\nu = \{ p; a_{v-1} < U_n^0(p) < a_v' \}, \quad \nu = 1, 2, \cdots, k_n + 1, (a_{k_n + 1}' \equiv \log M_n^0) \]

\[ D_n^\nu = \{ p; a_{v-1} < U_n^0(p) < a_v' \}, \quad \nu = 1, 2, \cdots, k_n + 1. \]

The modulus $\mu^\nu(p)$ of $D_n^\nu$ with respect to $\beta^\nu = \{ p; U_n^0(p) = a_{v-1}' \}$ and $\partial D_n^\nu - \beta^\nu$ is equal to $a_v' - a_{v-1}'$, since the function $U_n^0(p) - a_{v-1}'$ plays the role of $u_n(p)$ introduced in No. 3. Let $\mu^\nu(p)$ be the modulus of $D_n^\nu$.
with respect to $\beta^v$ and $\partial D^*_n - \beta^v$. Since $\mu^{(\gamma)} \geq \mu^{(\gamma)}$, we obtain, by (1),

$$
\sum_{n=1}^{k_n + 1} \log \mu^{(\gamma)} \geq \log M^0_n - 2^{-n}.
$$

We have decomposed $E^0_n$ into $k_n + 1$ subsets $D^*_n$. $E^*_n - E^0_n$ consists of components $H^*_n$ such that $\beta^\nu = \partial H^*_n + 1 \cap \partial E^0_n (\nu = 1, 2, \cdots, k_n)$. By decomposing $H^*_n$ into $k_n - \nu + 1$ slices, we obtain a decomposition of $E^*_n$ into $k_n + 1$ parts. It is possible to divide each of the other components of $F^*_n - \bar{F}^*_n$ into $k_n + 1$ pieces so that we get an exhaustion $\{F_n\}$ which is a refinement of $\{F^*_n\}$. $D^*_n$ plays the role of $E^*_n$ with respect to this exhaustion. Therefore, by (2), we get

$$
\sum_{n=2}^{\infty} \log \mu_n \geq \sum_{n=2}^{\infty} \log M^0_n - 1 = \infty.
$$

6. Remark. On a "schlichtartig" surface, every exhaustion is canonical. If $F$ is an arbitrary Riemann surface, the question arises whether or not Savage’s criterion is still necessary under the restriction that $\{F_n\}$ is canonical. The answer is given by

**Theorem 4.** There exist a $\gamma$ of an $F$ which is weak and such that

$$
\prod_{n=2}^{\infty} \mu_n < \infty \text{ for every canonical exhaustion.}
$$

Construction of $F$: In the plane $|z| < \infty$, consider the closed intervals

$$
I_k : [2^{k^2}, 2^{k^2} + 1] \quad (k = 2, 3, \cdots)
$$
on the positive real axis, and the circular arcs

$$
\alpha_v : |z| = \nu, |\arg z| \leq \frac{\pi}{2}
$$

$$
(\nu = 2^{k^2} + 2, 2^{k^2} + 3, \cdots, 2^{(k+1)^2} - 1; k = 2, 3, \cdots).
$$

Take two replicas of the slit plane ($|z| < \infty$) — $\cup_{k=2}^{\infty} I_k$ and connect them crosswise across $I_k (k = 2, 3, \cdots)$. From the resulting surface, delete all the $\alpha_v$’s on both sheets. This is a Riemann surface $F$ of infinite genus.

$F$ has an ideal boundary component $\gamma$ over $z = \infty$, which is evidently weak.

Let $\{F_n\}$ be an arbitrary canonical exhaustion. Consider $E_n$ corresponding to $\gamma$ (No. 3). The interval $I_k$ determines a closed analytic curve $C_k$ on $F$. Since $\gamma_{n-1} = \partial E_n \cap \bar{F}_{n-1}$ is a dividing cycle, the intersection number $\gamma_{n-1} \times C_k$ vanishes and, therefore, $\gamma_{n-1} \cap C_k$ consists of an even number of points whenever it is not void.* Take two consecutive points

* Added in proof. We should have mentioned the case where $\gamma_{n-1}$ tangents $C_k$. The following discussion covers this case if the number of the points of $\gamma_{n-1} \cap C_k$ is counted with the multiplicity of tangency and case $p=q$ is not excluded.
p and q in $\gamma_{n-1} \cap C_k$. There are two possibilities according as the arc $\overline{pq} \subset \gamma_{n-1}$ is homotopic to $\overline{pq} \subset C_k$ or not. If the latter case happens for at least one pair of p and q, we shall say that $\gamma_{n-1}$ intersects $C_k$ properly.

Since $\gamma_{n-1}$ is a closed curve separating $\gamma$ from $F_{n-1}$, there exists a number $k$ such that $\gamma_{n-1}$ intersects $C_k$ properly. If there is more than one $k$, we take the greatest one and denote it by $k(n)$.

To estimate $\mu_n$, let $\{c\}_n$ be the family of all cycles in $E_n$ separating $\gamma_{n-1}$ from $\partial E_n - \gamma_{n-1}$. We have mentioned that $\log \mu_n = 2\pi \lambda \{c\}_n$. Let $C_k$ be a curve for which there are numbers $n$ with $k(n) = k$. Evidently these $n$ are finite in number and consecutive. Let $n_k$ be the greatest.

I. If $k(n) = k$ and $n < n_k$ then $\gamma_{n-1}$ and $\gamma_n$ intersect $C_k$ properly. Since every $c \in \{c\}_n$ separates $\gamma_{n-1}$ from $\gamma_n$, it has a component which intersects $C_k$ and is not completely contained in the doubly connected region $A_k$ consisting of all points that lie over $\{z; 2^{k^2 - 1} < |z| < 2^{k^2 + 2}, |\arg z| < \pi/2\}$. Therefore, every $c$ contains a curve in $\{c'\}^{(k)}$ which is the family of all curves in the right half-plane connecting $I_k$ with the imaginary axis. Consequently

$$\sum_{\substack{n \neq n_k \ni k(n) \ni k}} \frac{1}{\lambda \{c\}_n} \leq \frac{1}{\lambda \{c'\}^{(k)}} .$$

II. $k(n) = k$ and $n = n_k$. Consider all the $\alpha_v (v \geq 2^{k^2} + 2)$ on the upper sheet. Let $G_{n-1}$ be the component of $F - \overline{F_{n-1}}$ such that $\partial G_{n-1} = \gamma_{n-1}$. For a sufficiently large $v$, $\alpha_v$ is an ideal boundary component of $G_{n-1}$. Let $v(k)$ be the least $v$ with this property. If $v(k) = 2^{k^2} + 2$, then every $c \in \{c\}_n$ separates $\gamma_{n-1}$ from $\alpha_{v(k)}$ and, therefore, it has a component intersects either $C_k$ or one of four line segments over $[2^{k^2 - 1}, 2^{k^2}]$ or $[2^{k^2 + 1}, 2^{k^2 + 2}]$. When $v(k) = 2^{k^2} + 2$ for some $l > k$, then $\gamma_{n-1}$ separates $\alpha_{v(k)}$ from $\alpha_{v(k)}$ and every $c \in \{c\}_n$ separates $\gamma_{n-1}$ from $\alpha_{v(k)}$, so that $c$ has a component with the above property. If $v(k)$ is not of the form $2^{k^2} + 2$, then, for the same reason, every $c \in \{c\}_n$ has a component which intersects the line segment on the upper sheet lying over $[v(k) - 1, v(k)]$, and is not contained in the simply connected region on the upper sheet consisting of all points over $\{z; v(k) - 1 < |z| < v(k), |\arg z| < \pi/2\}$. In any case, every $c \in \{c\}_n$ contains a curve in $\{c''\}^{(k)}$ which is the family of all curves in the right half-plane connecting $[v(k) - 3, v(k)]$ with the imaginary axis. Therefore,

$$\frac{1}{\lambda \{c\}_n} \leq \frac{1}{\lambda \{c''\}^{(k)}} .$$

By (3) and (4), we obtain

$$\sum_{n=2}^{\infty} \log \mu_n = 2\pi \sum_{n=2}^{\infty} \frac{1}{\lambda \{c\}_n} \leq 2\pi \sum_{k=2}^{\infty} \left( \frac{1}{\lambda \{c'\}^{(k)}} + \frac{1}{\lambda \{c''\}^{(k)}} \right) .$$
To show the convergence of $\sum_{k=2}^{\infty} 1/\lambda\{c^{(k)}\}$, we make use of the transformation $z \rightarrow z^2$. It is immediately seen that $\lambda\{c^{(k)}\}$ is equal to the extremal distance between $[-\infty, 0]$ and $I_k = [2^{2k^2}, (2^{2k^2} + 1)^2]$ with respect to the region $A = \{[-\infty, 0] \cup I_k\}^c$. Since $A$ is conformally equivalent to Teichmüller's extremal region $\{[-1, 0] \cup [P, \infty]\}^c$ where

$$P = \frac{2^{2k^2}}{(2^{2k^2} + 1)^2 - 2^{4k^2}},$$

we have (Teichmüller [8])

$$\lambda\{c^{(k)}\} \sim \frac{\log P}{2\pi} \quad (P \rightarrow \infty)$$

$$\sim \frac{k^2 \log 2}{2\pi} \quad (k \rightarrow \infty),$$

and, therefore, $\sum_{k=2}^{\infty} 1/\lambda\{c^{(k)}\} < \infty$. Similarly $\sum_{k=2}^{\infty} 1/\lambda\{c^{(k)}\} < \infty$ because $\nu(k) \geq 2^{2k^2} + 2$. We conclude that

$$\sum_{n=2}^{\infty} \log \mu_n < \infty.$$

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